# Time-Dependent Traffic Flow in a Rectangular City with Rectilinear Distance

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**Abstract.** This paper develops a method for deriving a time-dependent traffic flow over a rectangular city with a rectangular grid network, when a commuters' destination arrival time distribution is given by an arbitrary probability density function defined on a finite range of the time axis. The introduction of the time variable is a great extension of prior studies that deal only with the spatial distribution of traffic within a city. The model is designed for the morning peak period of commuter traffic during which the greatest overload of existing networks are observed. The results are useful in exploring the geometrical properties of rectangular grid networks and analyzing the impact of flexible working hours on commuter trip distributions.

## 1. Introduction

This paper presents the analytical and geometrical framework for constructing a timedependent traffic flow model based on a rectangular grid network. In the theory of continuous traffic flow modeling, the primary assumptions are that the distributions of endpoints of trips can be represented by continuous functions defined over the twodimensional plane and that continuous movement on the infinitely dense idealized networks is possible. These assumptions facilitate the analytical treatment of the problems and allow us to explore how spatial traffic patterns are influenced by (1) the shape of the city, (2) the geometrical arrangement of the network, and (3) the distributions of endpoints of trips over the city. The pioneering work was done by Reuben Smeed who emphasized the importance of the continuous approach to transport problems and laid the mathematical foundations for analyzing traffic patterns in idealized cities (SMEED, 1961, 1963, 1968). After his series of works, a number of studies that deal with the spatial distribution of traffic in idealized cities have appeared (HOLROYD, 1968; ANGEL and HYMAN, 1976; VAUGHAN, 1987). The basic ideas of these studies are that, by deriving traffic as an analytical function of position under given assumptions (network arrangement and distributions of endpoints of trips), the location of potential congestion areas resulting from these assumptions can be analyzed.

These studies, however, concern only with locational variations of the traffic flow and ignore completely the temporal fluctuations of the traffic flow. The prime aim of this paper is to extend prior studies of traffic flow models by incorporating the time variable explicitly. This great extension allows us to analyze spatial-temporal traffic patterns during the peak commuting period and provides us insights into how the commuters' destination arrival time distribution influences traffic over the urban transport infrastructure.

We assume that transportation network consists of rectangular grid networks running in two perpendicular directions and parallel to the sides of the rectangle. The shortest distance on this rectangular networks is called the *rectilinear distance* (also known as the Manhattan distance (BECKMAN, 1999) and the taxicab distance (KRAUSE, 1987)), and the assumption of movement on this network is extensively used in a variety of disciplines, such as urban economics (ANJOMANI, 1980), location theory (LARSON and ODONI, 1981; BUTT and CAVALIER, 1997; BECKMAN, 1999), computational geometry (REZEND et al., 1989), transport planning (VAUGHAN, 1987). Efficiency of movement on the rectangular networks in comparison with other types of metrics has also been studied in VAUGHAN (1987), KURITA (2001), and OIKAWA (2001). The spatial distribution of traffic over a square city with rectangular networks is treated in VAUGHAN (1987) and HOLROYD (1968). Rodney Vaughan, who is a major contributor to this field, summarized a number of works on various models of the spatial traffic distribution, including his own contributions, and examined the effects of various transport-network designs on the spatial traffic patterns. Holroyd considered an interesting problem of routing policy for relieving traffic congestion in a square city such that the number of travelers crossing each other's routes is minimized. Spatial traffic models assuming the radial-arc distance (TANAKA and KURITA, 2001), the direct distance (OHTSU and KOSHIZUKA, 1998), and the minimum time distance (ANGEL and HYMAN, 1976) have also been developed.

There have been very few studies that explicitly considered the time variation of traffic flow (PEARCE, 1975; TANAKA and KURITA, 2002). PEARCE (1975) considered a circular city with radial-arc road networks and derived the distribution of traffic that includes the time variable as well as the locational variables. TANAKA and KURITA (2002) developed a similar model in the case of a square city with rectangular grid networks. These studies, however, derived traffic distribution only in the very limited case in which all commuters arrive at their workplaces at the same time. This paper generalizes this assumption and assumes that the destination arrival time distribution is given by an arbitrary probability density function defined on a finite range of the time axis. This generalization gives us a new tool to analyze the relationship between the degrees of concentration of commuters' destination arrival time and transport demand in space and time. The model developed in this study can be used to obtain some policy implications of flexible working hours.

#### 2. Model Description

In this section, we describe a general setting, ranging from the assumptions about the city model to the assumptions about the movement of travelers. Let us consider a rectangular city with side lengths  $L_1$  and  $L_2$  as depicted in Fig. 1. The position of an arbitrary point is denoted by (x, y) by the Cartesian coordinates with the left bottom corner of the rectangle at the origin. For convenience, the positive side of Y axis is taken to be north with



Fig. 1. A rectangular city with side lengths  $L_1$  and  $L_2$  and routes used between home and workplace.

the road networks running in east-west and north-south directions. For ease of expression, one of the two endpoints of a trip will be called a home, the other a workplace, and their positions will be expressed as  $P(x_h, y_h)$  and  $Q(x_w, y_w)$  respectively.

