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The Relationship of the Cotangent Function to Special Relativity Theory, Silver Means, *p*-cycles, and Chaos Theory

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Abstract. The cotangent function is shown to play a key role in special relativity theory, silver mean constants, p-cycles for all positive integer values of p, and period doubling bifurcations from chaos theory. Geometry and properties of number play important roles in the analysis. A new theorem pertaining to periodic continued fractions is introduced.

1. Introduction

The cotangent function has been found to play a role in special relativity theory where the standard formulas are expressed in terms of the multiplier that determines the magnitude of the Doppler effect rather than the usual v/c where v is the velocity of a particle and c is the speed of light. Also a sequence of silver mean constants that arise naturally in dynamical systems theory are characterized by the cotangent. The cotangent is found to characterize a map that has periods of all lengths. Finally, the cotangent function is a natural way to express the values of the dynamical variable approaching chaos at a point of period doubling bifurcation.

2. Special Relativity and the Cotangent Function

According to special relativity theory, a body moving at velocity v experiences a shortening in length L by the factor $L' = L/\beta$, a dilation of time t by $t' = t\beta$ and an increase in mass m by the factor $m' = m\beta$ where β is the Lorenz coefficient,

$$\beta = \frac{1}{\left(1-\left(\frac{v}{c}\right)^2\right)^{1/2}}. \tag{1}$$

Also if two particles are moving on a straight line with respect to each other, the frequency

ω of the light that they emit varies according to $\omega' = \omega/k$ while the wavelength of that light varies by $\lambda' = \lambda k$. This is known as the Doppler effect and we refer to k as the Doppler coefficient. When the particles move in opposite directions then $k > 1$ and the frequency shift is towards the red whereas when the particles are moving towards each other $0 < k < 1$ and the shift is towards the blue where,

$$k = \frac{\left(1+\frac{v}{c}\right)^{1/2}}{\left(1-\frac{v}{c}\right)^{1/2}}. \tag{2}$$

Generally v/c is specified and all calculations follow. We consider the Doppler coefficient k to be the parameter with which to reference all other factors. If we set $k = \cot\theta$ for $0 \le \theta \le \pi/2$, then,

$$\cot 2\theta = \frac{1}{2}\left(k - \frac{1}{k}\right) = \frac{k^2-1}{2k} = \sinh \ln k. \tag{3}$$

It follows algebraically that

$$\beta = \cosh \ln k \tag{4}$$

and

$$v / c = \tanh \ln k. \tag{5}$$

This can be visualized both in terms of the geometry of a circle and that of a hyperbola. First consider the triangle shown in Fig. 1. From this,

$$\cos 2\theta = \frac{k^2-1}{k^2+1} = v / c \tag{6}$$

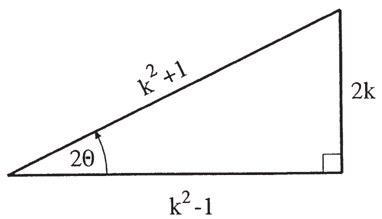

Fig. 1. A right triangle defines $\cos 2\theta$, $\sin 2\theta$, and $\cot 2\theta$ in terms of the parameter k.

$$\sin 2\theta = \frac{2k}{k^2+1} = 1/\beta. \tag{7}$$

In Fig. 2 these relationships are pictured in a circle. Setting $x = \cosh u$ and $y = \sinh u$ where $u = \ln k$ these relationships can be pictured on an hyperbola as shown in Fig. 3 where the area subtended by the curve and rays emanating from the origin is $A = u/2 = \ln \sqrt{k}$. The fact that the velocity is limited by the speed of light is reflected in the curve of tanh ln$k$ shown in Fig. 4 where $v \to c$ as $k \to \infty$ and $v \to -c$ as $k \to 0$.

