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Systematic Study of Convex Pentagonal Tilings, I: Case of Convex Pentagons with Four Equal-length Edges

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Abstract. At the beginning of the series of papers we present systematic approach to exhaust the convex pentagonal tiles of edge-to-edge (EE) tilings. Our procedure is to solve the problem systematically step by step by restricting the candidates to some class. The first task is to classify both of convex pentagons and pentagonal tiling patterns. The classification of the latter is based on the analysis of vertex patterns of pentagonal tiling. As the first step of the procedure, the candidates are restricted to the pentagons with four edges of the equal length. Furthermore, the analysis is restricted to the simplest category of node conditions. As a result, we obtained the result that in the above restricted tilings, 14 patterns are possible by the combinatorial analysis, the topological judgment and the geometric judgment.

1. Introduction

Packing, covering, and tiling are the central topics of the geometry of arrangement and have important applications in the consideration of ideal geometric models in a wide range of scientific fields. Tiling refers to coverage of the plane with polygons (tiles) without gaps or overlapping, and a single congruent polygon that tiles a plane is called a *prototile* (GRÜNBAUM and SHEPHARD, 1987), or a *polygonal tile*. The tiled arrangement of polygons on a plane is referred to as a *tiling pattern* in this study. Plane tiling with convex polygons has primarily been studied in an attempt to exhaust all the possible shapes that can tile a plane by a single congruent element. The current state of knowledge in this field can be summarized as follows.

(I) Any single triangle and quadrilateral, including concave quadrilaterals, are tileable (i.e., all prototiles), since the sums of the internal angles of triangles and quadrilaterals divide evenly into 360° and all vertices can be concentrated to one point when the corresponding edges are fitted together.

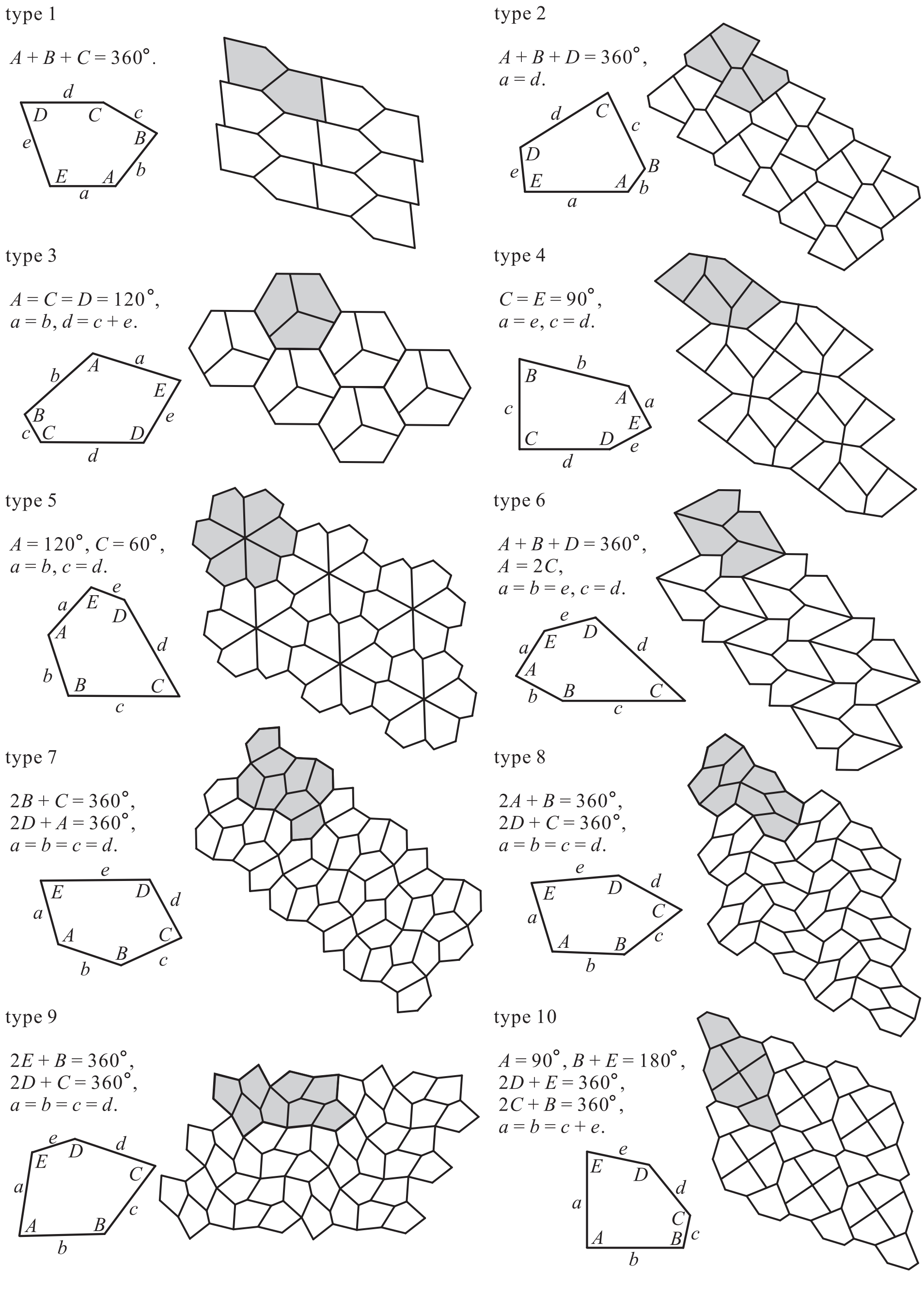

Fig. 1. Convex pentagonal tiles of type 1–10.

