

## Simulations of Sunflower Spirals and Fibonacci Numbers

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**Abstract.** Computer simulations to produce spiral patterns are made by an algorithm to put successive points on the 2D plane with a fixed angle increase around a center and a fixed distance increase from the center. After putting each point, it is connected to the nearest point among the points already put. When the angle increase is equal to that by golden section of  $2\pi$ , the curves made of the point connections become spirals growing from the center, and their number is one of the Fibonacci numbers. These spirals make branching as growing, and several Fibonacci numbers coexist. This situation is seen in real sunflowers. Next, a new algorithm is proposed to produce the Fibonacci sequence. In contrary to the conventional one, i.e. to add a new term  $a_n (=a_{n-2} + a_{n-1})$  successively at the right end, we add a new term  $a_1 (=1)$  at the left as the initial term and shift the other terms to the right. Under certain conditions neighboring terms are allowed to merge to a new term with summation of the two terms. Then, at discrete steps we have Fibonacci sequences growing successively. Relations between the productions of the sunflower spirals and the Fibonacci sequences are discussed.

### 1. Introduction

It is well known that small flower elements are arranged along spirals on a sunflower and that the number of spirals is found to be one of Fibonacci numbers (WEYL, 1952; KAPPRAFF, 1991, etc.). Mechanism of formation of these spirals is proposed by DOUADY and COUDER (1992). They made a numerical simulation and an experiment by the use of small magnetic balls. These studies show that, if flower elements appear at a position on a circle successively with a constant delay time under the influence of repulsive forces from those appearing earlier and if those flower elements shift towards the radial direction, then they form sunflower patterns similar to real ones with spirals of Fibonacci numbers. Since position of each new flower element is determined so that the potential energy associated with the repulsive force becomes minimum, this problem is analogous to that of phyllotaxis

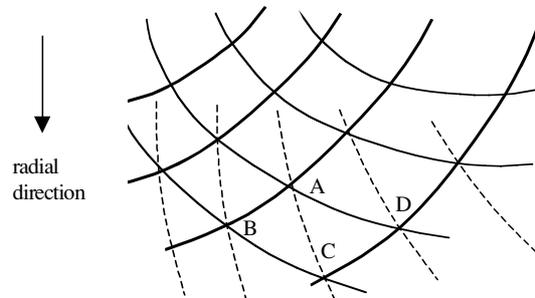


Fig. 1. Parallelograms formed by two sets of spirals.

which assures the most efficient configuration of leaves. Another mechanism for sunflower spiral pattern is shown to appear in Benard convection cells and bubble arrangement (RIVIER *et al.*, 1984; RIVIER and WEAIRE, 1984) and around a dislocation point in the elastic body (NIRI and RIVIER, 2002). This is also understood in terms of the minimum energy principle. Precise mathematical discussion on the nature of spiral is given by AZUKAWA and YUZAWA (1990) and AZUKAWA (1995).

The mechanism how the Fibonacci numbers come out for number of spirals can be explained as follows, which is also suggested by DOUADY and COUDER (1992). Suppose two sets of spirals begin to be formed on a 2D plane or a convex surface (see Fig. 1), whose numbers are  $a_{n-2}$  and  $a_{n-1}$ , respectively. Neighboring spiral curves in each set are assumed to keep a constant distance. Two curves from two sets of spirals form parallelograms, one of which is indicated as ABCD in Fig. 1. As the spirals are extended in radial direction (let radius from the center of spirals be denoted by  $r$ ), the parallelograms become more and more flattened, because, in the parallelogram ABCD, the distance BD grows in proportion to  $r$ , while another distance AC remain nearly constant. Then, according to the psychology of observers, they tend to recognize new spirals with number  $a_{n-2} + a_{n-1}$  extending to the direction of AC (indicated by the dashed lines), while one of the two sets of spirals directed more to radial direction corresponding to the larger number of spirals  $a_{n-1}$  (drawn by thicker lines) remains to be recognized. Hence, the new sets have  $a_{n-1}$  and  $a_n$  spirals, where

$$a_n = a_{n-2} + a_{n-1}. \quad (1)$$

As the radius is increased, these numbers increase with this summation rule, and we have Fibonacci numbers as the number of spirals.

