Photo 1 Castle Shirasagi, Himeji

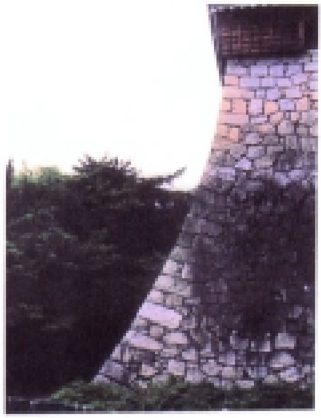

Photo 2 "Sori" curve of the jacket wall

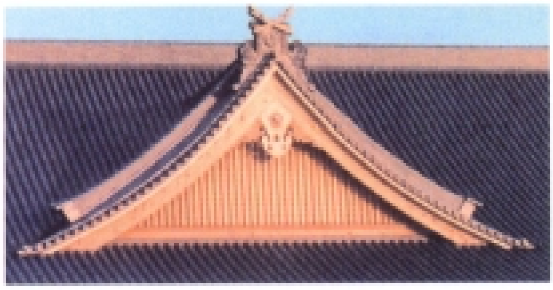

Photo 3 "Chidori" gable

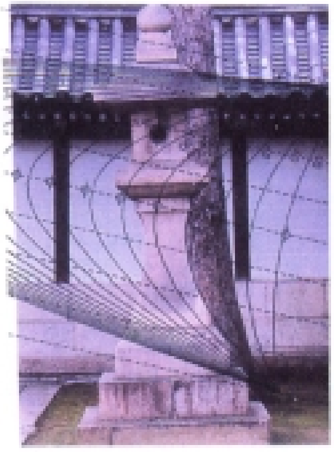

Photo 4 Curve of a stone lantern

Some examples of curves in Japanese traditional structures. For detail see the paper by H. Yanai (pp.177–186).

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Parabola Drawing Methods in Traditional Japanese Architecture —A Hypothesis in Technological Genealogy

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Abstract. A drawing procedure of the curves of stone jacket walls in Japanese castles is described in manuscripts of Gotoh family written in early 17th century. This procedure is based on constructing a parabola with horizontal axis as an envelope of a set of straight lines made by spanned strings. The origin of such a procedure is surmised to be in “Ying zao fang shi” published in 1078 in China in which an envelope is used to construct the curve of the wooden support between pillar and beam. The validity of this conjecture is to be discussed in this paper.

1. Introduction

There remain many feudal age castles in Japan. Some of them, like famous Shirasagi castle located at Himeji (Photo 1, see p. 175), are well conserved, while others are ruins. The castles with curved roofs, whitewashed donjon and also with curved jacket walls, together with the history, attract people as their tourism targets.

Most of the jacket walls of Japanese castles were built as stone covering on steepened slopes of hills upon which the keeps were settled. They are shaped into concave, peculiar curves called technically “sori” (Photo 2, see p.175).

Why are they curved? Some people seek for the reason in highly moisturized soil in rainy seasons which would imperil the physical stability of the walls. Some people seek for the aesthetic reasons: the curves of the jacket walls must harmonize with the curves of the roofs. There are also some others (YANAI, 1993, 1999) who seek for the metaphysical connotations. But all these subjects are beyond the scope of this paper.

In this paper, we concentrate our attention to a constructing, namely, drawing procedure of such curves. The method to be discussed in this paper is the one described in manuscripts of Gotoh family written in 17th century. Gotoh family was a samurai family which belonged to Kaga clan (now Kanazawa prefecture) as professionals in jacket wall building (KITAGAKI, 1987).

2. Procedure in Gotoh's Manuscripts

According to Gotoh's manuscripts, the stones are piled as follows in order to form the curved jacket walls.

Let the hypothenuse of right angled triangle ABC be the slope of the hill to be covered by stones. The real scale of the triangle shown as the fundamental example in those manuscripts is h = 8.2 joh ≈ 24.8 m (joh = Japanese classical unit of length) in height and

$$\overline{BC} = \alpha \times \overline{AB}, \quad \alpha = 0.73 \tag{1}$$

where the coefficient α is remarked as a "secret figure" bequeathed only for the descendants of the Gotoh family.