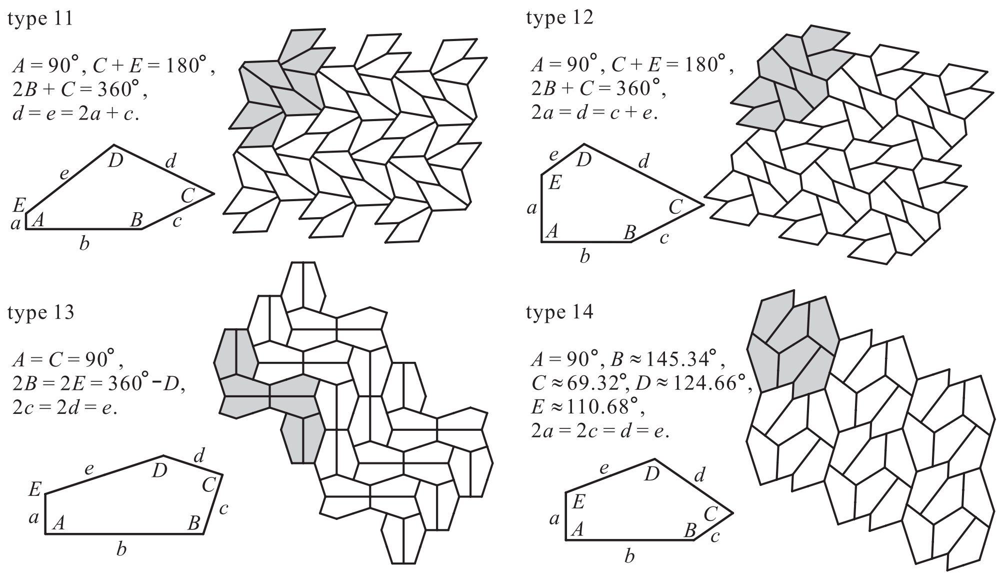

Fig. 2. Convex pentagonal tiles of type 11–14.

(II) Tiling is not necessarily possible with other convex polygons because of constraints on angles and edge lengths. For example, no tiling exists for a polygon if no combination of its interior angles sum to 360°.

(i) In the case of convex hexagons, prototiles can be categorized into three types (Reinhardt, 1918; Bollobás, 1963; Gardner, 1975; Grünbaum and Shephard, 1987).

(ii) For convex polygons with seven or more edges, no prototiles exist (Reinhardt, 1918; Grünbaum and Shephard, 1987).

(iii) For the convex pentagonal tiles, there are at present 14 classifications (see Figs. 1 and 2), but it remains unproven whether this is the perfect list of such pentagons (Reinhardt, 1918; Kershner, 1968; Gardner, 1975; Schattschneider, 1978, 1981; Hirschhorn and Hunt, 1985; Grünbaum and Shephard, 1987; Wells, 1991; Sugimoto, 1999; Sugimoto and Ogawa, 2000a). Note that, as shown in Figs. 1 and 2, each of convex pentagonal tiles is defined by some conditions between lengths of edges and magnitudes of angles; but some degrees of freedom remain. (However, only the pentagonal tile of type 14 does not have any degrees of freedom except size. For example, the exact value of C in pentagon of type 14 is $\cos^{-1}((3\sqrt{57} - 17)/16)$, and the values of angles B, D, and E can be obtained by C.) Then, unless a convex pentagonal tile is a new prototile, any convex pentagonal tile belongs to one or more of 14 types. The pentagonal case is the only unsolved problem in the study of tiling the plane with congruent convex polygons, and the problem has yet to be approached scientifically (Sugimoto and Ogawa, 2003a).

The solution to this sole remaining problem of tiling the plane with convex pentagons requires a systematic approach, that is, the problem should be divided into stages and study

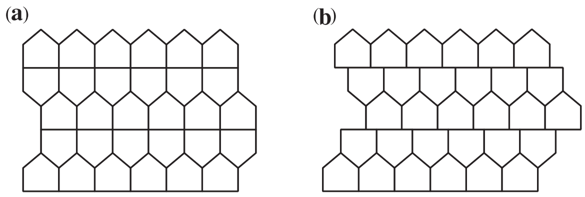

Fig. 3. Examples of (a) edge-to-edge tiling and (b) non-edge-to-edge tiling.

in each stage should be repeated from various perspectives. It is believed that this approach will eventually lead to the final result.

The discussion in the present paper is limited to the problem of convex pentagonal tiles. The problem of deriving a perfect list of all prototiles is separate from the problem of identifying tiling patterns, although the two are closely related (SUGIMOTO and OGAWA, 2003a). However, without classifying tiling patterns, it will be impossible to express the necessary and sufficient conditions for identifying prototiles. Dividing the problem based on clear criteria is an effective systematic approach to solving the problem. Thus, convex pentagons and tiling patterns are classified here into finite sets and each set is investigated for each classification to obtain the solution. Through such systematic examination, it is expected that the complete list of convex pentagon tiles will eventually be obtained.

The tilings themselves can be distinguished into two kinds: edge-to-edge (EE) tilings, and non-edge-to-edge (NEE) tilings (SUGIMOTO, 1999). In *EE tilings*, each edge of a polygon coincides with an equal-length edge of an adjoining polygon, that is, vertices may only contact other vertices (Fig. 3(a)). In *NEE tilings*, the vertices of polygons may contact the edges of adjoining polygons, that is, there is no restriction on how adjoining polygons meet (Fig. 3(b)). Eight of the 14 convex pentagon tiles have potential EE tilings. Thus, while the EE tilings are topologically pentagonal, NEE tilings are not; NEE tiling can be regarded as the case in which the convex conditions of hexagonal tiling are eased (SUGIMOTO and OGAWA, 2000b, 2003a; OGAWA *et al*., 2002).

The first problem to be solved by this systematic approach is the tiling of convex pentagons under an EE tiling restriction. Therefore, as long as cautions are unnecessary, EE tiling is written simply a tiling throughout the paper. In this study, pentagons are organized based on the edge length, and tilings are classified by the characteristic of vertex concentration.

As a first step in classifying pentagonal tilings, this report aims to present the perfect list of pentagon tiles with four equal-length edges under the simplest case of vertex concentration. Both convex and concave pentagons are included in this analysis. The method introduced here is expected to be applicable to a wide range of problems involving creation of a perfect list of prototiles.

2. Unique Characteristics of the Pentagon

The vertices and edges of pentagons will be referred to using the nomenclature shown in each pentagon of Figs. 1 and 2.

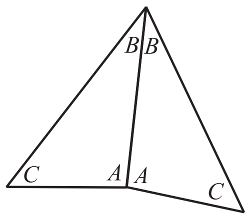

Fig. 4. Example of a regular triangular tiling pattern.

