

Hyplane—Polyhedral Models of Hyperbolic Plane

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Abstract. In this article we introduce a new category of polyhedra, called *Hyplane*. Hyplane is a polyhedral analogue of the hyperbolic plane in R^3 . This consists of faces of the same triangles, and looks like a horse saddle shape everywhere.

1. Introduction

In this article, we introduce a new category of polyhedra and we call them “Hyplane”. One of the most remarkable distinctive features of hyplane is that all faces are congruent triangles and that it is an analogue of hyperbolic plane in R^3 . See Fig. 1. This is an example of hyplane. We know that this polyhedron has a horse saddle shape locally. All faces of this fragmented polyhedron are congruent to each other.

If we consider an immersion of the hyperbolic plane into R^3 , it looks like a horse saddle locally. Such surfaces are called a $K = -1$ surface (or a negative constant Gauss curvature surface) and researchers of differential geometry and integrable systems are much interested in them. (About the definition of hyperbolic plane, see WEEKS (2001). Also see GRAY (1998), GPS PROJECT (1998), and INOUCHI *et al.* (2005)) Here we remark that $K = -1$ surfaces are different from hyperboloid, which is a surface of revolution obtained by rotating a hyperbola. Hyplane is a polyhedral analogue of $K = -1$ surface in R^3 .

The first hint of hyplane was in an exercise of a textbook of WEEKS (2001). The exercise is: Using many copies of the regular triangle, make a polyhedron such that there are just seven faces for each vertex. After various consideration we have new type of polyhedra such that faces are isosceles triangles with 63, 63, 54 degrees. Later we call it a (6,6,7)-hyplane. In 2000, the author developed a software named *Hyplane* (AHARA, 1999) where we could see this new polyhedron on a screen. Around 2002, the author originated mathematical definition of hyplane. Now it is known that hyplane is determined from a set of three positive integers, that hyplane is correspondent with a tiling of hyperbolic triangles, and that hyplane has subdivisional version (AHARA, 2004).

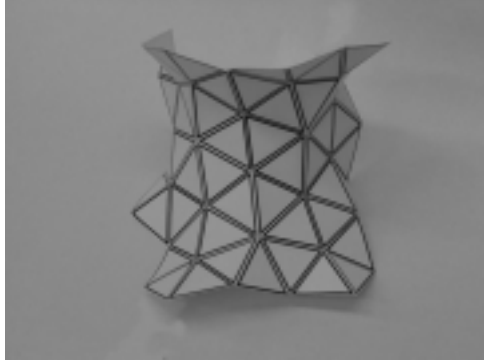


Fig. 1. An example of hyplane.

2. Definition of Hyplane

In this section we give the definition of hyplane. Let (a, b, c) be a triad of positive integers with the following two conditions (C1), (C2):

$$(C1) \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{1}{2}.$$

(C2) If a is odd then $b = c$. If b is odd then $c = a$. If c is odd then $a = b$.

We remark that there are infinite numbers of such triad. Let $\triangle ABC$ be defined by

$$\angle A = \frac{bc\pi}{bc + ca + ab}, \quad \angle B = \frac{ca\pi}{bc + ca + ab}, \quad \angle C = \frac{ab\pi}{bc + ca + ab}. \quad (1)$$

Under this assumption, (a, b, c) -hyplane is a polyhedron with the following three conditions (H1), (H2), and (H3):

(H1) All faces are congruent to $\triangle ABC$.

(H2) Any two of faces side by side are mapped to each other by a rotation of the side.

(H3) There are a faces meeting together at each vertex corresponding to A . In the same way, there are b faces at vertex B , and there c faces at vertex C .

The simplest example is $(7, 7, 7)$ hyplane. From the above formula (1), $\triangle ABC$ is the regular triangle. From the condition (H3), there are seven faces at each vertex. This is just the same as in the exercise of Weeks' Textbook.