It is noted here that we need one more assumption to obtain Fibonacci numbers, i.e. the number of spirals included in the initial sets appearing near to the center should be two successive Fibonacci numbers, such as 5 and 8. This is confirmed by DOUADY and COUDER (1992) by introducing the concept of minimum potential energy in flower arrangement. However, there seems to be no theoretical explanation why the minimum energy has lead to the Fibonacci numbers.

The above reasoning for production of spirals of Fibonacci numbers suggests inevitably

that the point configuration is composed of several concentric layers with different radii and different sets of spirals, where a layer with larger radius has larger Fibonacci numbers. This assertion contradicts to the usual habit owned by a lot of observers to admit only one layer in a sunflower with two sets of spirals. One of the present authors (Yuriko Ogiso) found in workshops for children that they had difficulty in counting the number of spirals within a sunflower, because it depended on which part of the sunflower they observed. Thus, the present authors are inclined to consider that the flower arrangement is a combination of several layers, each of which are associated with different Fibonacci numbers.

It should be stressed here that the recognition of spirals is a problem of psychology, and it is also possible to draw only two sets of spirals by extending them to far outer region, although it looks unnatural. Therefore, in order to consider the problem of recognition of spirals by human eye, we must introduce an objective criterion to define curves within 2D point. The simplest criterion might be that the curves should be drawn by connecting nearest points. This is the motivation of the first half of the present studies, which is explained in Sec. 2.

One of the present authors (Ryuji Takaki) had an idea of a new algorithm to produce the Fibonacci sequence. The conventional way to produce the Fibonacci sequence is to begin with initial two terms  $a_1 = 1, a_2 = 1$  and to add a new term  $a_n$ , which is a sum of the preceding two terms  $a_{n-2} + a_{n-1}$ . On the other hand, the Fibonacci numbers in the sunflower are realized while it grows from the center, i.e. new flower elements are created at the center and the preceding elements are pushed outwards. In this process the two Fibonacci numbers are merged by the mechanism shown in Fig. 1. It would suggest that the Fibonacci sequence can be produced by adding new initial terms at the left end of the sequence and merging preceding terms. This algorithm is introduced in Sec. 3. A discussion is given in Sec. 4 on the relation between this algorithm and the present simulation of sunflower.

## 2. Simulation of Sunflower Spirals

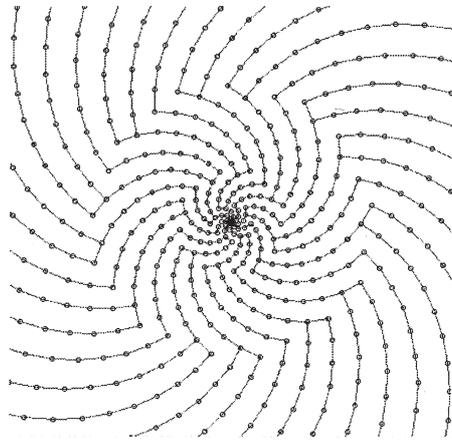
### 2.1. Method of simulation and results

In order to arrange points on the 2D plane the polar coordinate system  $(r, \theta)$  is introduced. Coordinates of successive points  $(r_n, \theta_n)$  are assumed to be given by

