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Curie Symmetry Principle in Nonlinear Functional Systems

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Abstract. The Curie symmetry principle states that the effect may occasionally have the same or a higher symmetry than the causes. But breaking this principle occurs in some nonlinear phenomena such as buckling of a cylinder. Sattinger's group theoretical method mathematically explains that these phenomena dominated by nonlinear functional equations break the Curie principle. The result of the group theoretical method manifests that both cases preserving and breaking the Curie principle occur in the same nonlinear system.

1. Introduction: The Curie principle

The concept "symmetry" began to be introduced in interpretations of physical phenomena by mineralogists and physicists in the middle of the 19th century (BRANDMÜLLER, 1986). For the symmetry of causes and effects of physical phenomena, CURIE (1894) proposed a causality relation, which is called "the Curie principle" nowadays. According to CURIE (1923), he generalized his principal statements to a general law composed of the following three parts:

- *If certain causes yield the known effects, the symmetry elements of the causes should be contained in the generated effects.*
- *If the known effects manifest certain dissymmetry (absence of symmetry elements), this latter should be contained in the causes which have generated those effects.*
- *The converse to these two previous propositions is not true, at least in practical: i.e., the effects may have higher symmetry than the causes which generate these effects.*

The Curie principle had been mentioned and restated by several researchers (JAEGER, 1920; PRIGOGINE, 1947; SHUBNIKOV, 1956; COTTON, 1963; NYE, 1969; CHALMERS, 1970; SHUBNIKOV and KOPTSIK, 1972; KOPTSIK, 1983; BRANDMÜLLER, 1986; ISMAEL, 1997; NAKAMURA and NAGAHAMA, 2000). The Curie principle is always correct in the phenomena dominated by linear process like the piezoelectric effect. Thus, the principle is still common assumption and powerful constraint to infer the composite causes (NYE, 1969).

On the other hand, several nonlinear phenomena, which are contradictory to the Curie principle, have been reported in relatively newer studies. STEWART and GOLUBITSKY (1992) discussed the symmetry breaking phenomena in which "*symmetric cause can have* *asymmetric effects*". They gave various examples opposing the Curie principle, such as a droplet (the cause) and a rising splash (the effect) like a milk crown, buckling of a cylinder loaded uniformly (the cause) and formed dimples (the effect) on it. But they did not state on breaking of the Curie principle by an explicit mathematical form. They, however, referred an instability of a nonlinear system as the cause of the symmetry breaking. Instability and symmetry change are mathematically discussed for a nonlinear system expressed in a functional form, in association with group theory (SATTINGER, 1977, 1978, 1979; FUJII and YAMAGUTI, 1980; GOLUBITSKY and SCHAFFER, 1985; GOLUBITSKY *et al.*, 1988).

In this paper, we show that breaking of the Curie principle can be derived from nonlinear functional equations mathematically by the use of Sattinger's group theoretical method. This does not deny an existence of the Curie principle, rather manifests both preserving and breaking the principle occurs in one nonlinear system.

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2. Sattinger's Nonlinear Functional Analysis

2.1. Functional

SATTINGER (1973) studied bifurcation and instability of solutions for a nonlinear functional equation. Nonlinear functional equations are represented in a following form:

$$F(\mu, w) = 0 \tag{1}$$

where F is a mapping (i.e. F: $\mathbf{R} \times B \to B$), $w = w(\mu)$, $\mu \in \mathbf{R}$ (real number) and $w \in B$ (Banach space). It denotes that w is defined on a Banach space B and depends on a real parameter μ, and that F maps w to new w with the change of μ.

Later SATTINGER (1979) extended his analysis in association with group theory. He investigated the functional equations whose mapping F is covariant under group G (an aggregate of transformations), i.e.

$$F(\mu, gw) = gF(\mu, w) \tag{2}$$

where g is an element of a set G.

In this paper, we consider a unitary group because the unitary group preserves the inner product as an invariant. Under this assumption, we have to analyze functional equations on Hilbert space H.

2.2. Standard decomposition

Let us consider a space on which the functional and its variables are defined. To analyze a functional by its composition, we have to decompose the space into subspaces. By applying the group representation theory, we can decompose the space H into subspaces

(SATTINGER, 1979). This decomposition is called “standard decomposition” in group representation theory. The standard decomposition generates the subspaces H^α which correspond to a representation of irreducible element $g_\alpha \in G$ and intersects orthogonally each other with symmetrical basis (SATTINGER, 1979; FUJII, 1980; FUJII and YAMAGUTI, 1980). Each representation of g_α handles elements in H^α and a projection P^α onto H^α can be defined (see for detail FUJII, 1980; FUJII and YAMAGUTI, 1980). Especially, in the case of $\alpha = 1$, g_α corresponds to an isotropy group and $T_g P^1 = P^1$ for the transformations T_g ($g \in G$). On the contrary, g_α ($\alpha = 2, 3, ..., q$) correspond to the other subgroups.