The polyhedron in Fig. 1 is an example of $(6, 6, 7)$ hyplane. From the above formula (1), we have $\angle A = \angle B = 7\pi/20 = 63^\circ$, and $\angle C = 3\pi/10 = 54^\circ$. For each vertex, there are 6 of 63° or 7 of 54° . In general, from the formula (1) we have the following:

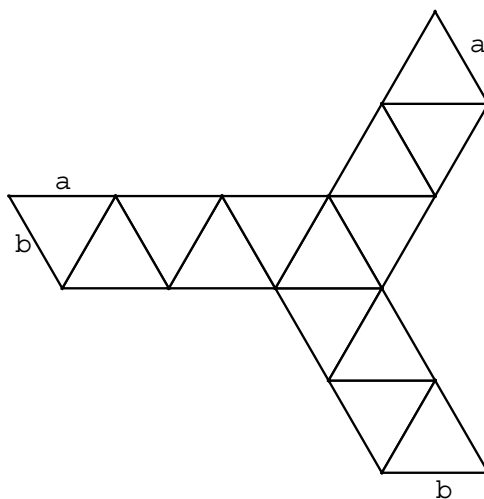


Fig. 2. (7,7,7) omusubi hyplane (developing figure).

(H4) The angle sum at each vertex is constant.

This is one of the important features of hyplane.

3. Omusubi-type and Chimaki-type

There are many configurations of hyplane. So it is an interesting problem to find out hyplanes with certain symmetry. In case (6,6,7) and (7,7,7) hyplane, we can construct Omusubi-type hyplanes and Chimaki-type hyplanes. Omusubi is a Japanese rice ball covered by nori (laver). Chimaki is a Chinese rice cake covered by bamboo leaves.

Omusubi-type (7,7,7) hyplane is given by the following developing figure (see Fig. 2).

In Fig. 2, if we paste edges a together and paste edges b together then we have a polyhedron like as in Fig. 3. This configuration has 120-degree-rotational symmetry. There are three boundaries and each boundary has four edges. See Fig. 4 left. This is a figure near a boundary. Along a boundary, there are vertices with three faces and vertices with four faces alternately. So we can paste two Omusubi hyplanes together as in Fig. 4. We can extend (7,7,7) hyplane in this way. In Fig. 5, we can see an example of the construction of Omusubi hyplanes.

This polyhedron is immersed in R^3 (that is, it may intersect itself in a process of extension), not embedded. But we can extend this polyhedron boundlessly and it follows that it is a complete polyhedron with infinite numbers of faces. Also we may consider this polyhedron as a covering space of a closed surface of genus 2 (or more).

Next, we introduce Chimaki-type (7,7,7) hyplane. Figure 6 gives a developing figure and a unit polyhedron is seen as in Fig. 7.



Fig. 3. $(7,7,7)$ omusubi hyplane (an unit).

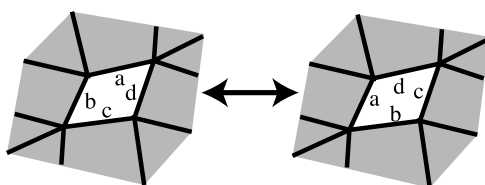


Fig. 4. How to paste omusubi hyplane.

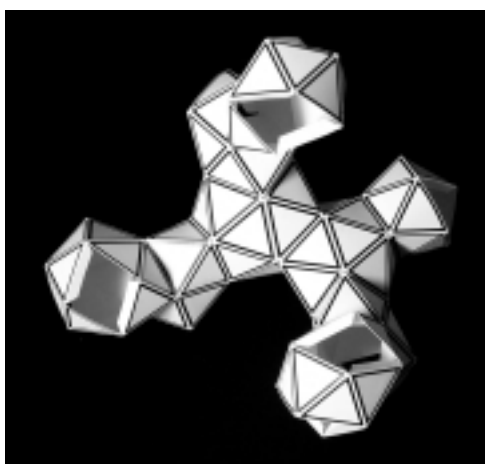


Fig. 5. $(7,7,7)$ omusubi hyplane (construction).