It is well known that when velocities are added relativistically their sum is

$$\frac{v}{c} = \frac{\frac{v_1}{c} + \frac{v_2}{c}}{1 + \frac{v_1}{c}\frac{v_2}{c}}.$$

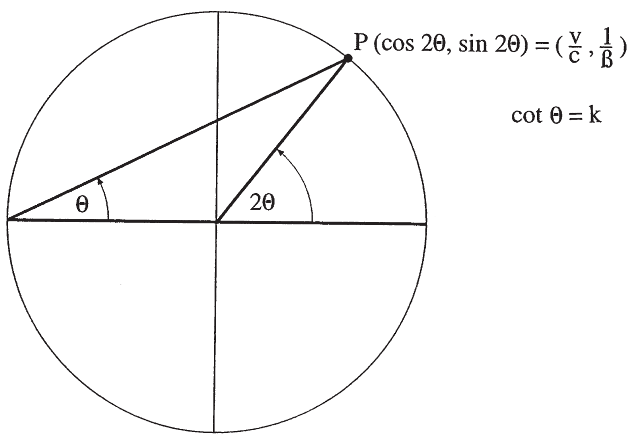

Fig. 2. The parameters v/c and β are represented by the geometry of a circle.

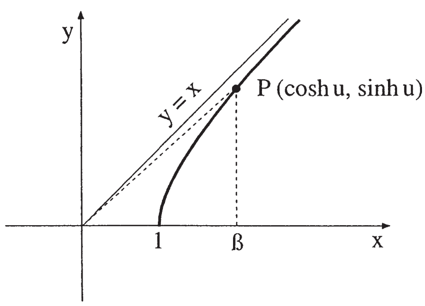

Fig. 3. The parameters β and k are represented by the geometry of an hyperbola.

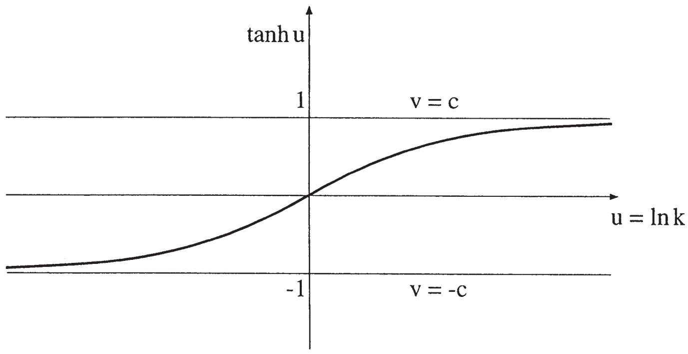

Fig. 4. The function v/c = tanh ln$k$ illustrates the speed of light c as the limiting velocity of a particle.

Adamson has generalized this formula to an arbitrary number of velocities,

$$\frac{v}{c} = \tanh\left(\tanh^{-1}\frac{v_1}{c} + \tanh^{-1}\frac{v_2}{c} + \tanh^{-1}\frac{v_3}{c} + \cdots\right). \tag{8}$$

3. Silver Means and the Cotangent Function

The solution to

$$x - \frac{1}{x} = N \text{ for positive integer } N \tag{9}$$

is the N-th silver mean of the first kind, $\mathrm{SM}_1(N)$ which arises naturally in dynamical systems theory (SCHROEDER, 1990). If $N = 1$ then $x = \mathrm{SM}_1(1) = (1+\sqrt{5})/2$, the golden mean, where as if $N = 2$, then $x = \mathrm{SM}_1(2) = 1 + \sqrt{2}$ referred to as the silver mean. As a result of Eq. (3) it follows that,

$$\mathrm{SM}_1(N) = e^{\sinh^{-1}\frac{N}{2}}. \tag{10}$$

Also if $k = \mathrm{SM}_1(N)$ then $v/c = N/\sqrt{4+N^2}$ and $\beta = \sqrt{4+N^2}/2$ follows from Eqs. (3), (4), (5), and (9).

4. p-cycles of the Cotangent Map

Consider the map,

$$x \to \sinh \ln x \tag{11a}$$

which has the property that

$$\cot\theta \to \cot 2\theta \tag{11b}$$

where now $x = \cot\theta$. Since $\sinh \ln x = (x^2-1)/2x$ it follows that this map is derived by applying Newton's method, $z \to z - f(z)/f'(z)$ for z complex, to the function $f(z) = z^2 + 1$. In other words it is the map that locates the root equal to the imaginary number i. If a seed $x_0 = \cot\pi/n$ for n an odd integer is taken in map 11a, then the trajectory is $x_0, x_1, x_2, ..., x_j, ...$ where $x_j = \cot(\pi 2^j/n)$ as a result of Eq. (11b). The trajectory is considered to be periodic with period p if $x_0 = x_p$ where p is the smallest integer for which $2^j = 1 \pmod n$. The trajectory is said to have a p-cycle. Likewise, the map $\cot\theta \to \cot m\theta$ has p-cycles when

$$m^p \equiv 1 \pmod n \tag{12}$$

for m relatively prime to n. This mapping can also be carried out by rational functions not shown here.