Unitary group is isomorphic with a general linear group (a set of linear mapping) and therefore each element of it can be represented as a matrix form. Thus, we consider Eq. (1) in a matrix form in this paper. This procedure does not loose generality.

2.3. Fréchet derivative

In the functional analysis the “Fréchet derivative” $F'(\mu_0,w_0) \equiv \partial F/\partial w$ is known to exist, if

$$\frac{\left\| F(\mu, w) - F(\mu_0, w_0) \right\|}{\left\| (\mu, w) - (\mu_0, w_0) \right\|} \to T \quad \text{for} \quad \left\| (\mu, w) - (\mu_0, w_0) \right\| \to 0 \tag{3}$$

where T is a finite linear operator, $(\mu_0,w_0) \in (\mu,w)$ denotes a point at which Fréchet derivative is defined, and $\|\cdot\|$ represents the norm standing for a distance. Fréchet derivative can be considered as representing a local mapping in the neighborhood of a point. If reducible $F'(\mu_0,w_0)$ exists, the implicit function theorem assures that Eq. (1) has a unique solution in the neighborhood of (μ_0,w_0). If F' is irreducible at a certain point, this point is singular and several solutions exist in its neighborhood. In other words, the solution path $w = w(\mu)$ of Eq. (1) is nonlinear at a singular point. For simplicity, we consider only a simple singular point in this paper. See also IKEDA *et al.* (1991) dealing with a multiple singular point.

Taking advantage of Sattinger’s analysis, we can consider both the symmetry preserving and breaking cases in nonlinear functional systems. The analysis is different according to whether the considered point is ordinary or singular.

3. Symmetry Preserving and Breaking in Functional Systems

3.1. Symmetry preserving in functional systems

First we consider the case with ordinary (nonsingular) points, i.e. F' is reducible. Since $T_g P^1 = P^1$, $T_g w = w$ for $w = w_1 \in H^1$, symmetry preserving of G-covariant mapping in the subspace is proved with this G-invariant property. This fact means that

$$F(\mu, w_1) = P^1 F(\mu, w_1) \quad (\forall w_1 \in H^1) \tag{4}$$

is derived from Eq. (2). Therefore,

$$P^{\alpha}F(\mu, w_1) = 0 \quad (\alpha = 2,\ 3,\ \ldots,\ q). \tag{5}$$

Since P^1 reflects the isotropy group, Eqs. (4) and (5) indicate that the solution path $w = w(\mu)$ of Eq. (1) is enclosed in the subspace H^1 unless F' becomes irreducible. In other words, the symmetry of the system does not change on each solution path consisting of ordinary points.

3.2. Symmetry breaking in functional systems

Next we consider the case with singular points, i.e. F' is irreducible. Fréchet derivative $F' \equiv \partial F/\partial w$ satisfies the following equation

$$F'\left(\mu, T_g w\right) \cdot T_g = T_g F'(\mu, w) \quad (\forall g \in G,\ \ \forall w \in H), \tag{6}$$

which is derived from Eq. (2). This relation is also valid for $\forall w_1 \in H^1$. Group representation theory enables us to derive

$$F'(\mu, w_1)P^{\alpha} = P^{\alpha}F'(\mu, w_1) \quad \left(\alpha = 1,\ \ldots,\ q, \quad \forall w_1 \in H^1\right) \tag{7}$$

after some calculation by the use of group characters. According to linear algebra, Eq. (7) denotes that F' can be represented as a diagonal matrix:

$$F'(\mu, w_1) = \begin{bmatrix} F_1' & & & \\ & F_2' & & \\ & & \ddots & \\ & & & F_q' \end{bmatrix} \quad \left(\forall w_1 \in H^1\right). \tag{8}$$

Since the basis represents the symmetry, each F_α' corresponds to a certain symmetrical element of the system. According to SATTINGER (1979), the symmetry corresponding to F_α' = 0 at a singular point dominates the new solution path, e.g. the path bifurcating from the singular point. The system follows the new path because the new solution path is more stable (FUJII and YAMAGUTI, 1980). In the case $F_\alpha' = 0$ ($\alpha = 2, \ldots, q$), the new symmetry of the system after the singular point is lower than old one before the singular point because of g_α ($\alpha = 2, 3, \ldots, q$) corresponding to the subgroups. So, the system is dissymmetrical. On the contrary, the symmetry of the system does not change in the case $F_1' = 0$ because of g_1 corresponding to the isotropy group.

