Random Sequential Covering of a Sphere
with Identical Spherical Caps

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Abstract. The coverage problem on the sphere is studied. We consider the sequential covering of identical spherical caps, so that none of them contains the center of another one. Set of the center points of such caps is said to form a Minkowski set. We give an algorithm of random sequential covering under the condition of Minkowski set.

Problems of packing or covering unit sphere with many identical spherical caps are applied to the globular histogenesis models of the biological organisms, to the uniform arrangement of observatories on the earth, to the information theories, and so on. These problems have different features from the similar problems with respect to the plane, and many of them are unsolved (G. FEJES TÓTH, 1969; L. FEJES TÓTH, 1972).

The coverage on the sphere is concerned mainly with the probability of complete coverage of the whole spherical surface. In the field of discrete mathematics, the interest is centered on the probability under the condition that the centers of spherical caps are chosen independently at random on the surface of the sphere (MAEHARA, 1988). A problem here is that the spherical caps may be put in the region already covered more than twice. Then, the coverage density often becomes too high.

In this paper, we consider a sequential covering of spherical caps, where none of them contains the centers of other ones. Set of such center points is said to form a Minkowski set (L. FEJES TÓTH, 1999). We propose a method to get a Minkowski set by putting centers on the perimeter of already placed spherical caps as in the following steps (see Fig. 1):

STEP 1: The center of the first spherical cap is put at $(x, y, z) = (0, 0, -1)$.

STEP 2: The center of the second spherical cap is chosen uniformly at random on perimeter of the first spherical cap. The center of the third spherical cap is chosen uniformly at random on the part of the perimeter of the first one which is not covered by the second one. Repeat this procedure, until the spherical caps cover the whole perimeter of the first spherical cap.
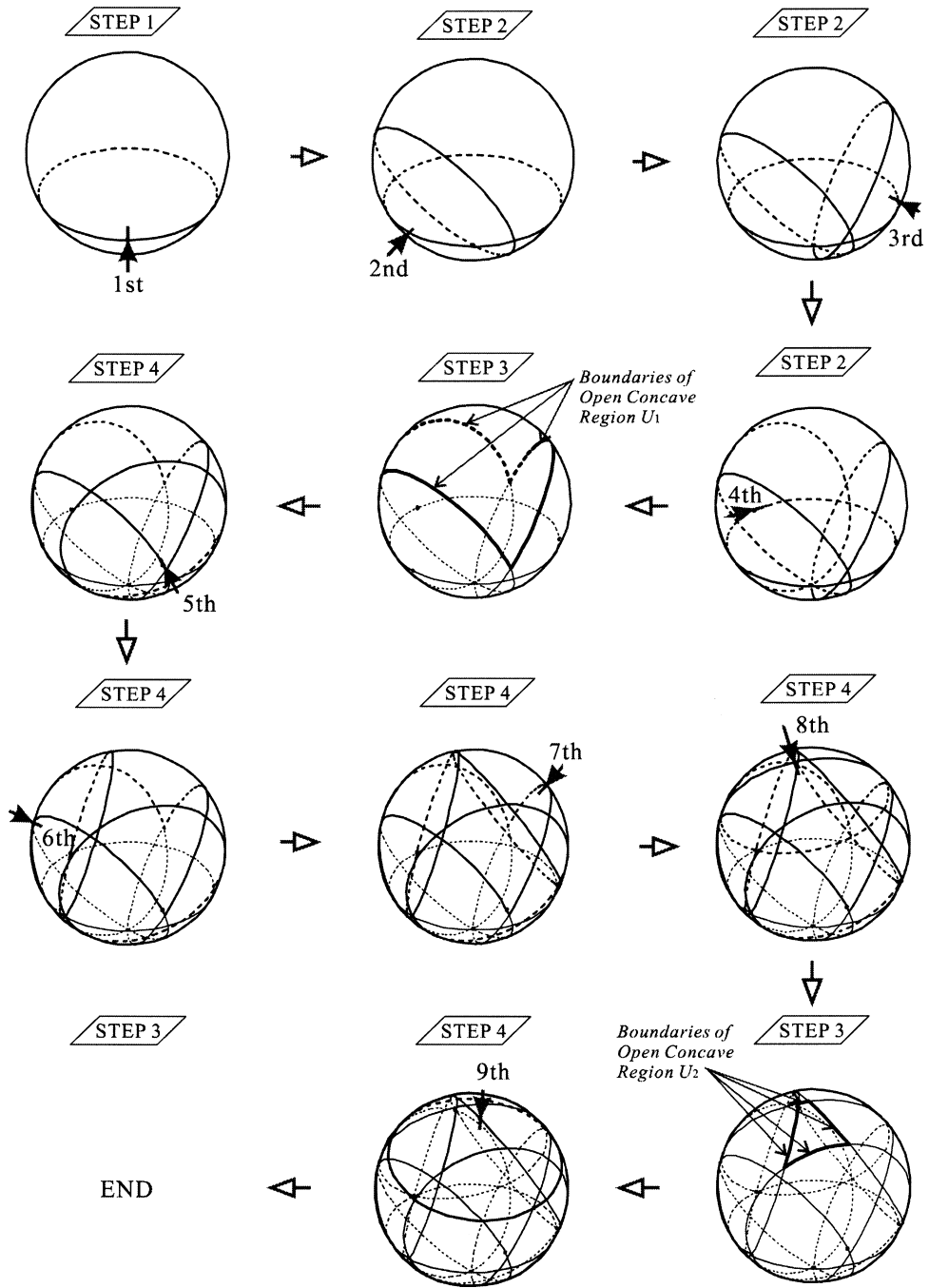
T. Sugimoto and M. Tanemura

STEP 1

STEP 2

STEP 2

1st

2nd

3rd

STEP 4

STEP 3

STEP 2

Boundaries of
Open Concave
Region U1

5th

4th

STEP 4

STEP 4

STEP 4

6th

7th

8th

STEP 3

STEP 4

Boundaries of
Open Concave
Region U2

STEP 3

END

9th

Fig. 1. The algorithm of our random sequential covering.