For n prime many p-values were tabulated by Beiler (BEILER, 1964). If n is an arbitrary odd integer the values of p are the result of the following Theorems.

Theorem 1 (Euler Fermat Theorem): If m and n are positive integers and m and n are relatively prime, then $m^{\phi(n)} \equiv 1 \pmod n$ where $\phi(n)$ is the Euler phi function (APOSTOL, 1976).
Remark: If n is prime then $\phi(n) = n - 1$ and Theorem 1 reduces to "Fermat's Little theorem."
Corollary 1: If m and n are relatively prime then m^j for $j = 0, 1, 2, 3, ...$ forms a p-cycle where p is a factor of $\phi(n)$.
Proof: The Corollary is an immediate consequence of Theorem 1 and Eq. (12).
Definition: The integer m is a primitive root modulus n if and only if for arbitrary $k \pmod n$ relatively prime to n there exists a unique integer j such that $m^j = k \pmod n$ where j is denoted by $j = \mathrm{ind}_m k$ (SCHROEDER, 1990).
Theorem 2: If m is a primitive root modulus n then $p = \phi(n)$.
Proof: If m is a primitive root modulus n, then $j = \mathrm{ind}_m k$ for each of the $\phi(n)$ values of k relatively prime to n.
Theorem 3: If m is not a primitive root modulus n then there is more than one p-cycle. The number T of p-cycles is given by $T = \phi(n)/p$.
How are these T, p-cycles found? The collection of points $\{x_k = \cot(\pi k/n)\}$ for k relatively prime to n are included in the T, p-cycles corresponding to n.

Let S be the set of indices relatively prime to n for $1 \le k < \phi(n)$,

$$S = \{k_0, k_1, ..., k_2, ..., k_{\phi(n)-1}\}.$$

Trajectory 1 has indices,

$$k_j^{(1)} = m^j\!: \; k_0^{(1)} = 1, \; m, \; m^2, ..., \; m^j, ..., \; m^p = 1 \pmod n$$

corresponding to the p-vector,

$$\overrightarrow{k^{(1)}} = \left(k_0^{(1)} = 1, k_1^{(1)}, \ldots, k_j^{(1)}, \ldots, k_{p-1}^{(1)}\right).$$

Let $k_0^{(i)}$ be the smallest integer from S not used in trajectories 1, 2, ..., i – 1. Then Trajectory i has indices,

$$k_j^{(i)} = k_0^i \cdot m^j:\ k_0^{(i)}, k_0^{(i)}m, k_0^{(i)}m^2, \ldots, k_0^{(i)}m^j, \ldots, k_0^{(i)}m^p = k_0^{(i)} (\bmod n)$$

corresponding to the p-vector $\vec{k}^{(i)} = (k_0^{(i)}, k_1^{(i)}, \ldots, k_j^{(i)}, \ldots, k_{p-1}^{(i)})$. The process continues until $i = T$ at which point all indices from S have been exhausted. These theorems are illustrated by the following three examples for the case of $m = 2$.

Example 1: $n = 11$ where 2 is a primitive root modulo 11. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. $k_j^{(1)} = 2^j$: 1, 2, 4, 8, 16 = 5, 32 = 10, 64 = 9, 128 = 7, 256 = 3, 512 = 6, 1024 = 1 (mod 11) so that $p = 10$ and $T = 1$ where $\overrightarrow{k^{(1)}} = (1, 2, 4, 8, 5, 10, 9, 7, 3, 6)$.

Example 2: $n = 7$ where 2 is not a primitive root modulus 7. $S = \{1, 2, 3, 4, 5, 6\}$. $k_j^{(1)} = 2^j$: 1, 2, 4, 8 = 1 (mod 7) so that $p = 3$ and $T = 2$ where $\overrightarrow{k^{(1)}} = (1, 2, 4)$. Therefore $k_0^{(2)} = 3$ and there is a second 3-cycle given by $k_j^{(2)} = 3 \cdot 2^j$: 3, 6, 12 = 5, 24 = 3 (mod 7) where $\overrightarrow{k^{(2)}} = (3, 6, 5)$.