#### 2.1. Movement of travelers

We make the following assumptions about the movement of travelers:

(i) Homes and workplaces are uniformly and independently distributed over the rectangular city.

(ii) There exist infinitely dense rectangular grid networks over the city.

(iii) Every commuter makes one's way from home to workplace by one of the two routes of the minimum trip length with only one turn (route I and II in Fig. 1).

(iv) Each commuter chooses the route I or II between any two points with equal probability.

(v) The speed on the network is a constant value v, irrespective of the position and time.

The assumption (i) of uniform origins and destinations is widely used in the fields of transportation planning, regional science, location theory, urban economics and so on. This idealized assumption, while not reflecting the actual situation in real cities, allows us to treat the problem analytically and to discover geometric and morphological properties of the network under investigation. In addition, a uniform model provides a first approximation of the more "real" model having nonuniform, location-dependent densities.

The minimum distance d(P, Q) between the point  $P(x_h, y_h)$  and  $Q(x_w, y_w)$  in the rectangular network is given by

$$d(\mathbf{P}, \mathbf{Q}) = |x_{\rm h} - x_{\rm w}| + |y_{\rm h} - y_{\rm w}|.$$
(1)



Fig. 2. An example of a probability density function of destination arrival time of commuters.

There exist, however, an infinite number of the minimum length routes between these two points. The natural assumptions are that every commuter chooses the minimum length route with the least turn with equal probability (assumption (iii) and (iv)).

#### 2.2. Destination arrival time distribution of commuters

To uniquely determine the traffic volume at a given point and time, the distribution of destination arrival time of commuters should be specified. This distribution is given by a probability density function (pdf) defined on a finite range of the time axis and is denoted by

$$f = f(t) \quad (t_0 \le t \le t_0 + a).$$
 (2)

We assume that all commuters follow the same destination arrival time distribution, f(t), irrespective of the position of home and workplace. An example of f(t) is given in Fig. 2. We can regard a time interval of length a as a measure of dispersion of destination arrival time of commuters. It should be noted that by expressing the arrival time distribution as an arbitrary pdf, it will be possible to analyze the relationship between the length of commuting duration a and spatio-temporal traffic patterns during the peak commuting period.

#### 3. Definitions of Traffic Flow and Traffic Flow Density

In this section, we introduce the function that describes traffic volume passing through a point (x, y) at a given time t, and this function is referred to as the *traffic flow density*. To derive the traffic flow density, we also introduce the function that describes the total traffic volume passing through a point (x, y), and this function is referred to as the *traffic flow*. Precise definitions of these two functions are given in the following subsections.



Fig. 3. Illustration of the definitions of traffic flow and traffic flow density.

## 3.1. Definition of traffic flow density

We denote the traffic flow density of easterly direction by  $p_E(x, y; t)$  which is the function of time *t* as well as position (x, y), and we define  $p_E(x, y; t)$  as follows: Let  $\alpha_E$  be the number of commuters crossing the line segment *C* connecting the point  $(x, y_1)$  and  $(x, y_2)$  during the time interval  $t \in [t_a, t_b]$  as shown in Fig. 3. The traffic flow density of easterly direction,  $p_E(x, y; t)$  is defined such that the following equation is satisfied:

$$\alpha_{\rm E} = \int_{t=t_a}^{t_b} \int_{y=y_1}^{y_2} p_{\rm E}(x, y; t) dy dt.$$
(3)

This definition of the traffic flow density of easterly direction gives that  $p_{\rm E}(x, y; t) dydt$  is the number of trips passing though the small line segment with length dy located at the point (x, y) during the small time period [t, t + dt], and its dimensions are given by number per length per time. The traffic flow densities of westerly, northerly, and southerly directions are similarly defined, and are denoted by  $p_{\rm W}(x, y; t)$ ,  $p_{\rm N}(x, y; t)$ , and  $p_{\rm S}(x, y; t)$ , respectively.

### 3.2. Definition of traffic flow

We denote the traffic flow of easterly direction by  $q_E(x, y)$  which is the function of position (x, y) only, and define  $q_E(x, y)$  as follows: Let  $\beta_E$  be the total number of commuters crossing the line segment *C* as shown in Fig. 3. The traffic flow of easterly direction,  $q_E(x, y)$ , is defined such that the following equation is satisfied:

$$\beta_{\rm E} = \int_{y=y_1}^{y_2} q_{\rm E}(x, y) \mathrm{d}y.$$
 (4)

This definition of the traffic flow of easterly direction gives that  $q_{\rm E}(x, y) dy$  is the total

number of commuters passing though the small line segment with length dy located at the point (x, y), and its dimensions are given by number per length. The traffic flows of westerly, northerly, and southerly directions are similarly defined, and are denoted by  $q_W(x, y)$ ,  $q_N(x, y)$ , and  $q_S(x, y)$ , respectively. The function defined by Eq. (4) was first proposed by HOLROYD (1968), and VAUGHAN (1987) calculated this function in a unit square city with rectangular grid networks. It should be noted that the traffic flow density we introduced in Eq. (3) is a natural (but great) extension of the function defined by Eq. (4). From Eqs. (3) and (4), we see that the traffic flow density is the temporal density of the traffic flow.