The upper 2/3, the interval AE is to be curved. Divide the interval AE into n equal subintervals, where n = 14 in the fundamental example in the original manuscripts, while, in Fig. 1, the case of n = 4 is shown for the sake of illustration. The horizontal straight line passing through the i-th node Q_i from E is represented by ℓ_i.

On the other hand, we put a horizontal plank AD at the top of the slope, on which n + 1 points R_0, R_1, ..., R_n are marked, where

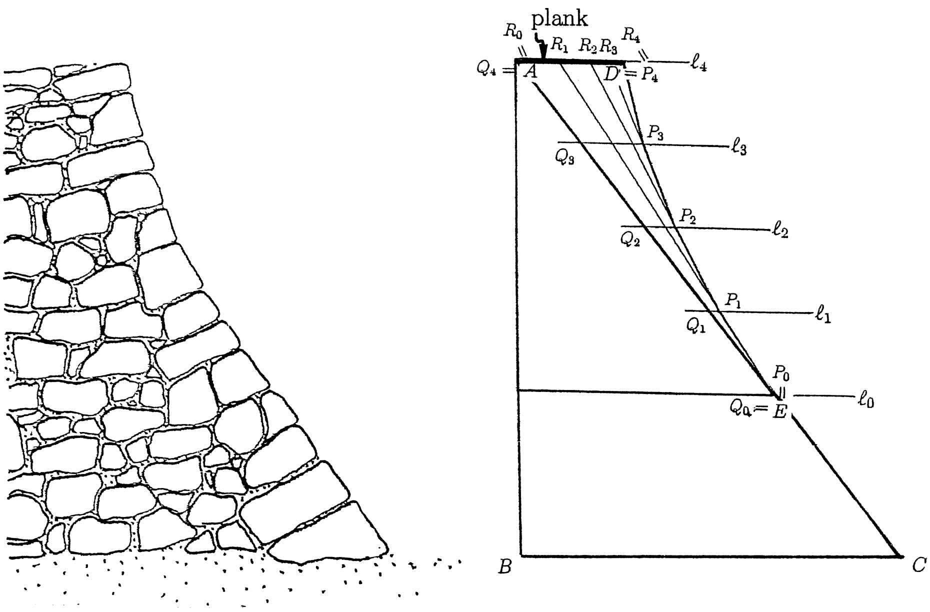

Fig. 1. Jacket wall building procedure in Gotoh's manuscripts.

$$\overline{AR_n} = \beta \times \overline{BC}, \quad \beta = 0.24. \tag{2}$$

The coefficient β is remarked as another secret figure in the manuscript. The distances between succeeding points constitute a diminishing arithmetic progression.

Now, stretch a string from R_1 to Q_0 = E. Represent by P_1 the point of intersection between this string and the level line ℓ_1. We pile up stones along this string one another until the point P_1 is attained.

Then we stretch a new string from P_1 to R_2. Again we pile the stones along this string until the point P_2 is attained, where P_2 is the point of intersection between this string and the level line ℓ_2.

Following such procedure until the highest level is attained, we complete the curvature of the whole wall, more exactly, the continuous line segments constituting the wall. However, if we increase the number n ad infinitum, these line segments will be regarded as a curve of envelope of the strings according to the concept of modern western mathematics.

Such an envelope is represented by a quadratic equation (YANAI, 1988),

$$y = \frac{bu}{h} + d\left(1 - \frac{3u}{2h}\right)^2, \tag{3}$$

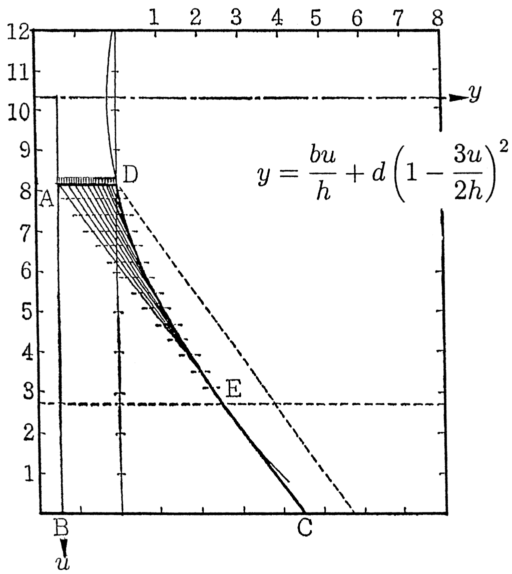

Fig. 2. Jacket wall curve as a parabola.