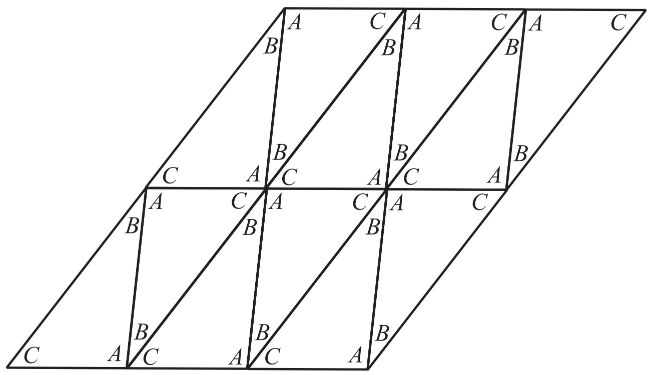

Fig. 5. Example of a reversed triangular tiling pattern

2.1. Nodes

The geometric definitions and tilings for the cases that have already been solved for triangles, quadrilaterals and hexagons are first explained briefly. When joining the edges *AB* of two identical polygons (e.g., for triangles *ABC*), the case where all vertices *A* of adjacent polygons are joined (and similarly for vertices *B*) is called an *AB-regular pattern* (i.e., mirror symmetry) (Fig. 4), and the case where vertices *A* and *B* are joined is called an *AB-reversed pattern* (point symmetry) (Fig. 5). Here, the concentration of three or more polygon vertices at a point in an EE tiling is called a *vertex concentration point*, or *node*, with a *valence* κ equal to the number of vertices comprising the node. For example, EE tilings of triangles are comprised of 6-valent nodes, as the sum of internal angles for any triangle is 180° (triangles have EE tilings when using the reversed pattern even if the three edges are of different lengths). In the case of quadrangles, even if concave, EE tilings have 4-valent nodes (using a repeated reversal configuration). Convex hexagonal tilings are always comprised of 3-valent nodes, even if the requirement for congruent hexagons is relaxed. The number of prototiles with 3-valent tilings is thus quite limited. For EE tilings with seven or more polygons, the mean valence of nodes in the tiling will be smaller than 3. Therefore, since the lowest possible valence of nodes in an EE tiling is 3, EE tilings with seven or more polygons are impossible.

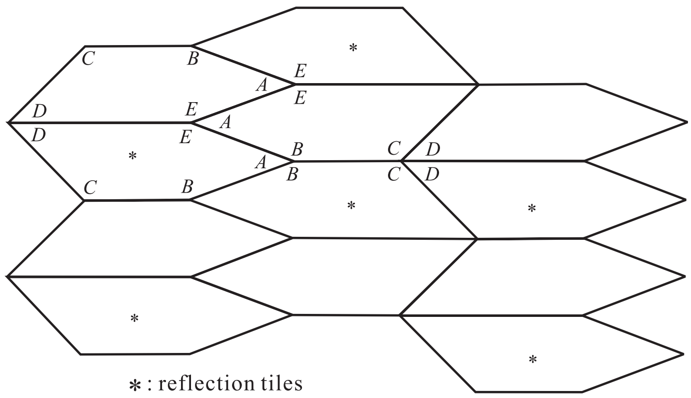

Fig. 6. Pentagonal tiling with two 3-valent nodes ($2B + A = 2E + A = 360°$) and one 4-valent node ($2C + 2D = 360°$).

In pentagonal EE tilings, the nodes have multiple valences. It will be shown that the mean valence ($\bar{\kappa}$) is 10/3. The sum of internal angles of any N pentagons is $3\pi N$, and thus the sum of internal angles of vertices meeting at these nodes is $2\pi V$, where V is the number of nodes. For a tiling pattern of congruent convex pentagons, the number of nodes along the periphery becomes much smaller than V as the pattern increases in size (large N and V). Therefore, for sufficiently large patterns, the number of nodes on the periphery can be neglected, yielding the relation $V = 3N/2$. The number S of lines connecting nodes is $5N/2$, which is equal to half the sum $\bar{\kappa}V$ of valences for the pattern. That is, $S = 5N/2 = \bar{\kappa}V/2$. Therefore, $\bar{\kappa} = 5N/V = 5N/(3N/2) = 10/3$ holds (SUGIMOTO and OGAWA, 2003a). Another approach is to imagine the entire tiling plane as the surface of a large polyhedron; the factor of 2 may then be neglected in Euler's relation of $V - S + N = 2$ at the infinite limit ($V \to \infty$, $S \to \infty$, $N \to \infty$).

If the nodes are limited to valences of 3 and 4 in pentagonal tiling, with the total number of nodes of valence κ written V_κ, we obtain the relations $3V_3 + 4V_4 = 2S$ and $V_3 + V_4 = V$, yielding $V_3 = N$ and $V_4 = N/2$ (SUGIMOTO 1999; SUGIMOTO and OGAWA, 2000a, 2003a). The simplest case would be two kinds of 3-valent nodes (including the case when the two kinds are identical), and one kind of 4-valent node. Figure 6 shows an example of this type of tiling, where the two kinds of 3-valent nodes are defined by $2B + A = 2E + A = 360°$, and the one kind of 4-valent node is defined by $2C + 2D = 360°$. Thus, the three kinds of nodes in this pattern are comprised of a total of $(2 \times 3) + (1 \times 4) = 10$ vertices of pentagons, that is, in the equations for the three kinds of nodes ($2B + A$, $2E + A$, $2C + 2D$), each of the five vertices (A, B, C, D, E) appear twice. Hence, $\bar{\kappa} = 10/3$ holds. In this case, all vertices of a pentagon have a statistically equal chance of appearing at a certain node. This tiling condition for convex pentagons, with two kinds of 3-valent nodes (including the case when the two kinds are identical) and one kind of 4-valent node, represents *the simplest node condition* (SUGIMOTO and OGAWA, 2003a).

Table 1. Combinations of vertices at nodes in pentagonal tilings.