#### 4. Formulation of Traffic Flow Density

This section describes the method for calculating the traffic flow density defined in the previous section. First, we formulate the relationship between the traffic flow density and the traffic flow. Then, derivation methods for these two functions are explained.

## 4.1. Relationship between traffic flow density and traffic flow

From Eqs. (3) and (4), the relationship between the traffic flow density of easterly direction and the traffic flow of easterly direction is given as follows:

$$q_{\rm E}(x, y) = \int_{t=T_0}^{T_1} p_{\rm E}(x, y; t) \mathrm{d}t,$$
(5)

where  $T_0$  and  $T_1$  represent the time that the first and the last commuter pass at the point (x, y). We introduce the probability density function that describes the distribution of commuters' passage time *t* at the point (x, y) and denote this function by  $\xi_{\rm E}(t|x, y)$ . The meaning of this pdf is that  $\xi_{\rm E}(t|x, y)dt$  is the proportion of the number of commuters passing at the point (x, y) during the time period [t, t + dt] in the easterly direction to the total number of commuters passing at the point (x, y). From Eq. (5) the traffic flow density can be expressed as follows:

$$p_{\rm E}(x, y; t) = q_{\rm E}(x, y) \cdot \xi_{\rm E}(t|x, y).$$
(6)

We can confirm the validity of Eq. (6) in the following manner: Let  $\alpha_E$  be the number of commuters passing though the small segment with length dy located at the point (x, y)during the small time period [t, t + dt] in the easterly direction. From Eq. (3),  $\alpha_E$  is given by  $p_E(x, y; t)$ dydt. On the other hand, this can also be expressed as the total number of commuters passing though this segment, times the proportion of commuters that pass though this segment during the time interval [t, t + dt]. The former is given by  $q_E(x, y)$ dy from Eq. (4) while the latter is equal to  $\xi_E(t|x, y)$ dt by the definition of the probability density function,  $\xi_E(t|x, y)$ . Thus, Eq. (6) can be obtained. The relationship defined by Eq. (6) indicates that the traffic flow density is the temporal density of the traffic flow. In the remainder of this section, we concentrate on developing methods for deriving  $q_E(x, y)$  and  $\xi_E(t|x, y)$ .



Fig. 4. Paris of acceptable homes and workplaces for crossing the segment located at the point (x, y): (a) the commuters turn after (x, y); (b) the commuters turn before (x, y).

## 4.2. Number of trips between two regions

To derive the traffic flow defined by Eq. (4), the methods for calculating the number of trips between any two regions in the city should be developed. We follow the method described in VAUGHAN (1987), which gives the areal densities of homes and workplaces, and we denote these densities by  $\lambda(x_h, y_h)$  and  $\mu(x_w, y_w)$  respectively. Let *n* be the number of commuters from a region  $S_P$  to a region  $S_Q$ . From the assumption of independence between a given home and workplace (assumption (i) in Sec. 2), the number of commuters, *n*, between these two regions is given by

$$n = N \int_{S_{\rm Q}} \int_{S_{\rm P}} \lambda(x_{\rm h}, y_{\rm h}) \mu(x_{\rm w}, y_{\rm w}) \mathrm{d}x_{\rm h} \mathrm{d}y_{\rm h} \mathrm{d}x_{\rm w} \mathrm{d}y_{\rm w}, \tag{7}$$

where *N* means the total number of commuters within the city. When the densities of homes and workplaces are given by the uniform distributions (assumption (i) in Sec. 2),  $\lambda(x_h, y_h)$  and  $\mu(x_w, y_w)$  are reduced to the following simple expressions:

$$\lambda(x_{\rm h}, y_{\rm h}) = \mu(x_{\rm w}, y_{\rm w}) = \frac{1}{L_1 L_2}.$$
(8)

## 4.3. Derivation of traffic flow

We explain the method for deriving the traffic flow in the easterly direction,  $q_E(x, y)$ , in line with the method described by VAUGHAN (1987). First, consider the commuters passing across the segment (y, y + dy) in the easterly direction at the point (x, y) as shown in Figs. 4(a) and (b). There are only two types of path the commuters can use (with equal probability in this case) to pass through the segment. The commuters turn after crossing this segment as in Fig. 4(a) or before crossing as in Fig. 4(b). In these figures shaded areas and gray areas represent the areas of acceptable homes and workplaces for the commuters to pass through this segment. Let us denote the traffic flow corresponding to Fig. 4(a) by  $q_E^{I}(x)$ .