where y is the abscissa, u is the ordinate which goes downwards,

$$b = \overline{BC}: \quad \text{the width of the base of the fundamental triangle,} \tag{4}$$

$$d = \overline{AR_n}: \quad \text{the length of the plank,} \tag{5}$$

$$h = \overline{AB}: \quad \text{the height.} \tag{6}$$

In the lower part of Fig. 2 the curve of this equation is overlaid on the original continuous line segments with 15 nodes according to Gotoh's manuscripts.

3. Juan Sha Method

Parabolas were so constructed in Gotoh's manuscripts as described above. It seems to be, indeed, a very sophisticated method, indeed, too elegant to arrive at such a method without knowledge of modern mathematics; the author has been wondering how they came up with such an elaborate idea.

There existed traditional Japanese mathematics at that time. Arithmetic and geometric progressions and also trigonometric functions were known, although we measured the angles by their tangents and cotangents. Determinant theory to solve systems of linear equations, differential and integral calculus were also developed. But since Japanese mathematics was unfortunately not attached with kinematics, the idea of differential equation was not known, much less the envelopes.

Hence, there arises a question how they are arrived at such an elaborate procedure without knowledge of modern mathematics. It might be more natural to presume that Gotohs had some knowledge not only of Japanese mathematics, but also about preceding techniques, than to presume, from purely logical point of view, that Gotohs, had created everything from nothing.

Recently, however, the author found a hint to set up a conjecture. The hint was in "Ying zao fang shi" by Lie Jie, a Chinese canonical textbook on architecture published in 1078, 11th century, under the permission of the Emperor Hui Zong of China (TAKESHIMA, 1970–1972; KITAO, 1999). In this book, a method called "juan sha" is described to use an envelope to construct curves of wooden support between pillar and beam. Equidistant points are marked along each of two rectangular edges of a wooden block. The straight lines connecting the points on the two edges are sawn to constitute a parabola as the envelope and a convex parabolic wooden support is obtained (Fig. 3).

The angle between the two edges of the original wooden block is rectangular in most of the cases. However, if we broaden the angle we obtain various parabolas of various curvatures, among which we would be able to find some suited for jacket walls. But since the curve of a jacket wall is a part of the half parabola, it is necessary to cut out a part from the envelope. And in order to map the straight lines to real scale of the jacket wall, the end points of the strings must be determined. Lower ends are equidistantly distributed in heights. As for the uppers, horizontal distances among the straight lines, as it is inferred, are measured at the height corresponding to the plank to know to constitute a diminishing

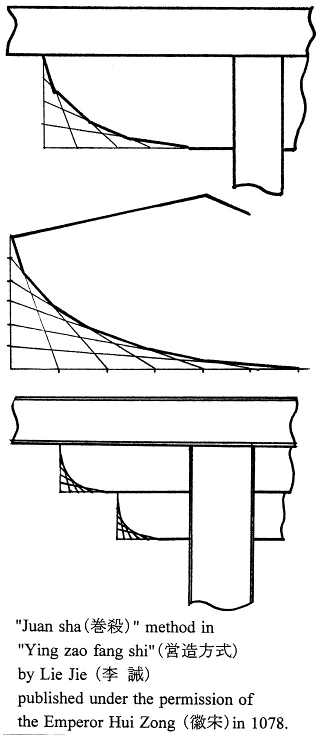

Fig. 3. "Juan sha" method.

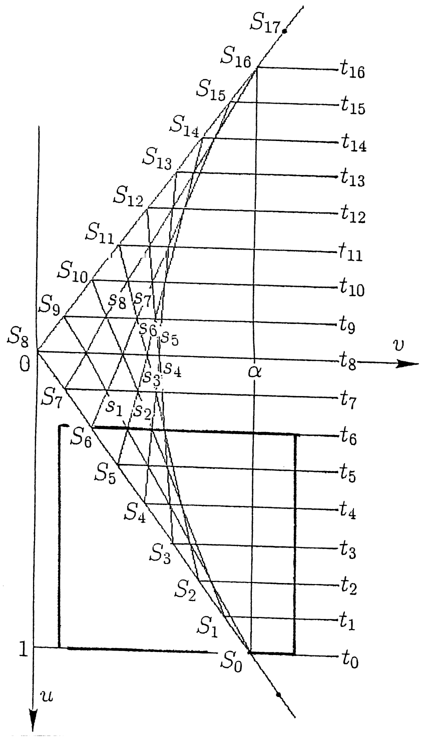

Fig. 4. Parabola as an envelope.

arithmetic progression (Fig. 4).