Classification[†1]	4-valent nodes	3-valent nodes		Number of vertex combinations	Note
22	1122[†2]	334	455	$10 \times 3 = 30$	
20	1122[†2]	345	345	$10 \times 1 = 10$	
12	1123[†3]	244	355	$30 \times 2 = 60$	
11	1123[†3]	445	235	$30 \times 2 = 60$	
10	1123[†3]	245	345	$30 \times 1 = 30$	
01	1234[†4]	155	234	$5 \times 4 = 20$	Exclusion is possible.
00	1234[†4]	125	345	$5 \times 6 = 30$	Exclusion is possible.
SUM				240	

[†1]Classification is given by xy, where x is the degeneracy (the number of vertices of the same kind appearing in one node) of the 4-valent node, and y is the degeneracy of the 3-valent nodes (20 represents the case of two identical 3-valent nodes).

[†2]The number of vertex combinations is 10.

[†3]The number of vertex combinations is 30.

[†4]The number of vertex combinations is 5.

2.2. *Classification of convex pentagons with equal edges*

Pentagons can be categorized by the number of equal-length edges and their positions, from figures with five unequal edges to those with five equal edges. Here, the edge lengths are designated symbolically in anticlockwise order, with identical symbols for edges of identical lengths, and descriptions of congruent shapes with different starting points or mirror-reflections are excluded. Beginning with equilateral pentagons, followed by those with four equal-length edges, etc., there are a total of 12 unique classifications: [11111], [11112], [11122], [11212], [11123], [11213], [11223], [11232], [12123], [11234], [12134], and [12345] (SUGIMOTO 1999; SUGIMOTO and OGAWA, 2003a). This study addresses in general terms the possible tilings for pentagons with four equal-length edges, [11112], under the simplest node condition (i.e., not the complete set of possible tilings for such pentagons).

2.3. *Classification of EE tiling patterns*

EE tiling patterns are classified here based on the valences of nodes. As a simple first case, the tilings are limited to those that satisfy the simplest node condition defined above (nodes with valence of 5 or more will be handled separately). This case consists of 240 possibilities, using every possible combination of symbols (SUGIMOTO and OGAWA, 2003a). The combinations are listed in Table 1, where the combinations are grouped without duplication and the numerals 1–5 denote the interior angles *A*–*E* of the pentagons (see Fig. 1 or 2). That is, combinations of 4-valent nodes are classified as a single set, without regard to the ordering or configurations of the vertices. For example, the two nodes *AADC* (1123) and *ADAC* (1213) in Fig. 7 are both classified as 1123. The maximum possible numbers of node combinations are listed in Table 1. In the case of four different

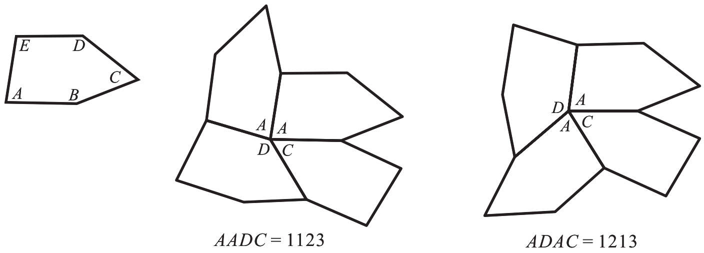

Fig. 7. Examples of pentagonal tilings with 4-valent nodes of 1123 and 1213.

interior angles concentrated at a node (i.e., 01 and 00 in the table), the fifth angle becomes 180°. For example, if the combination of the 4-valent node is $A + B + C + D = 360°$, the polygon is not a pentagon since $E = 540° - (A + B + C + D) = 180°$. Therefore, classifications 01 and 00 can be excluded entirely, while excludable cases in the other classifications need to be identified individually.

2.4. Degrees of freedom

The degrees of freedom (*DOFs*) of a polygon with n edges (≥3, integer) is generally given by $2n - 3$. For an n-edge polygon, there are $2n$ planar coordinate systems, attached to n vertices (each vertex is connected to an edge). However, if the location of any point (e.g., the centroid) in an n-edge polygon is known, the number of DOFs is reduced by 2, and a further 1 DOF is lost because the sum of interior angles is given ($(n - 2) \times 180°$). For example, a triangle may be uniquely defined by knowledge of three edge lengths, two edge lengths and the included angle, or two angles and the included edge length, corresponding to $2n - 3 = 3$ DOFs. Similarly, the addition of a vertex in place of an edge in an n-edge polygon results in a polygon with an additional 2 DOFs.

Figures such as a five-pointed star, the projected edges of which intersect, are also included in this generalization. Even if figures with intersecting edges are excluded, or further, if convexity is required, this condition merely imposes additional constraints on the shape of the polygon and does not reduce the available DOFs. Thus, this general theory states that pentagons have 7 DOFs. However, this definition includes size as a DOF (i.e., similarity, not congruency). For congruent polygons, the number of DOFs will therefore be $2n - 4$. Furthermore, the constraint of four equal-length edges, equivalent to 3 DOFs, reduces the number of DOFs for our case to 3 ($=2n - 7$) (SUGIMOTO and OGAWA, 2003a).

3. Limiting Pentagons to Those of Interest: Pentagons with Four Equal-length Edges

Hereafter, the study presented in this report is limited to pentagons with four equal-length edges. Note that, in this report, the *DE* edge ($=e$) of pentagon is the sole edge of different length. The EE tiling of pentagons with four equal-length edges requires the *DE* edges to be joined, in regular or reverse fashion, as shown in Fig. 8. The vertices *D* (or *E*) of adjoining pentagons are coincident in a *DE*-regular mirror-reflected pattern, while the vertex *D* of each pentagon meets the vertex *E* of the adjoining pentagon in a *DE*-reversed

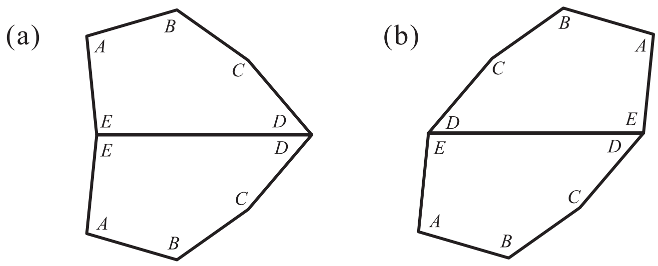

Fig. 8. (a) *DE*-regular and (b) *DE*-reversed patterns for pentagons with four equal-length edges.

Table 2. Combinations of vertices at nodes for tilings of convex pentagons with four equal-lengths edges.