y) and Fig. 4(b) by  $q_E^{II}(x, y)$ . Then, the traffic flow in the easterly direction  $q_E(x, y)$  is expressed as a combination of these two:

$$q_{\rm E}(x, y) = q_{\rm E}^{\rm I}(x, y) + q_{\rm E}^{\rm II}(x, y).$$
(9)

The acceptable areas of homes  $A_P$  and workplaces  $A_Q$  in Fig. 4(a) and  $B_P$  and  $B_Q$  in Fig. 4(b) are given as follows:

$$A_{\rm P} = \left\{ \left( x_{\rm h}, y_{\rm h} \right) \middle| 0 \le x_{\rm h} \le x, \ y \le y_{\rm h} \le y + \mathrm{d}y \right\},\tag{10}$$

$$A_{\rm Q} = \left\{ \left( x_{\rm w}, y_{\rm w} \right) \middle| x \le x_{\rm w} \le L_1, \ 0 \le y_{\rm w} \le L_2 \right\},\tag{11}$$

$$B_{\rm P} = \left\{ \left( x_{\rm h}, y_{\rm h} \right) \middle| 0 \le x_{\rm h} \le x, \ 0 \le y_{\rm h} \le L_2 \right\},\tag{12}$$

$$B_{\rm Q} = \left\{ \left( x_{\rm w}, y_{\rm w} \right) \middle| x \le x_{\rm w} \le L_1, \ y \le y_{\rm w} \le y + {\rm d}y \right\}.$$
(13)

The number of commuters passing through the small segment in Figs. 4(a) and (b) is given by  $q_E^{I}(x, y)dy$  and  $q_E^{II}(x, y)dy$  respectively by the definition of Eq. (4), and these can be expressed as follows:

$$q_{\rm E}^{\rm I}(x,y)dy = \frac{N}{2} \int_{y_{\rm w}=0}^{L_2} \int_{x_{\rm w}=x}^{L_1} \int_{y_{\rm h}=y}^{y+dy} \int_{x_{\rm h}=0}^x \lambda(x_{\rm h}, y_{\rm h}) \mu(x_{\rm w}, y_{\rm w}) dx_{\rm h} dy_{\rm h} dx_{\rm w} dy_{\rm w},$$
(14)

$$q_{\rm E}^{\rm II}(x,y)dy = \frac{N}{2} \int_{y_{\rm w}=y}^{y+dy} \int_{x_{\rm w}=x}^{L_1} \int_{y_{\rm h}=0}^{L_2} \int_{x_{\rm h}=0}^{x} \lambda(x_{\rm h}, y_{\rm h}) \mu(x_{\rm w}, y_{\rm w}) dx_{\rm h} dy_{\rm h} dx_{\rm w} dy_{\rm w}.$$
 (15)

Direct calculation of Eqs. (14) and (15) gives  $q_E^{I}(x, y)$  and  $q_E^{II}(x, y)$  as follows:

$$q_{\rm E}^{\rm I}(x,y) = q_{\rm E}^{\rm II}(x,y) = \frac{NL_2 x (L_1 - x)}{2 (L_1 L_2)^2}.$$
(16)

Thus, we can obtain  $q_{\rm E}(x, y)$  from Eqs. (9) and (16) as follows:

$$q_{\rm E}(x,y) = \frac{NL_2 x (L_1 - x)}{(L_1 L_2)^2}.$$
(17)

Similar procedures described above give  $q_W(x, y)$ ,  $q_N(x, y)$ , and  $q_S(x, y)$  as follows:

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$$q_{\rm W}(x,y) = \frac{NL_2 x (L_1 - x)}{(L_1 L_2)^2},$$
(18)

$$q_{\rm N}(x,y) = \frac{NL_1 y (L_2 - y)}{(L_1 L_2)^2},$$
(19)

$$q_{\rm S}(x,y) = \frac{NL_1 y (L_2 - y)}{(L_1 L_2)^2}.$$
 (20)

## 4.4. Derivation procedure for passage time distribution

We present the method for deriving the distribution of commuters' passage time at the point (x, y) when the destination arrival time distribution is given by Eq. (2). Let us denote this pdf corresponding to Fig. 4(a) by  $\xi_E^{I}(t|x, y)$ , and Fig. 4(b) by  $\xi_E^{II}(t|x, y)$  respectively. In the following, the derivation method for  $\xi_E^{I}(t|x, y)$  is explained. The meaning of  $\xi_E^{I}(t|x, y)$  is that  $\xi_E^{I}(t|x, y)dt$  is the proportion of commuters who have their workplaces within the area  $A_Q$  in Fig. 4(a) and pass the point (x, y) during a small time interval [t, t + dt]. To obtain  $\xi_E^{I}(t|x, y)$ , we introduce the function g(u|x, y) that is defined as the probability density function of travel time u from the point (x, y)du is the proportion of workplaces to which it takes travel time [u, u + du] from the point (x, y).