Example 3: $n = 9$ where 2 is a primitive root modulus 9. $S = \{1, 2, 4, 5, 7, 8\}$. $k_j^{(1)} = 2^j$: 1, 2, 4, 8, 16 = 7, 32 = 5, 64 = 1 (mod 9) so that $p = 6$ and $T = 1$ where $\overrightarrow{k^{(1)}} = (1, 2, 4, 8, 7, 5)$.

Depending on n, there exist p-cycles of all lengths related to the doubling of the argument of the cotangent function. Another sequence of p-cycles was shown to be related to the doubling of the argument of the cosine function. These were shown to correspond to the Mandelbrot map at the extreme left hand point on the real axis, a point of chaos (KAPPRAFF, 2002; KAPPRAFF and ADAMSON, 2004).

5. The Cotangent Function and Chaos Theory

Steven Strogatz has described the following general theory of period doubling bifurcations (STROGATZ, 1994). Let $f(x, \mu)$ be any unimodal map that undergoes a period-doubling route to chaos. Suppose that the variables are defined such that the period-2 cycle is born at $x = 0$ when $\mu = 0$. Then for both x and μ close to 0, the map is approximated by

$$x_{n+1} = -(1+\mu)x_n + ax_n^2 + \cdots \tag{13a}$$

since the eigenvalue is –1 at the bifurcation. If the variable x is rescaled by $x \to x/a$, then Eq. (13a) can be rewritten as,

$$x_{n+1} = -(1+\mu)x_n + x_n^2 + \cdots. \tag{13b}$$

By definition of period-2, p is mapped to q and q to p. Therefore Eq. (13b) yields,

$$p = -(1+\mu)q + q^2 \quad \text{and} \quad q = -(1+\mu)p + p^2.$$

By subtracting one of these equations from the other, and factoring out $p - q$, and then multiplying these two equations together and simplifying yields

$$p + q = \mu \tag{14a}$$

and

$$pq = -\mu. \tag{14b}$$

From which it follows that

$$p = \frac{\mu + \sqrt{\mu^2 + 4\mu}}{2} \tag{15a}$$

and

$$q = \frac{\mu - \sqrt{\mu^2 + 4\mu}}{2}. \tag{15b}$$

Since the map $x \to f^2(x)$ has the same form as Eq. (13) it continues to hold as p and q bifurcate.

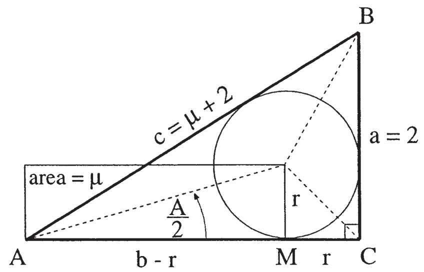

Fig. 5. The geometry of a right triangle represents the trajectory values p and q after a period doubling bifurcation where the radius of the inscribed circle $r = -q$, line segment AM = p, and the area of the rectangle formed by r and AM equals μ .

We have found a simple geometric representation of Eq. (15), once again using the cotangent function. Consider the right triangle ABC in Fig. 5 with sides a, b, and c.

The area of triangle ABC is computed by the equation,

$$A_{\text{triangle}} = sr \tag{16}$$

where s is the semi-perimeter and r the radius of the inscribed circle. From Eq. (16) it follows that,

$$r = \frac{ab}{a+b+c}. \tag{17}$$

From Fig. 5 it is clear that,

$$\cot\frac{A}{2} = \frac{b-r}{r}. \tag{18}$$

Using Eqs. (16) and (17) it follows after some algebra that cot(A/2) satisfies the equation,

$$x + \frac{1}{x} = \frac{2c}{a} \tag{19}$$

where c is the length of the hypothenuse of the triangle ABC. The solution $x > 1$ to this equation is referred to as the silver mean of the second kind SM_2 when $a = 2$ and c is an integer, i.e., the solution to $x + (1/x) = n$ is

$$x = \mathrm{SM}_2(n) = \frac{n + \sqrt{n^2 - 4}}{2} \quad \text{for} \quad x > 1. \tag{20a}$$