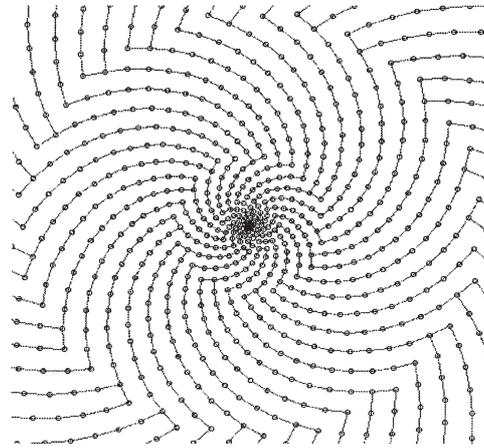
$$r_n = Vn, \quad \theta_n = \phi_n, \quad (2)$$

where  $V$  and  $\phi$  are constants. The unit of length is determined in relation to the pixel size in graphic image, i.e. 1 pixel corresponds to the length 15. These equations do not contain constant terms, but they are not losing generality, because the value of  $n$  for the initial point can be chosen arbitrarily and the origin of the angle  $\theta$  can be shifted arbitrarily. The typical point arrangement is the case of the golden division, where  $\phi = 360^\circ/(1 + 1.618) = 137.51^\circ$ . The points were put up to  $n = 2000$ .

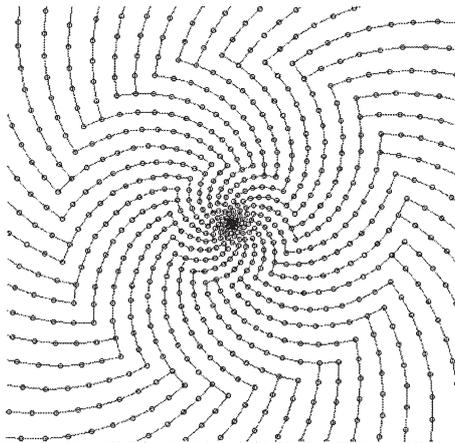
As  $n$  is increased, the region occupied by points grows outwards. This region is called "inner region" when looked from a new point. Instead of putting the  $n$ -th points at the radius  $r_n$ , we can put points always on a circle with fixed radius and shift all points in the radial direction with a constant speed; this method corresponds to the real process in the



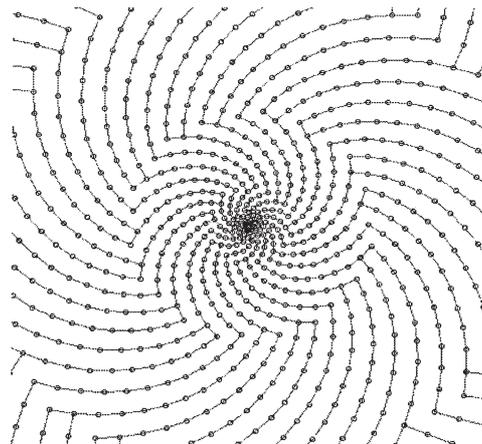
(a)  $V = 8, \phi = 137.51^\circ$  (golden division)  
(3, 5, 8, 13, 21, 34, ...)



(b)  $V = 5, \phi = 151.13^\circ$   
(5, 7, 12, 19, 31, 50, ...)



(c)  $V = 5, \phi = 99.5^\circ$   
(4, 7, 11, 18, 29, 47, ...)



(d)  $V = 5, \phi = 47.25^\circ$   
(7, 8, 15, 23, 38, 61, ...)

Fig. 2. Results of point arrangements and spiral formations. Numbers of spirals are given in parentheses arranged from inner to outer layers.

sunflower. However, resulting point arrangements due to these methods are essentially the same.

Spiral curves for resulting point arrangement are drawn in the following way. After putting each new point, it is connected by a line segment to the nearest point among those in the inner region. In this way the ambiguity of fixing spirals is avoided. On the other hand, this way of drawing curves allows appearance of kinks and branching.

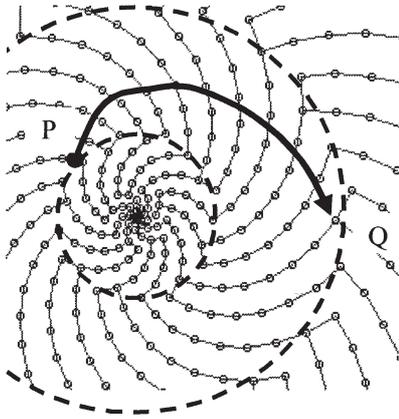


Fig. 3.