4. Discussion

4.1. Curie principle in functional analysis

We consider the Curie principle in functional analysis. We can derive another Fréchet derivative $\dot{F} \equiv \partial F/\partial\mu$ by the same procedures described above. Because Eq. (1) denotes the relation $w = w(\mu)$, we can refer μ as the cause and w as the effect. Then, $\dot{F}$ and F' represent the symmetries of the cause and the effect, respectively. Once F_α', which does not correspond to the existing component of $\dot{F}$, becomes to 0 at a singular point, the symmetry of the cause is no longer preserved in the symmetry of the effect. This case is just the "breaking of the Curie symmetry principle". On the contrary, if F_α' corresponding the existing component of $\dot{F}$ becomes to 0, the symmetry of the cause is held in the effect. The principle is preserved in this case. The principle is of course preserved at ordinary points. In this sense, both phenomena preserving and breaking of the Curie symmetry principle exist in the same system.

4.2. The Curie principle in real phenomena dominated by functionals

In the functional dominating the buckling of spherical shells with uniform loading, μ and w represent the intensity of uniform loading and the displacement of the shell, respectively (FUJII and YAMAGUTI, 1980). This fact demonstrates that symmetry break down in functional analysis is not only the concept in mathematics but also applied to science and engineering. We can find other phenomena whose dominant equation for equilibrium state is expressed by G-invariant functional, e.g. Landau theory of phase transitions, equilibrium shape of rotating fluid, elementary-particle theory (CICOGNA, 1981). However, we cannot apply the Curie principle easily to all these phenomena, because the meanings and the definition of the cause and the effect give rise to further problems, e.g. those in quantum mechanics (ISMAEL, 1997).

To judge whether the principle preserves or not is also important when we introduce the concept of the Curie principle to real phenomena. It's because both preserving and breaking the principle occur in nonlinear systems seen above. Therefore, we must select an appropriate range of quantities to consider the Curie principle in nonlinear phenomena.

4.3. Instability and the Curie principle

We can refer the relation between instability and the Curie principle through functional analysis. In a functional system, instability occurs when one of the eigenvalue F_α becomes to 0 at an singular point. As is described above, functional systems do not have a unique solution at an singular point, i.e. we can derive several solutions. Each solution represents a mathematically possible symmetry of the effect.

In the real world, "*tiny disturbance*" (such as imperfection) in the system can trigger the system to emigrate from an original unstable solution path to a new stable one (STEWART, 1990; STEWART and GOLUBITSKY, 1992). Since the symmetry of the new stable path (that is the symmetry of the effect) always has other symmetry than the original path (the symmetry of the cause), it means that breaking of the symmetry of the system occurred. However, should we state that the breaking of the symmetry triggered by "a tiny fluctuation in an otherwise perfectly symmetric setting" (STEWART and GOLUBITSKY, 1992) breaks the Curie principle? The answer is yes, because not the symmetry of the cause, i.e. the basis of μ, but the intensity of it determines the stable symmetry of the effect in such a case. In other words, tiny disturbance is not necessary in determining how many points of milk crown are generated (whereas the disturbance may be related with which direction axes of the realized crown is preferred).

The mathematical theory of course enable us to derive one of the unstable solution path holding the same symmetry of the cause and to trace it in “quasi-static” system. But in the real world, such ideal state cannot be realized because of existence of tiny disturbance. In this sense, “Curie was right in asserting that symmetric systems have symmetric states—but he failed to address their stability.” (STEWART, 1990; STEWART and GOLUBITSKY, 1992).

Even if one assert the symmetry of tiny disturbance will present the symmetry of the effect, we are able to introduce the case that instability does not trigger symmetry breaking. FUJII and YAMAGUTI (1980) mathematically explained the case that a solution path holding the same symmetry is nonlinear and therefore becomes unstable in part through the analysis of functional equation dominating the buckling of arch structure. With tiny disturbance, the system jumps over the unstable part of its solution path, but the symmetry of the system does not change. In practice, instabilities not triggering the symmetry breaking, i.e. obeying to the Curie principle, is observed in the buckling experiment for spherical shell structures with uniform loading. This fact is also proved mathematically because the dominant equation of the buckling of spherical shells is functional and therefore is able to be analyzed by Sattinger’s group theoretical method (FUJII and YAMAGUTI, 1980). Therefore, we can state that the tiny disturbance does not affect the symmetry of the effect.

5. Conclusions

The nonlinear systems dominated by functional equations can break the Curie principle. We can apply the mathematical analysis constructed by Sattinger to the practical nonlinear phenomena, when we consider the appropriate quantities as the cause and the effect. Furthermore, Sattinger’s method proves both preserving and breaking the principle in such nonlinear systems. Therefore we have to consider appropriate range of quantities in judging whether the Curie principle preserves or not in nonlinear phenomena.

Tiny disturbance is necessary to realize the symmetry of stable solution path. But Sattinger’s method denotes that tiny disturbance does not affect in determining which symmetry is stable and therefore present in the real world.

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