Classification	4-valent nodes	3-valent nodes		Number ofcombinations‡	
				Cases of *DE*-regular	Cases of *DE*-reversed
22	1122	334	455	$18 \rightarrow 10$	$3 \rightarrow 2$
20	1122	345	345	$1 \rightarrow 1$	$4 \rightarrow 3$
12	1123	244	355	$18 \rightarrow 9$	0
11	1123	445	235	$6 \rightarrow 3$	$6 \rightarrow 3$
10	1123	245	345	0	$3 \rightarrow 2$
SUM				$43 \rightarrow 23$	$16 \rightarrow 10$

‡Arrow indicates the reduction in the number of combinations when symmetry for the simultaneous exchange of $D \leftrightarrow E$ and $A \leftrightarrow C$ is considered.

point-symmetric pattern. That is, in the tiling of pentagons with four equal-length edges, the pentagonal pair (i.e., in this study, *DE*-regular and *DE*-reversed) is always the foundation of tiling. Therefore, the tiling is bisected by the regular pattern and the reversed pattern (SUGIMOTO and OGAWA, 2003a).

The characteristics of nodes in a pentagonal tiling were described in some detail above. The average number of pentagons meeting at nodes in a pentagonal tiling is 10/3, indicating that it is physically impossible to form a tiling of pentagons with a single valence of node. At least three kinds of nodes must be used to construct a tiling for pentagons, except for the case of two identical kinds of 3-valent nodes as described above. Hence, it is assumed that three or more vertices meet at each node, as the simplest node condition, requiring two kinds of 3-valent nodes (including the case when the two kinds are identical) and one kind of 4-valent node. Counting the combinations using symbols as in Table 1 yields 240 cases that satisfy this simplest node condition. For the case of pentagons with four equal-length edges (i.e., edge *DE* is special), the number of possible combinations is reduced. For EE tilings, the *DE* edge must be used in either a regular or reversed manner (see Fig. 8). Then, considering symbolic symmetry in the simultaneous $D \leftrightarrow E$ and $A \leftrightarrow C$ reflections within the range allowed by the symbols, the cases of possible tiling are reduced to a total of 33 (see Table 2), consisting of 23 regular patterns and 10 reversed

Table 3. Potential tilings for DE-regular patterns of convex pentagons with four equal-length edges.

Classification	Cases	4-valent nodes	3-valent nodes		Comments
22	1	$DDEE$	AAC	BBC	*
	2	$DDEE$	CCB	AAB	
	3	$DDAA$	BBC	EEC	*
	4	$DDAA$	CCB	EEB	*
	5	$DDBB$	AAC	EEC	*
	6	$DDCC$	AAB	EEB	*
	7	$DDCC$	BBA	EEA	*
	8	$DDBB$	CCA	EEA	*
	9	$AABB$	DDC	EEC	*
	10	$CCAA$	DDB	EEB	
20	11	$DDEE$	ABC	ABC	
12	12	$DDAB$	EEA	CCB	*
	13	$DDAB$	EEB	CCA	*
	14	$DDBC$	EEB	AAC	*
	15	$DDBC$	EEC	AAB	*
	16	$DDCA$	EEC	BBA	*
	17	$DDCA$	EEA	BBC	*
	18	$AABC$	DDB	EEC	*
	19	$BBCA$	DDC	EEA	*
	20	$CCAB$	DDB	EEA	*
11	21	$DDAB$	EEC	ABC	*
	22	$DDBC$	EEA	ABC	*
	23	$DDCA$	EEB	ABC	

*Congruent with simultaneous $D \leftrightarrow E$ and $A \leftrightarrow C$ reflections.

patterns (SUGIMOTO and OGAWA, 2003a). These patterns are listed in Tables 3 and 4.

For the patterns having a congruent equivalent, the case listed is that for $D \leq E$ where it is possible to judge easily. It should be noted that the tiling patterns satisfying the node restriction (the relations of angles among three nodes which are given from the simplest node condition) include not only periodic cases, but also non-periodic cases. The 33 cases listed in these two tables consider both the tiling pattern (with respect to node combinations) and the shape of the pentagons.

4. Investigation 1: Topological Judgment

In this section and next sections, each of the 33 cases shown in Tables 3 and 4 are evaluated by (i) topological judgment (graph theory) to investigate the possibility of tiling using symbolized notation without breaking down the order of pentagonal meshes, and (ii) geometric judgment to investigate the possibility of the existence of the convex pentagon in Euclidean space. This geometric judgment considers the range of possible pentagons that

Table 4. Potential tilings for *DE*-reversed patterns of convex pentagons with four equal-length edges.

Classification	Cases	4-valent nodes	3-valent nodes		Comments
22	1	*DDEE*	*AAC*	*BBC*	*
	2	*DDEE*	*CCB*	*AAB*	
20	3	*DDEE*	*ABC*	*ABC*	
	4	*AABB*	*CDE*	*CDE*	*
	5	*CCAA*	*BDE*	*BDE*	
11	6	*AADE*	*BBC*	*CDE*	*
	7	*AADE*	*CCB*	*BDE*	*
	8	*BBDE*	*AAC*	*CDE*	*
10	9	*AABC*	*BDE*	*CDE*	*
	10	*BBCA*	*CDE*	*ADE*	

*Congruent with simultaneous $D \leftrightarrow E$ and $A \leftrightarrow C$ reflections.

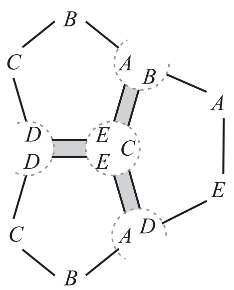

Fig. 9. *DE*-regular 3: *DDAA*, *BBC*, *EEC*.

satisfy the node restrictions based on quantitative relationships between edge length and angles (see Sec. 5). Then, for each pentagon, it is investigated whether a tiling is possible, and whether the pentagon is convex. Through this procedure, the range of possible prototile and tiling patterns for each classification can be determined. As a general problem of logic, the creation of a perfect list involves both confirmation and exclusion to satisfy a necessary and sufficient condition.