More precisely, let OS_0S_{2m} be an isosceles triangle with the vertical base; the lower side OS_0 is parallel to the slope of the hill. Selecting three numbers *m*, *n* and *k*, such that

$$m = 2n + k, \tag{7}$$

we divide the side S_0O and OS_{2m} into *m* subinterval of the equal length respectively and denote the node by S_i ($m = 8$, $n = 3$, $k = 2$ in Fig. 4). Connecting S_{i-1} and S_{2m+i} to construct the network of lines s_i's in the triangle, we obtain a parabola with horizontal axis as the envelope; the mesh sizes in the network diminishes arithmetically from the left to the right. The lower part of the network from S_0 up to the node S_{2n} constitutes the structure of the curve of the jacket wall.

4. A Counter Hypothesis

Since this book, "Ying zao fang shi" was widespread and "juan sha" method was of course known among Japanese architects in 17th century, the author set up a hypothesis that

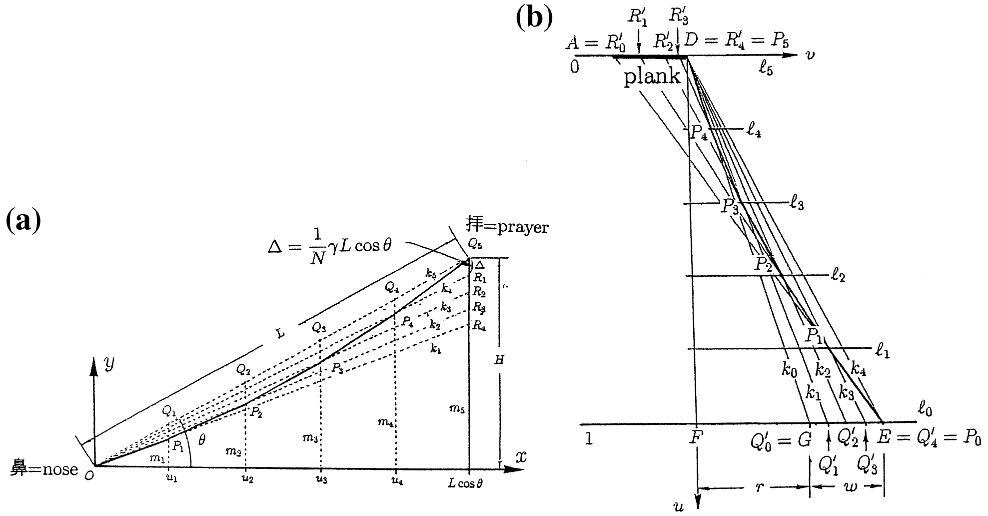

Fig. 5. a) Drawing procedure for Chidori gable. b) Used also for jacket walls?

Gotohs came up to the idea of constructing parabolas in this way. However, this hypothesis is nothing more than a conjecture and there is no way to give an evidence today. It is necessary to try a debate with counter hypothesis.

A possible counter hypothesis which seems to be strong enough is a method to construct curves of "Chidori gable" which forms also a parabola by, so to say, a "gradation" method (MORINAGA, 1975, Photo 3, see p. 175, Fig. 5(a)).

Described in terms of western mathematics, the first procedure begins with setting the coordinate system with horizontal abscissa and vertical ordinate with the origin at the lower end of the curve, which is technically called the "nose". According to the scale of the gable, we determine the beeline length L, the height H and thus the corresponding summit which is technically called the "prayer".

Divide the basis by N equally spaced vertical straight lines m_1, m_2, ..., m_N, numbered from the "nose", where N is say 5. While, beginning at the "prayer", the summit, take equally spaced points R_0, R_1, ..., R_{N-1} downwards, where the distance between the neighbouring points is

$$\Delta = \frac{1}{N}\gamma L\cos\theta \tag{8}$$

with a parameter γ. Then draw straight lines connecting the "nose" O and these points. The straight lines are represented by k_1, k_2, ..., k_{N-1} numbered in reverse order.