We first derive cumulative distribution function (cdf) of g(u|x, y) and denote this by G(u|x, y). This cdf can be obtained by calculating the proportion of workplaces within the area  $A_Q$  that can be reached within travel time u from the point (x, y). Let  $\Omega_Q$  be the area in which it is possible to reach within a certain travel time u from the point (x, y). Then, G(u|x, y) is expressed by

$$G(u|x,y) = \frac{\int_{\Omega_Q} \mu(x_w, y_w) dx_w dy_w}{\int_{A_Q} \mu(x_w, y_w) dx_w dy_w}.$$
(21)

When workplaces are uniformly distributed (assumption (i) in Sec. 2), Eq. (21) reduces to the following simple expression:

$$G(u|x, y) = \frac{\int_{\Omega_Q} dx_w dy_w / L_1 L_2}{\int_{A_Q} dx_w dy_w / L_1 L_2} = \frac{|\Omega_Q|}{|A_Q|},$$
(22)

where  $|\Omega_Q|$  and  $|A_Q|$  are the area of  $\Omega_Q$  and  $A_Q$  respectively. This indicates that the problem of obtaining G(u|x, y) is reduced to the problem of calculating the area of  $\Omega_Q$ . The area  $|\Omega_Q|$ 



Fig. 5. (a) Region to which travel time is in the interval [u, u + du], and (b) equi-travel-time contours from the point (x, y).

| can be derived by drawing a contour of travel time u from the point (x, y) as depicted in Fig. 5(a) and calculating the area of the inner region of the contour. This diamond-shaped travel time contour, as is widely known in the literature, is 45° rotated square (ANJOMANI, 1981; OKABE *et al.*, 2000) and can be obtained directly from the following equation:

$$|x - x_{w}| + |y - y_{w}| = u \cdot v, \tag{23}$$

where v is the travel speed of commuters. The analytical expression of  $|\Omega_Q|$  changes when the contour collides with the five points on the boundary of  $A_Q$  (illustrated as filled squares in Fig. 5(b) and the time this collision occurs is referred to as the *collision time*). In the case of Fig. 5(b), the order of appearance of the collision time is given by,

$$\frac{L_2 - y}{v} < \frac{y}{v} < \frac{L_1 - x}{v} < \frac{L_2 - y}{v} + \frac{L_1 - x}{v} < \frac{y}{v} + \frac{L_1 - x}{v},$$
(24)

so that G(u|x, y) is defined on the interval  $[0, y/v + (L_1 - x)/v]$ . In the following, the maximum value of u is denoted by  $u_{max}$  (in the case of Fig. 5(b),  $u_{max} = y/v + (L_1 - x)/v$ ). It should be noted that the order of the collision time varies depending on the position (x, y). Consequently, we have to consider all possible orders of the collision time u with each producing different analytical expression of G(u|x, y). This leads to the partitioning of the collision time u. In the case of a square city, there exist ten such subregions. By differentiating G(u|x, y) with respect to u, we can derive g(u|x, y) as a function of (u, x, y). See Appendix A for a more detailed description of the derivation procedure for g(u|x, y) and examples of its analytical expressions.

Having obtained g(u|x, y), we next explain the method for deriving  $\xi_{\rm E}^{\rm I}(t|x, y)$  by using



Fig. 6. (a) Arrival time distribution of commuters; and (b) passage time distribution at the point (x, y).

this function. Consider the commuters whose workplaces are located within the gray area in Fig. 5(a) to which travel time from the point (x, y) is in the small interval [u, u + du]. Let h(t|u) be the conditional probability density function of the passage time t for the commuters who have workplaces to which travel time from the point (x, y) is u. In order for these commuters to reach at their workplaces at time t, they have to pass the point (x, y) at time t - u as illustrated in Figs. 6(a) and (b). Thus, h(t|u) can be obtained by shifting the arrival time distribution of commuters by u toward the negative direction of t axis:

$$h(t|u) = f(t+u). \tag{25}$$

From a similar argument, the set of points (t, u) for the commuters who have workplaces within  $A_Q$  corresponds to the parallelogram (as illustrated by the bold line in Fig. 7) in the *T*-*U* plane. Therefore, the proportion of commuters that can pass the point (x, y) by the time *t*, which is the cdf of  $\xi_E^{I}(t|x, y)$  by its definition, is equal to the proportion of commuters whose (t, u) points in the *T*-*U* plane are included in the gray area, *D*, as shown in Fig. 7. This proportion can be obtained by integrating the joint probability density function of (t, u) over the region *D*. This joint pdf can be expressed as  $h(t|u) \cdot g(u|x, y)$ , considering that the proportion of commuters whose travel time from the point (x, y) is in the interval [u, u + du] and who pass the point (x, y) during the time period [t, t + dt] is given by

$$h(t|u)dt \cdot g(u|x, y)du = f(t+u)dt \cdot g(u|x, y)du.$$
(26)