By the symmetry of this equation the other solution is

$$x = \frac{1}{\mathrm{SM}_2(n)} = \frac{n - \sqrt{n^2 - 4}}{2}. \tag{20b}$$

Let $a = 2$ and $c = N + 2$ in which case we have shown that,

$$\cot\frac{A}{2} = \mathrm{SM}_2(N+2). \tag{21}$$

Similar to Eq. (10),

$$\mathrm{SM}_2(N) = e^{\cosh^{-1}\frac{N}{2}}. \tag{22}$$

Using Eqs. (17) and (18) with $a = 2$, it is an exercise in algebra to show that,

$$p = \mathrm{SM}_2(N+2) - 1 \tag{23a}$$

and

$$q = \frac{1}{\mathrm{SM}_2(N+2)} - 1 = -r \tag{23b}$$

where $\mu = c - 2$ which equals N when $c = N + 2$. This is seen geometrically in Fig. 5 where the rectangle with sides r and AM has area, $A_{\text{rect}} = r(b - r) = N$.

From Eq. (14a), $p + q = N$ when $\mu = N$ an integer. However, it can be easily shown that

$$p^n + q^n = f_n(N) \tag{24a}$$

and

$$\frac{p^n + q^n}{p - q} = g_n(N) \tag{24b}$$

for n a positive integer and $f_n(N)$ and $g_n(N)$ integer functions of N where $f_n(N) = N$, $N(N + 2)$, and $(N + 2)^2(N - 1) + 4$ for $n = 1, 2$, and 3 respectively and $g_n(N) = 1$ and N for $n = 1$ and 2 respectively. Equation (24a) is proven by replacing p and q in Eq. (24) by their values from Eqs. (23) and using the fact that

$$\mathrm{SM}_2(N+2) + \frac{1}{\mathrm{SM}_2(N+2)} = N + 2.$$

Equation (24b) is proven by replacing p and q in Eq. (24) by their values in Eq. (23) and using Eq. (20) with $n = N + 2$. Equation (24) are generalized forms of Binet's formula given by,

$$\alpha^n + \beta^n = L_n(N) \quad \text{and} \quad \frac{\alpha^n + \beta^n}{\alpha - \beta} = F_n(N)$$

where $\alpha = \mathrm{SM}_1(N)$ and $\beta = -\{1/[\mathrm{SM}_1(N)]\}$, both satisfying Eq. (9), $L_n(N)$ are generalized Lucas numbers with $L_n(1) = 1, 3, 4, 7, 11, \ldots$ for $n = 1, 2, 3, \ldots$, and $F_n(N)$ are generalized Fibonacci numbers with $F_n(1) = 1, 1, 2, 3, 5, \ldots$ for $n = 1, 2, 3, \ldots$

Also it can be shown that,

$$\tan\frac{A}{2} = \frac{r}{s-a}, \quad \tan\frac{B}{2} = \frac{r}{s-b}, \quad \text{and} \quad \frac{C}{2} = \frac{r}{s-c}. \tag{25}$$

The silver means also have a close connection to continued fractions with periodic indices as shown in Appendix. Based on this connection we state the following theorem the proof of which is given in Appendix.

Theorem: For a and b positive integers, if $[a, b, a, b, \ldots] = \left[\overline{a,b}\right]$ is a continued fraction with repeated indices of period 2, then

$$\frac{\overline{[a,b]}}{\overline{[b,a]}} = \frac{b}{a} \quad \text{and} \quad \frac{1}{\overline{[b,a]}} - \overline{[a,b]} = ab.$$

If $a = 1$ and $b = N$ then this yields,

$$\frac{\overline{[1,N]}}{\overline{[N,1]}} = N \quad \text{and} \quad \frac{1}{\overline{[N,1]}} - \overline{[1,N]} = N.$$

It follows from Eq. (14) that p and q for $\mu = N$ an integer can be expressed as the continued fractions,

$$\frac{1}{p} = [N,1,N,1,N,1,\ldots] = \left[\overline{N,1}\right] \quad \text{and} \quad q = -[1,N,1,N,1,N,\ldots] = \left[\overline{1,N}\right]. \tag{26}$$

The continued fraction expansions of p and q for $\mu = N$ can be verified by the computational procedure given in Appendix.