Any collection of points joined by lines is called a graph. The lengths of the lines and the angles between the lines are not relevant to this topological investigation. From the viewpoint of graph theory, tiling patterns are considered to be graphs that can fill the plane (WILSON, 1996). Pentagonal tiling patterns are planar graphs in which all the spaces between lines are pentagons. Here, in each pentagonal mesh, the vertices are labeled in the same order. Specific examples will make this easier to understand.

Table 3, *DE*-regular patterns, Case 3 describes the case of a *DDAA* 4-valent node and *BBC* and *EEC* 3-valent nodes. Figure 9 shows an array with two pentagons on the left and one on the right. Only one node, an *EEC* node, is shown for simplicity (i.e., Fig. 9 is a simplified graph that does not aim to reproduce pentagonal forms correctly). Since the combination of two vertices *EE* can only be concentrated at a node with vertex *C*, the position of *C* for the pentagon on the right can be decided easily. The neighboring vertices *B* and *D* may be reversed, but in either case, both are concentrated with vertex *A* and thus the situation is the same. However, as *A* can be concentrated only with *D*, the upper concentration of *B* in Fig. 9 does not fit the node restrictions. In this case, it is the conclusion that there is no tiling pattern that satisfies this node restriction. Following this clearly negative result, there is no further need to investigate this case. When affirmative results are confirmed, or a clear negative result cannot be confirmed, the pattern is investigated to a greater range by continuing this process of combining vertices symbolically as long as valid 3- or 4-valent nodes are achieved. Once it becomes impossible to assign a vertex, the case is eliminated as impossible, but it is sometimes necessary to distinguish between different possible arrays during the process up to that point in order to avoid missing possible successful arrays. As the number of successful nodes increases, the prospect of tiling (graph) will arise. Many of these cases are periodic patterns, and as long as some kind of regularity can be found, the possibility for a tiling remains open. Logically, however, there are many possibilities for such combinations, and the conclusion of elimination or confirmation cannot be drawn easily. Nevertheless, for the 33 cases examined here, it turned out to be rather simple to confirm or eliminate these arrays. An affirmative result refers to cases in which a unique assignment of vertex labels is identified. The pentagons forming these arrays therefore have the potential to be prototiles. However, from the relations among edges and interior angles of tile, some of these potential pentagons will not be prototiles because some shapes do not allow convex (or concave) pentagons. For example, a shape with a valid pattern is not pentagon when the length of edge *DE* is proven to be zero by geometric analysis. An example of this case is *DE*-regular 19 (Case 19 in Table 3). This will be shown in the geometric considerations in Sec. 5. It is also important to note that some cases have multiple tilings. Any case that has at least one uninterrupted set of combinations has the potential to be tiling.

4.1. DE-regular pattern

In *DE*-regular patterns, *DD* and *EE* always occur as a pair. Table 5 shows the classification of Cases 1–23 in Table 3 focusing on pairs with 3- or 4-valent nodes.

In group 2 of Table 5, where only one of the two *DD* or *EE* pairs forms part of a 3-valent node, the process of evaluation is simpler. Cases 3–6, 8, 13, 15, 17 and 23 can be readily eliminated at this stage.

In group 3 of Table 5, where both the *DD* and *EE* pairs form part of 3-valent nodes, Cases 9 and 18 cannot be eliminated without further investigation, but are in fact eliminated eventually. Theoretically, if one type is eliminated, no investigation of the other is necessary. However, for Case 10, which also has 3-valent nodes for both *DD* and *EE*, both nodes meet the conditions, and the third vertex in both cases is *B*. It is somewhat difficult to distinguish between the different cases here, but it soon becomes clear upon further investigation of Case 10 how the tiles assemble uniquely, yielding a periodic pattern.

The remaining problems are Cases 1, 2 and 11 in the *DDEE* 4-valent node class in group 1 of Table 5. Investigation of these cases requires creation of a module with an

Table 5. Regrouping of cases 1–23 in Table 3 by valence of nodes involving *DD* and *EE*.

Group	*DD*	*EE*	Cases in Table 3
1	4-valent	4-valent	1, 2, 11
2	4-valent	3-valent	3, 4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 21, 22, 23
	3-valent	4-valent	
3	3-valent	3-valent	9, 10, 18, 19, 20

Table 6. Regrouping of cases 1–10 in Table 4 by valence of nodes involving *DE*.

Group	*DE*		Cases in Table 4
1	4-valent	4-valent	1, 2, 3
2	4-valent	3-valent	6, 7, 8
3	3-valent	3-valent	4, 5, 9, 10

infinite, periodically repeating pair, such that tiles are connected in a *DE*-regular pattern. All the remaining nodes are 3-valent (i.e., *ABC*), in which case the "jagged" edges of the modules consist of convex vertices *B* and concave vertices *A* and *C*. The convex vertex *B* fits in one concave vertex in one module, and the concave vertex fits to one convex vertex *B* of another module, forming *ABC* 3-valent nodes. Case 11 showed some regularity, while Cases 1 and 2 could be eliminated.

The cases in which regularity (or certain periodicity) was found are 7, 10–12, 14, 16, and 19–22. Of these, problems were found with Cases 19 and 22, as will be explained in Sec. 5.

4.2. *DE-reversed pattern*

DE-reversed patterns always contain *DE* pairs. Cases 1–10 in Table 4 are classified into three groups according to how the *DE* pairs are distributed over the 3- or 4-valent nodes, as shown in Table 6. In *DE*-reversed patterns, the point-symmetric pairs of tiles are the basic element for tiling. There is a minor additional difference between groups 1 and 3 in Table 6, but the task of confirmation and elimination remains the same as for the *DE*-regular patterns.

As above, when only one of the two nodes is 3-valent (group 2 of Table 6), the case is easy to evaluate. In group 2, only Case 8 was eliminated. For group 3, it is necessary to distinguish between multiple possible patterns, and ultimately only Case 9 was eliminated. In group 1, a module with an infinite, periodically repeating pair (with tiles connected in a *DE*-regular pattern) could be formed, but it was not always necessary to use a simple parallel transformation as each module could be reversed to obtain the same transformation. On the basis of this investigation, only Case 1 was eliminated, while Cases 2 and 3 were indicated as a possible prototile. Therefore, Cases 2–7 and 10 exhibit regularity (certain periodicity). It is also still possible to mix some reversals with other transformations, but that technique does not fulfill the simplest node condition and will not be considered here.