From the above argument, the cdf of  $\xi_E^{I}(t|x, y)$  (we denote this by  $\Xi_E^{I}(t|x, y)$ ) can be expressed as follows:

$$\Xi_{\rm E}^{\rm I}(t|x,y) = \iint_D f(t+u) \cdot g(u|x,y) \mathrm{d}t \mathrm{d}u. \tag{27}$$

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Fig. 7. The diagram for the derivation of  $\xi_{\rm E}^{\rm I}(t|x, y)$ .

Analytical expression of  $\Xi_{\rm E}^{\rm I}(t|x, y)$  can be obtained as a function of the position (x, y) and time t. This derivation, however, involves a quite complicated procedure since  $\Xi_{\rm E}^{-1}(t|x, y)$ must be expressed differently, depending on the four variables x, y, t, and a. By differentiating  $\Xi_{\rm E}^{\rm I}(t|x, y) \text{ with respect to } t, \text{ we can obtain } \xi_{\rm E}^{\rm I}(t|x, y). \text{ See Appendix B for a more detailed description of the derivation procedure for } \xi_{\rm E}^{\rm I}(t|x, y).$ The derivation procedure for  $\xi_{\rm E}^{\rm II}(t|x, y)$  is similar to that described above. Having obtained  $\xi_{\rm E}^{\rm I}(t|x, y)$  and  $\xi_{\rm E}^{\rm II}(t|x, y)$ , we can finally obtain the traffic flow density in the

easterly direction,  $p_{\rm E}(x, y; t)$ , as follows:

$$p_{\rm E}(x, y; t) = q_{\rm E}^{\rm I}(x, y) \cdot \xi_{\rm E}^{\rm I}(t|x, y) + q_{\rm E}^{\rm II}(x, y) \cdot \xi_{\rm E}^{\rm II}(t|x, y).$$
(28)

The traffic flow densities for the other three directions can be similarly derived. In a rectangular model, however, the traffic flow density in the westerly direction,  $p_{W}(x, y; t)$ , is directly obtained without repeating the similar calculation developed so far, by substituting  $L_1 - x$  into x in  $p_E(x, y; t)$  by using the symmetry of the model. The same is true in the case of the relationship between  $p_N(x, y; t)$  and  $p_S(x, y; t)$ .

## 5. Numerical Examples

In this section, we present some numerical examples of the traffic flow and traffic flow density formulated in the above sections. Throughout this section, a square city of side length L is assumed and parameter values of N = 1, L = 1, and v = 1 are adopted.

## 5.1. Traffic flow

Figure 8 shows the traffic flow of (a) east-west direction; (b) north-south direction; and (c) combination of all direction,  $q_T(x, y)$ , in a unit square city. Figure 8(c) indicates that the city center has the maximum value even when the endpoints of trips are uniformly



Fig. 8. Traffic flow in a unit square city: (a) east-west direction; (b) north-south direction; (c) all directions.



Fig. 9. Illustration of (a) a uniform arrival time distribution; and (b) its numerical examples.

distributed over the city. This implies that the city center has the greatest potential of traffic congestion.

# 5.2. Temporal distribution of traffic flow density

In the present and next subsections, some numerical examples of the combined (total) traffic flow density,

$$p_{\rm T}(x, y; t) = p_{\rm E}(x, y; t) + p_{\rm W}(x, y; t) + p_{\rm N}(x, y; t) + p_{\rm S}(x, y; t)$$
(29)

are provided. In these two subsections, the arrival time distribution of commuters is given by the uniform density function f(t) centered at k and defined on the time interval [k - a/2, k + a/2] as illustrated in Fig. 9(a); and f(t) is given as follows:

$$f(t) = \frac{1}{a} \quad (k - a / 2 \le t \le k + a / 2). \tag{30}$$

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Fig. 10. Temporal distribution of combined traffic flow densities: (a) at (x, y) = (0.5, 0.5) and (b)at (x, y) = (0.75, 0.75) when the arrival time distributions are given by Fig. 9(b).

The value of k can be seen as the average arrival time of commuters, while a can be regarded as the measure of dispersion of the arrival time distribution.