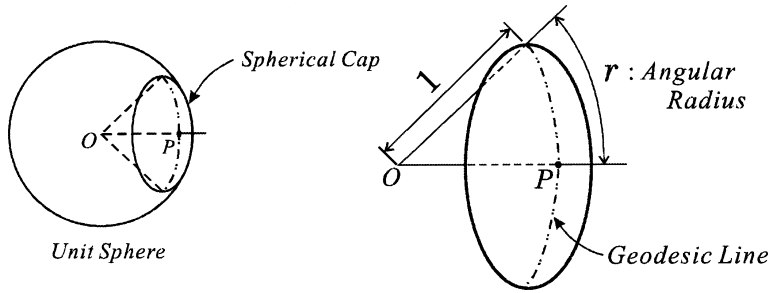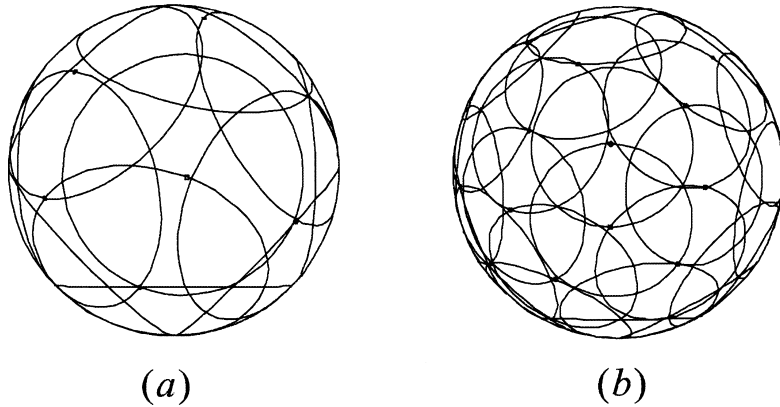Fig. 2. Spherical cap of the angular radius r.

Fig. 3. View point: (−10, 0, 0). (a) $r = 0.800$ rad, $N = 17$. (b) $r = 0.500$ rad, $N = 43$.

STEP 3: If the whole sphere has been covered, the procedure ends; otherwise go to the next step. We call the uncovered regions open concave regions. Denote them by U_i, $i = 1$, ..., k; k is the number of open concave regions in this step.

STEP 4: The center of a new spherical cap is chosen uniformly at random on the boundaries of the open concave regions (U_1, ..., U_k) (Let us note that the boundary curves consist of the perimeter of spherical caps and that they are not geodesic curves.) Repeat this process until the boundaries of the open concave regions (U_1, ..., U_k) are all covered. Go to the STEP 3.

Note that in the present method, new centers are chosen on the boundaries of concave regions (STEP 3). There is another possibility of choosing centers; namely, centers of spherical caps are chosen uniformly at random from all boundaries of the open concave regions.

We obtained (theoretically) the angular radius and the positions of spherical caps such that the economical efficiency is the worst, in other words, a Minkowski set with the highest density. Incidentally, the solutions for spherical cap numbers 1, 2, 3, 4, 5, 6, 7, 10, 12 and
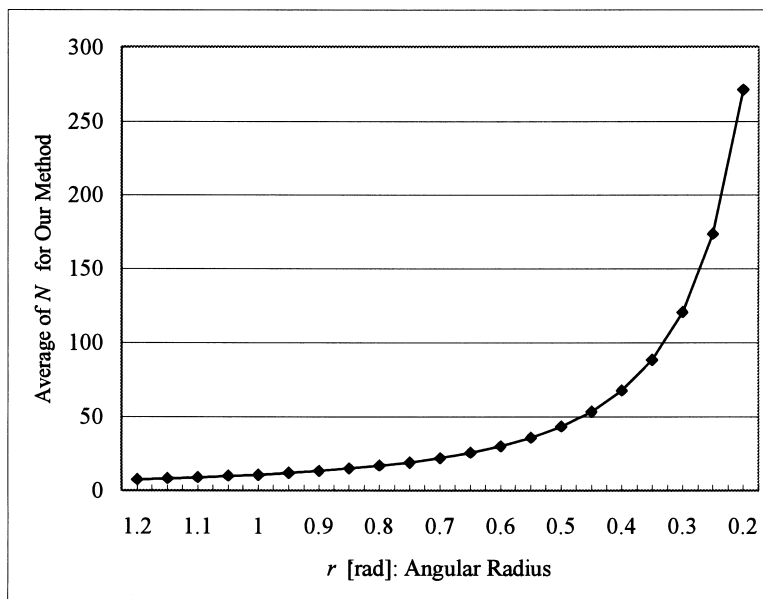
Fig. 4. The result of our computer simulation.

14 were derived and proved when the economical efficiency is the best in the sequential covering.

In our computer simulation, the angular radius *r* of spherical caps (see Fig. 2) is fixed. The number of spherical caps *N* which cover the whole spherical surface is obtained. Sample size is 10,000.

Figure 3 shows that the example of computer simulation according to our method for *r* = 0.800 and 0.500 rad.

Figure 4 demonstrates the result of our computer simulation. The horizontal and vertical axes of Fig. 4 report the angular radius and the average of *N* for our method, respectively.

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