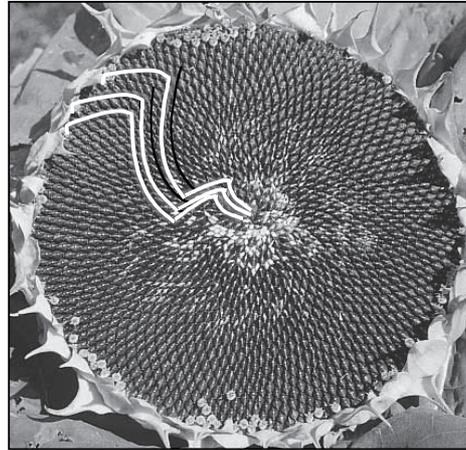


Fig. 4.

Fig. 3. Correspondence of the kink of a spiral in the inner layer and the branch point of a spiral in the intermediate layer.

Fig. 4. An example of drawing spirals on a real sunflower. We can see 89, 55, 34, 21 spirals in the layers (from outer to inner).

Some examples of resulting point arrangements and spiral curves are shown in Fig. 2. The spiral curves drawn by connecting nearest points have kinks and branching, which contribute to dividing the whole region into several layers. Numbers of spirals counted for these layers are shown below each figure within parenthesis. Note that these numbers come from the denominator  $q_n$  of the  $n$ -th approximant  $r_n$  in the continued-fraction expansion of  $\phi/360^\circ$  (pp. 172–173 of AZUKAWA and YUZAWA, 1990). Figure 2(a) is the result for the golden division, and the numbers of spirals are Fibonacci numbers as is expected. On the other hand, for other cases with other division ratios (Figs. 2(b), (c), (d)) spiral patterns look similar to that with the golden section, but the numbers of spirals are different from Fibonacci numbers. However, the same rule (1) as that for Fibonacci sequence is satisfied for any of these cases.

Close observation of these spiral patterns reveals (see Fig. 3) that, if a spiral ending with a kink (point A) in an inner layer is extended outwards into the outer layer, it comes to a branch point (point Q) of a spiral in the intermediate layer. Its reason is not yet clear, but this fact means that the numbers of spirals in the outer layer is the sum of numbers in the inner and the intermediate layers, hence the sum rule (1) is valid for any division ratio.

Moreover, several trials proved that the types of resulting spiral patterns do not depend on the value of  $V$ , i.e. patterns with different values of  $V$  are geometrically similar to each other. This fact can be understood in the following way. If the value of  $V$  is increased, points are more separated in the radial direction. But, as the value of  $n$  is increased (hence,  $r$  is increased), distances between points in the circumferential direction increases in proportion to  $r$ , and we have a similar situation as that without increasing  $V$ . Therefore, in the present