Now, beginning with the "nose" O, we connect the points of intersection between the vertical line m_i and the inclined straight line k_i to draw line segments. The curve of the gable is bow down by thus obtained continuous line segments.

The limit of this continuous line segments with N, the number of nodes, ad infinitum is a parabola with a vertical axis, which is represented by a quadratic equation of x (YANAI, 1991),

$$y(x) = \frac{x}{\cos\theta}\left(\sin\theta - \frac{(L\cos\theta - x)}{NL}\right), \tag{9}$$

where θ is the angle of elevation of the beeline. Figure 5(a) shows the parabola overlaid on the original continuous line segments with $N = 5$. Either of them seems to coincide each other.

It is to be remarked also, that this procedure is regarded as a graphical solution of a differential equation

$$\frac{d^3y}{dx^3} = 0 \tag{10}$$

with boundary conditions at the both ends, the nose and the prayer,

$$y(0) = 0, \tag{11}$$

$$y(L\cos H) = H, \tag{12}$$

and with an initial condition as to the inclination at the nose,

$$y'(0) = \frac{1}{\cos\theta}\left(\sin\theta - \frac{1}{N}\cos\theta\right). \tag{13}$$

Rotating this scheme by 90° clockwise, we obtain a parabola with a horizontal axis, from which curves of jacket walls can be obtained. In order to obtain the points on the plank, however, the vertices of the equi-spaced end-points at the bottom must be connected and extrapolated to the level of the plank, as shown also in Fig. 5(b). The parabola drawing method for the second hypothesis is thus obtained.

5. Comparison of Hypotheses

We are now in the stage to compare these two hypothesis, whether Gotohs came up to their idea of constructing curves of jacket walls by “juan sha” or by “gradation” method.

5.1. Technological genealogy

First point of view is technological genealogy. The year of publication of “Ying zao fang shi” is exactly known to be in 1078; widespread as the canonical text book. It is quite natural to presume the Japanese architects in 17th century knew the technique. It is also possible to presume that, even before that book, this procedure was widely known among East Asian carpenters, since “Ying zao fang shi” was an authorized canonical textbook and could have been a collection of the procedures followed in those days. Moreover, Gotoh family belonged originally to ancient “Anoh” clan specialized in masonry and presumably of Korean origin which spread to many samurai clans as professionals in the Middle-Ages (KITAGAKI, 1987).

On the other hand, the literature about the gradation method in “Chidori gable” is rather new so far as the author is aware of. The book in which this method was shown (MORINAGA, 1975) is a reprint of a publication whose original could not be identified; the publisher whom the author visited directly knew nothing about the author. It is inferred, however, that the original publication is about 80–90 years old estimated from its style. This procedure was also searched among older textbooks (e.g. KOBAYASHI, 1857) published in Edo era (1600–1867) but in vain. Anyway, the philological genealogy could not be traced, although it is highly possible that the origin of the method can be traced at least in Edo era which was one of the cultural flowering period in Japan. But even so, there is no evidence for this method to be old enough that Gotohs found the hint to devise their procedure.

5.2. Naturalness in the way of thinking

The next point of view is the naturalness in the way of thinking. Simple gradation is a more natural way of thinking in forming concave curves than envelopes, although 90° rotation is necessary to set up the curve in the right position for the jacket walls. Indeed, we can find such procedures in standard textbooks of higher mathematics (SMIRNOV, 1965).

“Juan sha” hypothesis, on the other hand, has some difficulty to be created originally. But if the original envelope drawing method is known, it remains only to broaden the angle between two sides and cut out a part.

5.3. Avoidance of “Kensaki”

In some part of the Gotoh’s manuscripts it is written that “kensaki” are to be avoided in building jacket walls of castles. The “kensaki” is the crossing point of the extensions of the curves of the jacket walls in the upper space. It is stressed in the manuscript, although its metaphysical connotation is not to be well understood nowadays.

If the stones of the jacket wall were piled up along straight lines with a common inclination, the extension will cross in the upper space with that of the jacket wall in the opposite side of the castle. Hence, if Gotohs followed “juan sha” method, it is well understood that they made efforts to avoid “kensaki” in this way, while if the gradation method was followed, it is rather weak to explain the relation between the aim and mean.