5. Investigation 2: Geometric Judgment

Following from the symbolic and graph theory investigations above, the investigation is extended here to Euclidean geometry. When the geometric conditions are complex, the investigation can be rendered more tractable by reducing the degrees of freedom (DOFs), which simplifies the equations describing the geometric relationships. A pentagon has six DOFs, but a pentagon with four equal-length edges has just three DOFs (see Subsec. 2.4). Pentagons with four equal-length edges can be divided into three triangles; two isosceles triangles *EAB* and *BCD*, and a third triangle *BDE* (Fig. 10). The base angles of the two isosceles triangles *EAB* and *BCD* are denoted α and β ($0 < \alpha < 90°$, $0 < \beta < 90°$), and the peak angles of vertices *D* and *E* of triangle *BDE* are denoted δ and ε ($180° - \alpha - \beta < \delta + \varepsilon < 180°$), respectively. The interior angles of the pentagon can then be expressed as follows (SUGIMOTO and OGAWA, 2003b).

$$\begin{cases} A = 180° - 2\alpha, \\ B = \alpha + \beta + 180° - \delta - \varepsilon, \\ C = 180° - 2\beta, \\ D = \beta + \delta, \\ E = \alpha + \varepsilon. \end{cases} \tag{1}$$

Four variables have been introduced to represent the angles, but generally, only two of the three equations such as $2C + 2D = 360°$, $2B + A = 360°$, are independent under the node restriction (all nodes subtend 360°). Thus, at this stage, there are actually just two independent variables. The exceptions are the *DE*-regular 11 (Case 11 in Table 3) and the *DE*-reversed 3 and 4 (Cases 3 and 4 in Table 4), all of which have an additional DOF.

In reference to Fig. 10, the lengths of four equal-length edges are defined as 1 (i.e., $a = b = c = d = 1$), and the length of the fifth edge *DE* (=e) is designated ℓ to facilitate the analysis below. Additionally, the lengths of the other edges of triangle *BDE* are $\overline{BE} = 2\cos\alpha$ and $\overline{BD} = 2\cos\beta$. The half heights can thus be expressed by the relation

$$\cos\alpha \sin\varepsilon = \cos\beta \sin\delta. \tag{2}$$

Since this is a transcendental equation, an investigation of the case is required to identify possible solutions. Apart from this equation, however, there remains only one (or exceptionally two) case involving two independent angular variables. Thus, the problem is not particularly difficult.

If the pentagon is convex, the interior angle μ must be in the range $0 < \mu < 180°$. Even if the pentagon is allowed to be concave ($180° < \mu < 360°$), μ may not equal 180°. In any case, if there exists a solution, the length of the fifth edge *DE* is defined by

$$\ell = 2(\cos\alpha\cos\varepsilon + \cos\beta\cos\delta) \tag{3}$$

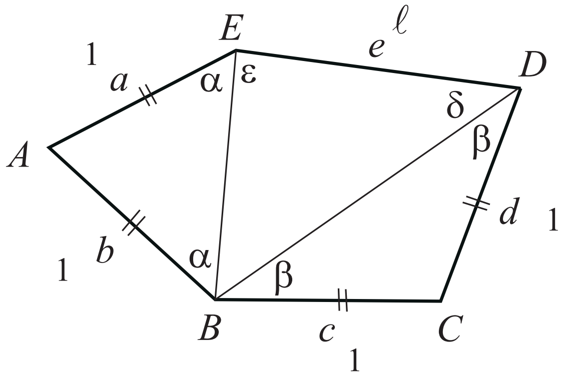

Fig. 10. Division of a pentagon with four equal-length edges.

where $\ell > 0$ is required (SUGIMOTO and OGAWA, 2003b). The essential points here are that Eq. (1) was derived on the assumption that the initial polygon is a pentagon with interior angles summing to 540°, and that the division of the pentagon into the three triangles shown requires a convex pentagon.

This geometrical investigation was performed for the cases exhibiting regularity in the investigation described in Sec. 4. The cases eliminated from the affirmative cases of previous investigation are considered below (*DE*-regular 19 and 22, *DE*-reversed 10).

- *DE-regular 19 (Case 19 in Table 3)*

The three vertex concentration types for this case (*BBCA*, *DDC*, *EEA*) can be described by the relations $2B + C + A = 360°$, $2D + C = 360°$, and $2E + A = 360°$. This leads to the result that $\delta + \varepsilon = 180°$, that is, $\ell = 0$. Thus, the shape is a quadrilateral (*D* merges with *E*), not a pentagon.

- *DE-regular 22 (Case 22 in Table 3)*

The three vertex concentration types for this case (*DDBC*, *EEA*, *ABC*) can be described by the relations $2D + B + C = 360°$, $2E + A = 360°$, and $A + B + C = 360°$. Thus, $\alpha + \beta = 180°$, that is, the shape is not a pentagon.

- *DE-reversed 10 (Case 10 in Table 4)*

The three vertex concentration types for this case (*BBCA*, *CDE*, *ADE*) can be described by $2B + C + A = 360°$, $C + D + E = 360°$, and $A + D + E = 360°$. Thus, $\delta + \varepsilon = 180°$, that is, $\ell = 0$ and the shape is a quadrilateral.

As the establishment of an index in this study involves eliminating candidates for tiling, it is not necessary to carry out geometrical investigations for any other cases. However, to demonstrate some examples of confirmed solutions, two cases for which solutions were found are provided below (*DE*-regular 7 and *DE*-reversed 3).

- *DE-regular 7 (Case 7 in Table 3)*

The three vertex concentrations for this case (*DDCC*, *BBA*, *EEA*) can be described by the relations $2D + 2C = 360°$, $2B + A = 360°$, and $2E + A = 360°$. Thus, Eq. (1) becomes

Table 7. Results of topological judgment and geometrical judgment.

Under node restrictions	Topological judgment	Geometric judgment	Corresponding cases (Cases in Tables 3 and 4)
Pentagonal tiles exist.	Y	Y	*DE*-regular 7, 10, 11, 12, 14, 16, 20, 21. *DE*-reversed 2, 3, 4, 5, 6, 7.
Pentagonal tiles do not exist.	Y	N	*DE*-regular 19, 22. *DE*-reversed 10.
	N	Y	*DE*-regular 1, 2, 3, 4, 5, 8. *DE*-reversed 1, 8.
	N	N	*DE*-regular 6, 9, 13, 15, 17, 18, 23. *DE*-reversed 9.