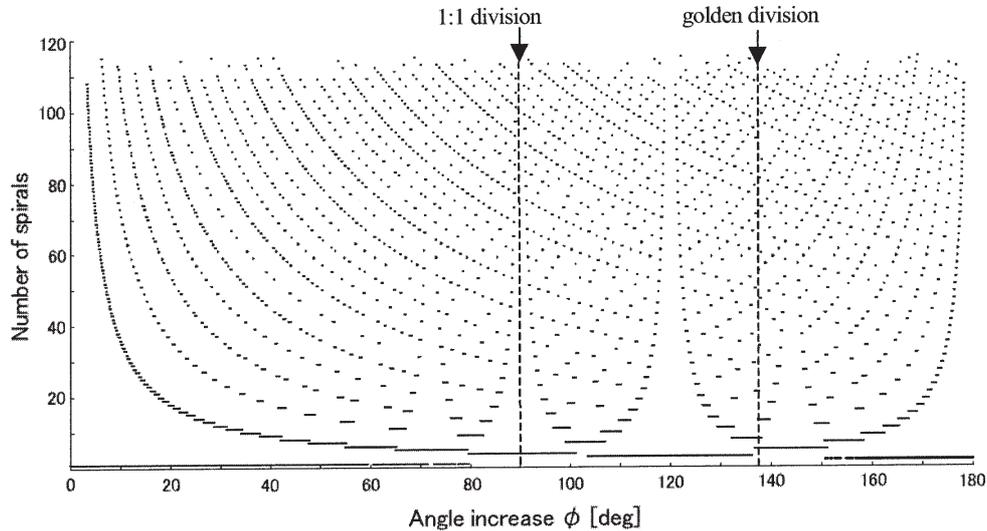


Fig. 5. Dependence of the numbers of spirals on the division angle  $\phi$  in the case with  $V = 20$ . The two cases with  $\phi = 90^\circ$  and  $137.5^\circ$  are indicated by vertical fines. In the former case the vertical fine intersects with one curve with value 2 for the number of spirals. In the latter, the vertical line inter-sects with many curves with numbers of spirals corresponding to the Fibonacci numbers.

simulations values of  $V$  are chosen so that points are not crowded too much.

An example to draw spirals on a real sunflower is shown in Fig. 4, where three kinked curves are drawn. We can see 89, 55, 34, 21 spirals in the layers (from outer to inner), while spirals in the fifth layer are difficult to count.

## 2.2. Dependence of the number of spirals on the parameter $\phi$

Production of spiral patterns was made for various values of  $\phi$  in  $(0, 180^\circ)$ , and numbers of spirals in layers were counted. Its results are shown in Fig. 5. The curves in this figure are expressed by arrays of dots, but they should be continuous curves. Two typical cases with  $\phi = 90^\circ$  and  $137.5^\circ$  are indicated by vertical lines. In the former case, where points are arranged along two straight radial lines, the vertical line in the figure intersects only once with a curve at the value 2 for number of spirals. In the case of the golden division the vertical line intersects with many curves with numbers of spirals corresponding to the Fibonacci numbers. The horizontal curve with value 1 is not important except near the origin ( $\phi \doteq 0$ ), where we have only one spiral. This horizontal curve is the result of connection to the first point from the second.

The region in this figure occupied by curves has a rather complicated shape. Above the lowest curves the region is separated into many parts by vertical cracks. Locations of these cracks are at angles with divisions of  $360^\circ$  by integers, which produce those numbers of radial lines as are shown by the lowest curve (monotonically decreasing. In the similar way divisions of  $360^\circ$  by rational numbers are considered to make cracks, although it is rather

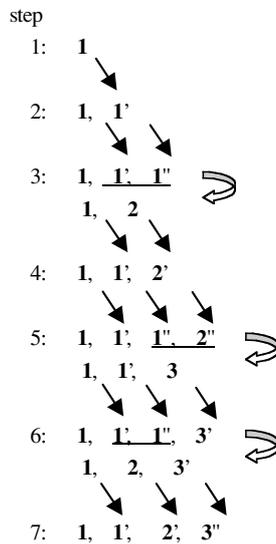


Fig. 6. First several steps of producing Fibonacci numbers. The single and double primes indicate ages 1 and 2, respectively. The oblique arrows indicate the shifts of terms. The round arrows indicate the mergers of two terms with underlines.

difficult to recognize in this figure. The case of  $\phi = 145^\circ = 360^\circ \div (5/2)$  can be seen to give a crack above the lowest curve and the second lowest curve (located between  $120 < \phi < 180$ ). The present authors have a guess that a value of  $\phi$  given by a division of  $360^\circ$  by a rational number, i.e.  $360^\circ \div (m/k)$ , makes a crack above  $k$  continuous curves.

### 3. A New Algorithm for Production of Fibonacci Sequence

#### 3.1. Rules in the algorithm

The following two rules are assumed for production of Fibonacci sequence. The sequence is produced step by step by applying these rules.

**Rule 1:** An initial term with value 1 is added at each step, and other terms are shifted to the right.

**Rule 2:** Neighboring two terms merge to make a term, whose value is the sum of these terms, if the conditions below are satisfied. This process occurs without increasing the step number.