This subsection explores temporal distribution of the traffic flow density at specified observation points. Figure 10 shows temporal distributions of  $p_T(x, y; t)$  (a) at the city center (x, y) = (0.50, 0.50) and (b) at a suburban point (x, y) = (0.75, 0.75). In Figs. 10(a) and (b), f(t) is given by the uniform density functions of four different arrival time durations: a = 0.5, a = 1.0, a = 1.5, and a = 2.0 (as shown in Fig. 9(b)), plus the Dirac delta function, i.e. all commuters arrive at their workplaces at the same time. In Fig. 9(b), the average arrival time k is taken at the same point among five cases and the origin of the time axis is defined so that t = 0 coincides with the time of the first commuter leaving his or her home in the case of a = 2.0. With the increase of the value of the arrival time duration a, we see that the distribution of traffic is gradually dispersed. The maximum value of the traffic flow density, and the time this value is observed are the very important measures for assessing the impact of flexible working hours on commuter trip patterns. This approach becomes possible only when incorporating the time variable explicitly as described so far.





Fig. 11. Spatial distribution of traffic flow density when all commuters arrive at their workplaces at the same time, t = 2.0.

## 5.3. Spatial distribution of traffic flow density

We next consider the spatial distribution of traffic over the city, a snapshot in time, when a time is specified. Figure 11 shows the spatial distribution of the traffic flow density,  $p_{\rm T}(x, y; t)$ , assuming that all commuters arrive at their workplace at the same time, t = 2.0 (the origin of the time axis is defined so that t = 0 coincides with the time of the first commuter leaving his or her home), and the graphs are drawn from t = 0.1 up to t = 1.9 by a step of time interval 0.1. Flow is first observed at the four corners of the square, since the maximum travel time of the commuters has the maximum value at these four points (journey length between the endpoints of the diagonal takes the largest value, 2L/v = 2). It

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Fig. 12. Spatial distribution of traffic flow density for the uniform arrival time distribution in the case of a = 1.0.

is interesting to note that the maximum value of traffic at a given time is not always observed at the city center.

We next present an example of the spatial distribution of traffic when a uniform arrival density defined on a finite length *a* is adopted. Figure 12 shows the spatial distribution of the traffic flow density,  $p_T(x, y; t)$ , in the case of a = 1.0 with arrivals occurring in the interval [2.0, 3.0] (the origin of the time axis is defined so that t = 0 coincides with the time of the first commuter leaving his or her home), and the graphs are drawn from t = 0.1 up to t = 2.9 by a step of time interval 0.1. In Fig. 12, the discontinuities observed in Fig. 11 are smoothed out, by the derivation procedure for the passage time distribution as explained in Fig. 7. By comparing these two figures, we can observe the effect of the dispersion of commuters' destination arrival time on spatial-temporal traffic patterns.

#### 6. Discussion and Conclusion

In this paper, we presented an analytical method for deriving time dependent traffic flow based on a rectangular grid network. With the introduction of the time variable, we succeeded in describing the effect of commuters' arrival time duration on the spatialtemporal traffic patterns. We conclude this paper by examinig possible further work.

First, similar models assuming other types of metrics on the continuous plane can also be developed. By comparing the results derived from various network assumptions, geometrical characteristics of each network can be analyzed. We can also consider extending the method developed on a continuous plane to the method defined on a network. This extension allows us to analyze the spatio-temporal traffic patterns over the actual network.

Second, generalization of the shape of the city from a rectangular case should be considered. When we analyze traffic patterns in an actual city, the assumption of rectangular city is rather restrictive. The relaxation of this assumption, however, is not so straightforward a task, since there do not always exist two minimum-distance routes with only one turn between two arbitrary points, when the shape of the city is extended from a rectangular case. It is also interesting to consider the effect of the presence of geographical barrier, such as rivers and lakes, across which a journey cannot be made. TANAKA and KURITA (2001) considered a sector-shaped city with radial-arc networks to model the city that located near the bay area and derived the spatial distribution of traffic that exhibits features of the city with a geographical barrier. It is of interest to consider spatio-temporal traffic patterns using city models with some barriers.

Finally, the assumption of independence between a home and workplace should be relaxed, and the introduction of spatial interaction into our framework should be explored. Spatial interaction models have been widely studied in the field of transportation studies that concern with description and prediction of flows of people or commodities, etc. between different locations in the city, with locations represented by discrete zones. ANGEL and HYMAN (1976) formulated the continuous models of spatial interaction by extending many of the concepts which have been restricted to the network representation of space. By incorporating this notion into our spatio-temporal framework, we can obtain further insights into spatio-temporal phenomena of transportation.

Appendix A. Examples of g(u|x, y)