We have found another application of Eqs. (15a) and (15b). Schroeder has described the characteristic impedance of the infinite electrical ladder circuit shown in Fig. 6 (SCHROEDER, 1990). The impedance Z_0 is given by the formula,

$$Z_0 = \frac{1}{2}\left(R + \sqrt{R^2 + \frac{4R}{G}}\right), \tag{27}$$

where R and G are the resistances of the circuit. However if we let $Z_0 = 1/r$, where $r = -q = \mu$, then since $-q = \mu/p$, from Eq. (14b) we have,

$$Z_0 = \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{\mu}}\right).$$

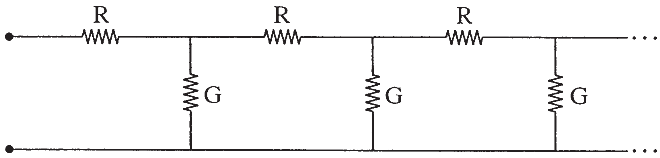

Fig. 6. An infinite ladder circuit with resistances R and G.

This corresponds to a normalized circuit with R = 1 and $G = \mu$ in Eq. (27). If $\mu = N$, an integer, then using Eq. (26), the following concise formula is obtained,

$$Z_0 = \frac{1}{\left[\overline{1,N}\right]}.$$

6. Conclusion

The cotangent function has been shown to span the areas of special relativity theory and the silver means of the first and second kind. It is useful for dynamical systems theory, leading to a characterization of periodic orbits of all lengths, and formulas governing the period doubling bifurcations of chaos theory. The period doubling bifurcations are seen to be directly connected to silver means and lead to a generalization of Binet's formula for expressing Fibonacci numbers. The cotangent function also arises in electrical circuit theory. It is significant that these connections are closely related to both geometry and the theory of numbers.

Appendix

Any real number can be expressed as a continued fraction of the form,

$$\alpha = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}} = \left[a_1; a_2 a_3 \ldots a_k \ldots\right]$$

where the indices a_1, a_2, a_3, ... are positive integers.

If α is a number between 0 and 1 then the leading integer can be eliminated. If the indices repeat with period p then this is indicated by the following notation,

$$\alpha = \left[a_1 a_2 \ldots a_p a_1 a_2 a_3 \ldots a_p a_1 \ldots\right] = \left[\overline{a_1 a_2 a_3 \ldots a_p}\right].$$

Consider a real number $0 < \alpha < 1$ expressible as a continued fraction with periodic indices $\left[\overline{n_1 n_2 n_3 \ldots n_k \ldots n_{p-1} n_p}\right]$. The approximants obtained by truncating the continued fraction at the k-th position is given by a_k/b_k for k = 1, 2, 3, ..., p, i.e.,

$$\begin{array}{cccccc} n_1 & n_2 & n_3 & n_k & n_{p-1} & n_p \\ \dfrac{a_1}{b_1} & \dfrac{a_2}{b_2} & \dfrac{a_3}{b_3} & \dfrac{a_k}{b_k} & \dfrac{a_{p-1}}{b_{p-1}} & \dfrac{a_p}{b_p} \end{array}$$

where

$$\frac{a_1}{b_1} = \frac{1}{n_1}, \quad \frac{a_2}{b_2} = \frac{n_2}{n_1 n_2 + 1}, \quad \frac{a_3}{b_3} = \frac{a_2 n_3 + a_1}{b_2 n_3 + b_1}, \quad \frac{a_k}{b_k} = \frac{a_k n_{k+1} + a_{k-1}}{b_k n_{k+1} + b_{k-1}} \quad \text{for} \quad k \geq 3.$$

Consider the matrix,

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

where $a = a_{p-1}$, $b = b_{p-1}$, $c = a_p$, and $d = b_p$.

We can now state the following theorem.

Theorem: Let a/b and c/d be the (p–1)-th and p-th approximant to the continued fraction expansion of α where $0 < \alpha < 1$ and has periodic indices of period p. If

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

then $\alpha = \left[\overline{n_1 n_2 n_3 \ldots n_k \ldots n_{p-1} n_p}\right] = c/(\lambda - a)$ where $\lambda = \mathrm{SM}_1(\mathrm{Tr}M)$ for p odd and $\lambda = \mathrm{SM}_2(\mathrm{Tr}M)$ for p even.

Proof: We outline the proof with details left to the reader.