(A) Terms have ages (steps after birth) equal to or more than their values.

(B) Terms with the same values do not appear except for the initial and the second terms.

Figure 6 shows how these rules work. In the first step we have only one term with value 1. In the second step this term is shifted to the right, as indicated by an oblique arrow in Fig. 6, and a new term with value 1 is added at the left. The second term has an age 1 (1 step after

Table 1. First 20 steps of producing Fibonacci numbers. The first and the second numbers in each parenthesis in the table indicate the value of the term and the age after birth, respectively. Products by merger are shown with arrows. Sequences with bold faces indicate intermediate stages of Fibonacci sequence production.

Step	Produced sequence and the results of merger
1	{1,0}
2	<b>{1,0}, {1,1}</b>
3	{1,0}, {1,1}, {1,2} → {1,0}, {2,0}
4	<b>{1,0}, {1,1}, {2,1}</b>
5	{1,0}, {1,1}, {1,2}, {2,2} → {1,0}, {1,1}, {3,0}
6	{1,0}, {1,1}, {1,2}, {3,1} → {1,0}, {2,0}, {3,1}
7	<b>{1,0}, {1,1}, {2,1}, {3,2}</b>
8	{1,0}, {1,1}, {1,2}, {2,2}, {3,3} → {1,0}, {2,0}, {5,0}
9	{1,0}, {1,1}, {2,1}, {5,1}
10	{1,0}, {1,1}, {1,2}, {2,2}, {5,2} → {1,0}, {1,1}, {3,0}, {5,2}
11	{1,0}, {1,1}, {1,2}, {3,1}, {5,3} → {1,0}, {2,0}, {3,1}, {5,3}
12	<b>{1,0}, {1,1}, {2,1}, {3,2}, {5,4}</b>
13	{1,0}, {1,1}, {1,2}, {2,2}, {3,3}, {5,5} → {1,0}, {1,1}, {3,0}, {8,0}
14	{1,0}, {1,1}, {1,2}, {3,1}, {8,1} → {1,0}, {2,0}, {3,1}, {8,1}
15	{1,0}, {1,1}, {2,1}, {3,2}, {8,2}
16	{1,0}, {1,1}, {1,2}, {2,2}, {3,3}, {8,3} → {1,0}, {2,0}, {5,0}, {8,3}
17	{1,0}, {1,1}, {2,1}, {5,1}, {8,4}
18	{1,0}, {1,1}, {1,2}, {2,2}, {5,2}, {8,5} → {1,0}, {1,1}, {3,0}, {5,2}, {8,5}
19	{1,0}, {1,1}, {1,2}, {3,1}, {5,3}, {8,6} → {1,0}, {2,0}, {3,1}, {5,3}, {8,6}
20	<b>{1,0}, {1,1}, {2,1}, {3,2}, {5,4}, {8,7}</b>

birth), which is indicated by a prime. In the third step these two terms are shifted to the right getting ages 1 and 2 (single and double primes), and a new term is added at the left. Then, since the ages of the second and the third terms are either equal to or more than their values 1, they are allowed to merge to a term with value 2 (as indicated by the round arrow in Fig. 6). By repeating these processes we can produce further terms.

### 3.2. Results of applying the algorithm

Up to the first 7 steps, as shown in Fig. 6, we have the first 4 terms of the Fibonacci numbers. Table 1 shows the result up to the 20-th step, where values and ages of terms are shown in parentheses. As is seen from this Table we have first several terms of Fibonacci numbers at the second, 4-th, 7-th, 12-th, and 20-th steps. Intervals between these steps are equal to the maximum Fibonacci numbers in the sequences, which have appeared after respective intervals. Sequences at the next of these steps are composed of Fibonacci numbers chosen each from every two terms of Fibonacci sequence.

Further processes were followed by the use of computer. Results at discrete steps, where intermediate stages of Fibonacci sequence are obtained, are shown in the Appendix. In the next steps of those steps, all terms have enough ages to merge with neighbors, so that in the 21-th step, for example, we have a sequence {1, 2, 5, 13}. The same is true for the 34-th step, 55-th step, etc.