We present analytical expressions for g(u|x, y) the pdf of travel time u from the point (x, y) to the workplaces uniformly distributed within the area  $A_Q$ . Examples corresponding to the square case of Fig. 4(a) are provided. Figure A1(a) illustrates ten subregions with each corresponding to one possible order of the collision times. When the center of the travel time contour, (x, y), is in the gray area in Fig. A1(a), the order of appearance of the collision time becomes,

$$\frac{L-y}{v} < \frac{y}{v} < \frac{L-x}{v} < \frac{L-y}{v} + \frac{L-x}{v} < \frac{y}{v} + \frac{L-y}{v},$$
(A.1)

as shown in Fig. A1(b). Consequently, there exist five different expressions of g(u|x, y). As we have already shown in Eq. (22), G(u|x, y), the cdf of g(u|x, y), can be obtained by calculating the inner area of the contour in Fig. A1(b), divided by the area of  $A_Q$  in the case of uniformly distributed workplaces. By differentiating G(u|x, y) with respect to u, we obtain g(u|x, y) as a linear function of travel time u. When the center of the travel time contour (x, y) is in the gray area in Fig. A1(a), g(u|x, y) can be obtained as follows:

$$\left|\frac{2v^2u}{L(L-x)}, \qquad 0 \le u < \frac{L-y}{v}, \qquad (A.2)\right|$$

$$\frac{v(vu+L-y)}{L(L-x)}, \qquad \qquad \frac{L-y}{v} \le u < \frac{y}{v}, \tag{A.3}$$

$$g(u|x,y) = \begin{cases} \frac{v}{L-x}, & \frac{y}{v} \le u < \frac{L-y}{v}, \end{cases}$$
(A.4)

$$\frac{v(3L-2vu-2x)}{L(L-x)}, \qquad \frac{L-x}{v} \le u < \frac{L-y}{v} + \frac{L-x}{v},$$
(A.5)

$$\frac{v(L-vu-x+y)}{L(L-x)}, \qquad \frac{L-y}{v} + \frac{L-x}{v} \le u < \frac{y}{v} + \frac{L-x}{v}.$$
(A.6)

Figure A2 shows (a) G(u|x, y) and (b) g(u|x, y) in the case of the unit square city and the center of the travel time contour is taken at the point (0.2L, 0.6L) (L = 1, v = 1).

# Appendix B. Examples of $\xi(t|x, y)$

We present examples of the passage time distribution formulated in Subsec. 4.4. Examples of  $\xi_{\rm E}^{I}(t|x, y)$  corresponding to the square case of Fig. 4(a) are provided. First, we show the derivation procedure when the commuters' destination arrival time distribution is given by the Dirac delta function centered at  $t = t_0$ , namely, all the commuters arrive at

their workplaces at  $t = t_0$ .

Let us consider the commuters who have their workplace to which it takes travel time u from the point (x, y). In order for the commuters to reach at their workplace at time  $t = t_0$ , they have to pass the point (x, y) at time  $t = t_0 - u$ . The pdf of travel time u from the point (x, y), however, is given by g(u|x, y), so that the pdf of passage time is given by



Fig. A1. (a) Ten subregions each corresponding to one possible order of collision times; (b) travel time contour from the point (0.2L, 0.6L), which is located in gray area in (a).



Fig. A2. (a) The cdf and (b) the pdf of travel time u from the point (0.2L, 0.6L) to the workplaces uniformly distributed within the area  $A_Q$ .

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Fig. B1. An example of the passage time distribution,  $\xi_{\rm E}^{\rm I}(t|x, y)$  for the Dirac delta function, when the observation point is given by (x, y) = (0.2L, 0.6L).

$$\xi_{\rm E}^{\rm I}(t|x,y) = g(t_0 - t|x,y). \tag{B.1}$$

Figure B1 shows the example of the passage time distribution,  $\xi_{\rm E}^{\rm I}(t|x, y)$ , when the observation point is given by (x, y) = (0.2L, 0.6L), and the origin of the time axis is defined so that t = 0 coincides with the time of the first commuter leaving his or her home, namely,  $t_0 = 2L/v$  (L = 1, v = 1).

We next present examples of  $\xi_{\rm E}^{\rm I}(t|x, y)$  when the commuters' arrival time distribution, f(t), is given by the uniform density function defined on the time interval  $[t_0, t_0 + a]$ . As we have already shown in Subsec. 4.4, the cdf of  $\xi_{\rm E}^{\rm I}(t|x, y)$  can be obtained by Eq. (27). The derivation of Eq. (27), however, is quite complicated since different analytical expressions are obtained depending on the position (x, y), and time t, and the length of the destination arrival time interval a.

We present some examples of  $\xi_{\rm E}^{\rm I}(t|x, y)$  when the observation point is given by (x, y) = (0.2L, 0.6L). Figure B2(a) shows five examples of f(t) with different values of a. In Fig. B2(b), five examples of  $\xi_{\rm E}^{\rm I}(t|x, y)$  corresponding to the examples of f(t) in Fig. B2(a), plus  $\xi_{\rm E}^{\rm I}(t|x, y)$  shown in Fig. B1 are presented (L = 1 and v = 1). From Fig. B2(b), we can analyze how the dispersion of arrival time duration affects the temporal distribution of  $\xi_{\rm E}^{\rm I}(t|x, y)$ .

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- Fig. B2. (a) Examples of the uniform destination arrival time distributions, and (b) its corresponding passage time distributions along with the one shown in Fig. B1, when the observation point is given by (x, y) = (0.2L, 0.6L).
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