Y: confirmed. N: eliminated.

$$\begin{cases} A = 180° - 2\alpha, \\ B = 90° + \alpha, \\ C = 180° - 2\beta, \\ D = 2\beta, \\ E = 90° + \alpha. \end{cases}$$

Since the above indicates that $\delta = D - \beta = \beta$ and $\varepsilon = E - \alpha = 90°$, Eqs. (2) and (3) become $\cos\alpha = \sin\beta\cos\beta = \sin(2\beta)/2$ and $\ell = 2\cos^2\beta$, respectively. Thus, the ranges for these angles and ℓ are $60° \le \alpha < 90°$, $0° < \beta < 90°$, and $0 < \ell < 2$, respectively. This has, as expected, one DOF. An equilateral pentagon with $\ell = 1$ is obtained when $\alpha = 60°$ and $\beta = 45°$.

- *DE-reversed 3 (Case 3 in Table 4)*

The three vertex concentrations for this case (*DDEE*, *ABC*, *ABC*) can be described by $2D + 2E = 360°$, $A + B + C = 360°$. In this case, it is not always necessary to rely upon Eq. (1). Instead, it is more convenient to regard the pentagon as the sum of an isosceles triangle *ABC* and a parallelogram *ACDE*. All the interior angles are required to be convex, but this condition only specifies the range of angles. Restricting the length of *DE* to $0 < \ell < 2$ determines the shape of the isosceles triangle *ABC*. The lengths of the edges of the parallelogram are all predetermined, but there remains an angular range permitted for the parallelogram. In other words, this case retains two DOFs. *DE*-regular 11 is similar to this case. Furthermore, if the 4-valent node has been determined, the remaining nodes under the simplest node condition (i.e., two 3-valent nodes) are no longer independent (whether identical or not). The restrictions on the 4-valent node are generally given by the sum of the relations describing the two 3-valent nodes. Note that the angular equations for each pair of 3-valent nodes is not independent if and only if the two 3-valent nodes are identical.

The cases eliminated by the topological judgment investigation in Sec. 4 were also examined by geometrical investigation. This geometric analysis eliminated *DE*-regular cases 6, 9, 13, 15, 17, 18 and 23, and DE-reversed case 9.

The results of the graph theory analysis in Sec. 4 and the geometric analysis in this section are summarized in Table 7.

6. Conclusions

The analyses above have afforded a list of pentagonal tiles with four equal-length edges under the simplest node condition. The part in Sec. 3 discussed the tiling patterns and employed a symbol-based investigation to classify and identify unique patterns, producing the list of 23 *DE*-regular patterns in Table 3 and 10 *DE*-reversed patterns in Table 4 (SUGIMOTO and OGAWA, 2003a). In next parts, these 33 cases were investigated by topological judgment (Sec. 4) and by geometric judgment (Sec. 5). The tiling cases identified by this investigation as being the only possible tilings of this class are *DE*-regular cases 7, 10, 11, 12, 14, 16, 20, and 21 (eight kinds), and *DE*-reversed cases 2–7 (six kinds), representing a total of 14 tilings (SUGIMOTO and OGAWA, 2003b). These tilings (both periodic and non-periodic) will be investigated in more detail in another report.

The tiles eliminated by topological judgment investigation but confirmed by geometric judgment cannot be immediately eliminated as invalid prototiles (pentagonal tiles), as the conclusions reached in this investigation are for specific node restrictions. If those restrictions are relaxed, some of the cases eliminated here may become valid in topological judgment. For example, the convex pentagons of *DE*-regular cases 1 and 2 and *DE*-reversed cases 1 and 8 have the possibility of tilings under different node restrictions according to topological judgment. Tilings may also be possible for *DE*-regular cases 3, 4, 5 and 8 if tiles are limited to equilateral pentagons, in which case the tiles will be of the same shape as the equilateral pentagonal tiles of *DE*-regular cases 10 or 12. These cases will be described in another report.

In the investigation of the 33 cases in this classification, it is not necessary to consider other node restrictions. That is, for pentagons with four equal-length edges, the prototiles will always appear among the 33 cases identified in this study (SUGIMOTO and OGAWA, 2003b). The complete range of pentagons meeting each node restriction must therefore be investigated, as will be presented in a future report.

7. Summary

This report presented a complete list of pentagonal tiles with four equal-length edges under the simplest node condition of one 4-valent node and two 3-valent nodes (including the case when the two kinds are identical). These tilings consist only of convex pentagons; no concave pentagons were found under this condition. Since the average number of pentagons meeting at a node (vertex concentration point) in an EE tiling of convex pentagons is 10/3, most patterns are expected to satisfy the simplest node condition (SUGIMOTO, 1999; SUGIMOTO and OGAWA, 2000a, 2003a). Thus, there are very few exceptional EE tilings that do not satisfy the simplest node condition, and these will be considered in future stages of this study.

At first glance, the simplest node condition appears to apply three constraints on the relations among the interior angles of pentagons. However, only two independent conditions in fact apply in this case. Furthermore, when the 4-valent node is *DDEE*, there is only one independent condition. As a result, the pentagon has only one or two degrees of freedom. The next-most simple node conditions are the cases involving a total of six kinds of nodes (allowing the case when the some nodes are identical): two 4-valent nodes and four 3-valent nodes, or one 5-valent node and five 3-valent nodes. In all of these cases, each of the vertices of the pentagon appears four times at the six nodes. A maximum of six conditions (restrictions) can thus be imposed in these situations. For cases in which there are more than two independent conditions among these six, there are no longer any DOFs, and only special solutions are permitted. For example, for the case of the 4-valent nodes *DDDD* and *EEEE*, it is required that 3-valent nodes *AAB* and *CCB* are repeated use together, i.e., this is degenerate case leaving only one DOF. This is clearly a valid case for tiling the plane, and it is considered possible to prove a generalization regarding pentagons with one DOF. The list of tilings including special solutions for the cases with zero DOFs will be a topic of future research.

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