5.4. Selection of parameters

Let us now trace the way of thinking in Gotohs manuscripts in selection of the parameter n, the number of horizontal lines or the number of the scaffolds. Japanese people at that time were very cautious about selecting numbers. Round and neat numbers were generally preferred as we do today. Small odd numbers like 1, 3, 5, 7 were recognized to be good numbers; as an exception, 8 was also a good number. Generally, numbers resembling ominous words in sound were carefully avoided, and so forth. It resembles that of the western culture in which 13 is avoided. These might be superstitions. But we can use this cultural anthropological point of view, in order to check the naturalness of the algorithm.

In the fundamental example in Gotoh manuscripts n is selected to be 14 for the case of the slope of height h = 8.2 joh ≈ 24.8 m, which corresponds to the vertical distance between succeeding stages of the scaffold to be 1.2 m. This distance, 1.2 m, is a reasonable height considering human heights to pile up the stones. So there is no unnaturalness from engineering point of view.

The number 14, however, seems to be a rather clumsy selection—why not 15? But, if we think of their numbering custom, the horizontal lines must have been counted from 1, not from 0 as mathematicians do today. According to the former, the plank is at the 15th horizontal line, which is quite natural and can be well accepted.

In this sense, we infer that there must have been some reasons for Gotohs to have concentrated on the selection of number of the subintervals. This is not so significant if gradation method is followed; we can simply select "good" numbers as parameters. But, if the method of "juan sha" is followed, selection of a parameter is linked to other parameters. So it is necessary to select the parameter very carefully so that all other linked parameters would be "good" numbers, too. We can examine our hypothesis also from this point of view.

More precisely, in the case of "juan sha", the number of scaffolds is connected not only to the total number of the end points of the strings, but also to the total number of the lines in the fundamental scheme of the network (Fig. 4). Although the explanation of the process is slightly complicated, we can calculate back (YANAI, 2001) the number m, the total number of the meshes, from n the number of the scaffolds, α and β, the parameters to determine the proportion of the fundamental triangle.

For n = 14, the corresponding m is calculated to be m = 39 and k = 11, which means the vertex S_m is the 40th node counted from the bottom, which seems to be also a natural and practical selection. It should be reminded here, however, that the division by 39 is not necessary in the draughting process, we simply need to measure a same distance 39 times, although division by 39 was not so difficult if an abacus is applied. The condition k = 11 means that the plank is on the 12th mesh line counted from the vertex S_m downwards. The number 12 was not included in the set of bad numbers (cf. the 12 signs of the Chinese and Japanese zodiacs).

5.5. Draughting feasibility

Presumably, Gotohs might have found out the arithmetic rule of diminution in the sizes of the meshes by directly measuring the distances on the draught. Precise drawing is necessary, although large numbers for m are not always necessary in order to find the fundamental rule. High precision is obtained easier by interpolations as in "juan sha" method, than extrapolations of line segments for long stretches as necessary in gradation method, especially if only primitive plotters like inked strings, thin brushes and bamboo pens are available.

5.6. *Masonry arts*

Now, on the other hand, we can see many parabola or parabola-like curves in Japanese traditional masonry work, stone lanterns for example as in Photo 4 (see p. 175). The author presumes that such curves were also drawn originally by the envelopes. If gradation method is followed, we have to draw the curve also to the opposite side and it is rather difficult to image the whole curve. If the former is the case, however, it is highly possible that Gotohs as masonry professionals were well acquainted with envelopes, which would support "juan sha" hypothesis.

Indeed, the author inquired with a help by K. Miyazaki (Professor of Kyoto University), not only many masonry shops and studios in Kyoto, the most traditional city of Japan, but also museums specialized in masonry arts. But the existence of a metal template was all what was confirmed; it was not known how the template was shaped. Today, many masons shape their lantern after old ones. However, although any physical evidence is not found, the author has not yet given up this presumption.

6. Concluding Remark

From all those reasons, comparing two hypotheses, the author supports the hypothesis that Gotohs are come up with the idea of their procedure based on "juan sha" method described in "Ying zao fang shi", although it still remains as a hypothesis.

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