Consider n iterates of a periodic continued fraction $\left[\overline{n_1 n_2 n_3 \ldots n_k \ldots n_{p-1} n_p}\right]$ with period p

$$\begin{array}{ccccccc} n_1 & n_2 & n_{p-1} & n_p & n_{p(n-1)+1} & n_{np-1} & n_{np} \\ \dfrac{a_1}{b_1} & \dfrac{a_2}{b_2} & \dfrac{a_{p-1}}{b_{p-1}} & \dfrac{a_p}{b_p} & \dfrac{a_{p(n-1)+1}}{b_{p(n-1)+1}} & \dfrac{a_{np-1}}{b_{np-1}} & \dfrac{a_{np}}{b_{np}} \end{array}$$

where we let $a = a_{p-1}$, $b = b_{p-1}$, $c = a_p$, and $d = b_p$.

Let

$$M = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

It can be shown by induction that

$$\begin{bmatrix} a_{np-1} & c_{np} \\ b_{np-1} & d_{np} \end{bmatrix} = M^n .$$

Therefore

$$\left[\overline{n_1 n_2 n_3 \ldots n_k \ldots n_{p-1} n_p}\right] = \lim_{n\to\infty} \frac{a_{np}}{b_{np}} .$$

The characteristic equation of M is given by: $\det(M - \lambda I) = \lambda^2 - \lambda \mathrm{Tr}M + \det M$. But $\mathrm{Tr}M = a + d$ and for continued fraction expansions it is well known that $\det M = 1$ when p is even and -1 when p is odd. Therefore the eigenvalues satisfy the equation, $\lambda \pm 1/\lambda = \mathrm{Tr}M = a + d = t$ and $\lambda = \mathrm{SM}_2(t)$ for p even and $\lambda = \mathrm{SM}_1(t)$ for p odd. It follows that

$$\lambda_1 = \frac{t + \sqrt{t^2 \mp 4}}{2}$$

and that $\lambda_2 = 1/\lambda_1$ for p even and $\lambda_2 = -(1/\lambda_1)$ for p odd where $\lambda_1 > 1$ and $|\lambda_2| < 1$. Since there is a complete set of eigenvalues, M is diagonalizable. Diagonalizing M, it follows that

$$M^n = V \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} V^{-1}$$

where V is a matrix of eigenvectors of M. Letting $\lambda_1 = \lambda$ and recognizing that $\lambda_2 = \pm(1/\lambda_1)$,

$$V = \begin{bmatrix} c & -c\lambda \\ \lambda - a & \pm\lambda a - 1 \end{bmatrix}$$

and

$$V^{-1} = \frac{1}{c\left(\lambda^2 - 1\right)} \begin{bmatrix} \pm\lambda a - 1 & c\lambda \\ a - \lambda & c \end{bmatrix}$$

where the positive sign corresponds to p odd and the negative sign to p even. After a lengthy computation, we find that

$$a_{np} = c^2\lambda^{n+1} \frac{c^2}{(\pm)^{n-1}\lambda^{n-1}} \quad \text{and} \quad b_{np} = c\lambda^{n+1}\left(\lambda - a\right) + \frac{(\lambda a - 1)c}{(\pm)^n \lambda^n} .$$

Since $\lambda > 1$, it follows that,

$$\left[\overline{n_1 n_2 n_3 \ldots n_k \ldots n_{p-1} n_p}\right] = \lim_{n\to\infty} \frac{a_{np}}{b_{np}} = \frac{c}{\lambda - a}.$$

Corollary: $\dfrac{\left[\overline{a,b}\right]}{\left[\overline{b,a}\right]} = \dfrac{b}{a}$ and $\dfrac{1}{\left[\overline{b,a}\right]} - \left[\overline{a,b}\right] = ab.$

Proof: For $\left[\overline{a,b}\right]$ it follows that

$$M_1 = \begin{bmatrix} 1 & b \\ a & ab+1 \end{bmatrix}$$

and therefore, $\left[\overline{a,b}\right] = b/(\lambda_1 - 1)$. Likewise it can be shown that for $\left[\overline{b,a}\right]$,

$$M_2 = \begin{bmatrix} 1 & a \\ b & ab+1 \end{bmatrix}$$

and $\left[\overline{b,a}\right] = a/(\lambda_2 - 1)$. Since $\lambda_1 = \lambda_2$ the first part of the corollary follows. The second part follows in a similar manner making use of the fact that $\lambda = \mathrm{SM}_2(ab + 1)$.

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