#### 4. Conclusions and Discussion

In this paper we have shown that the successive points with golden division of one turn around the center and constant radial shift produces a point arrangement which are composed of layers, in each of which spirals of different Fibonacci numbers are recognized. An example of spiral recognition for a real sunflower is also shown. This finding is consistent with the conventional understanding in the past, if a limited part of a sunflower is observed. For other divisions spirals of other numbers appear. The senses of rotation of spirals in these layers are reversed in neighboring layers.

Next, a new algorithm to produce the Fibonacci sequence is proposed. In this algorithm a new term is continuously added at the left of the sequence as the first term and mergers of terms are made when certain conditions are satisfied. This algorithm is shown to produce the Fibonacci sequence at discrete steps.

These two works are made independently, but they seem to be related each other. A discussion is given here as to this relation. The algorithm to produce the Fibonacci sequence is considered to match well to the spiral pattern of sunflower, because in the latter new flower elements always appear at the center (corresponding to adding the initial term continuously in the former).

In formation of real sunflower patterns a process of rearrangement of flower elements in each layer is necessary. As the elements are shifted in radial direction, they must be arranged along a circumference with larger radius, hence the rearrangement to a pattern with larger Fibonacci number becomes inevitable. Time interval required for this rearrangement is considered to be proportional to the number of flower elements to be rearranged, which correspond to the interval steps until the intermediate stages of Fibonacci sequence are attained (see Table 1 and Appendix). Note that in this interval steps the larger Fibonacci numbers do not change, while smaller ones make frequent changes. This situation seems to correspond to the continuous rearrangement of flower elements just after created at the center.

Rules in the algorithm can be also understood based upon the above notions. The Rule 1 would reflect the fact that new flower elements are created at the center, and that they push the other elements in the radial direction. The Rule 2 is a translation of the psychological effect to recognize a group of spirals in layers in a sunflower, whose number is the sum of spiral numbers of the two sets in the inner layer, as shown in Fig. 1. The condition (A) in Rule 1 would mean that the flower elements need some steps in order to make the next merger, during which they are making rearrangements. The condition (B) is posed in order to obtain more reasonable arrangements, i.e. outer layers should have more number of flower elements for uniform packing of elements.

After all, even if the two works presented in this paper look quite different from each other, they are related closely each other. They may contribute together to better understanding of the sunflower pattern and the Fibonacci numbers.

The present authors would like to express their cordial thanks to Professor J. Kappraff and Professor T. Ogawa for their helpful comments on this work.

## Appendix. Fibonacci sequenses at discrete steps after the 20-th step.

Step	{Values of terms, ages}
20	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}
33	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}
54	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}
88	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}
143	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 43}, {55, 54}
232	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}
376	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}
609	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}
986	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}
1596	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}
2583	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}
4180	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}
6764	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}
10945	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}
17710	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}
28656	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}, {10946, 10945}
46367	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}, {10946, 10945}, {17711, 17710}
75024	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}, {10946, 10945}, {17711, 17710}, {28657, 28656}
121392	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}, {10946, 10945}, {17711, 17710}, {28657, 28656}, {46368, 46367}
196417	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}, {10946, 10945}, {17711, 17710}, {28657, 28656}, {46368, 46367}, {75025, 75024}
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832039	{1, 0}, {1, 1}, {2, 1}, {3, 2}, {5, 4}, {8, 7}, {13, 12}, {21, 20}, {34, 33}, {55, 54}, {89, 88}, {144, 143}, {233, 232}, {377, 376}, {610, 609}, {987, 986}, {1597, 1596}, {2584, 2583}, {4181, 4180}, {6765, 6764}, {10946, 10945}, {17711, 17710}, {28657, 28656}, {46368, 46367}, {75025, 75024}, {121393, 121392}, {196418, 196417}, {317811, 317810}
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