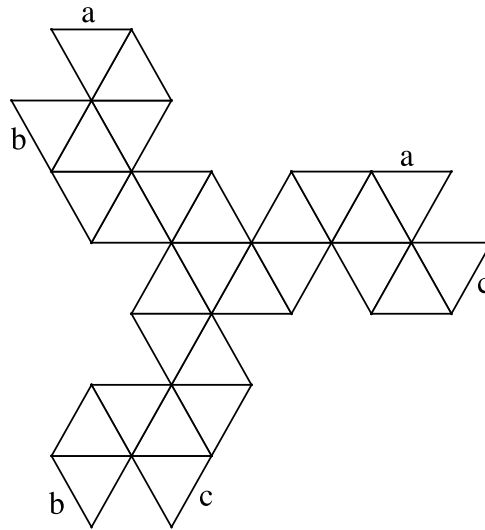


Fig. 6. $(7,7,7)$ chimaki hyplane (developing figure).



Fig. 7. $(7,7,7)$ chimaki hyplane (an unit).

We call this configuration Chimaki type hyplane. It has four boundaries and it has tetrahedral symmetry. Also in this case, we can paste two Chimaki type hyplane along their boundary and extend infinitely. See Fig. 8.

Next we introduce Omusubi and Chimaki type $(6,6,7)$ hyplane. In $(6,6,7)$ case, Omusubi type is defined by a configuration with 120 degree rotational symmetry and with

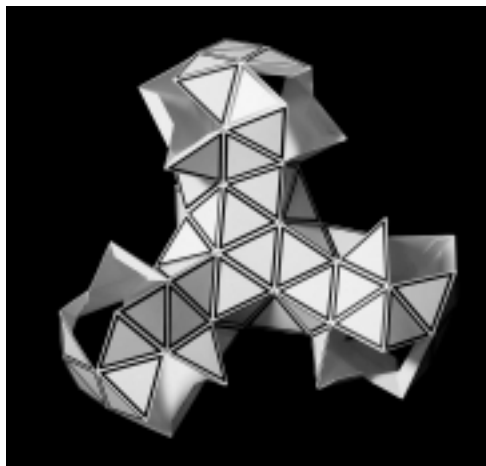
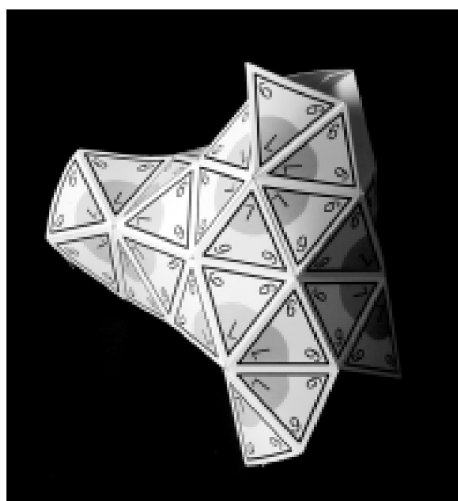
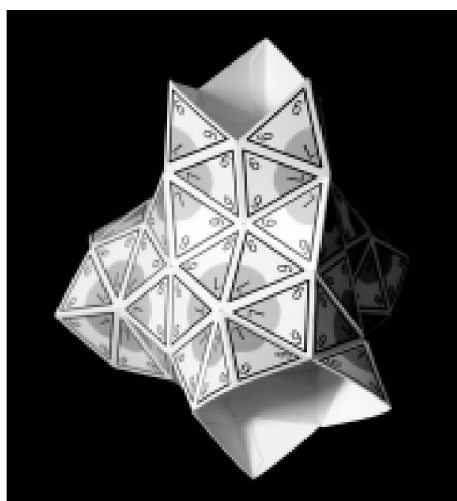


Fig. 8. (7,7,7) chimaki hyplane (construction).



Omusubi



Chimaki

Fig. 9. (6,6,7) hyplanes (construction).

three boundaries. Chimaki type is defined by a configuration with tetrahedral symmetry and with four boundaries. See Fig. 9.

Here we show a developing figure of Omusubi (resp. Chimaki) type (6,6,7) hyplane. See Fig. 10. These are not strict developing figures but their topological figure. In these figures, shaded angles are 54 degrees.

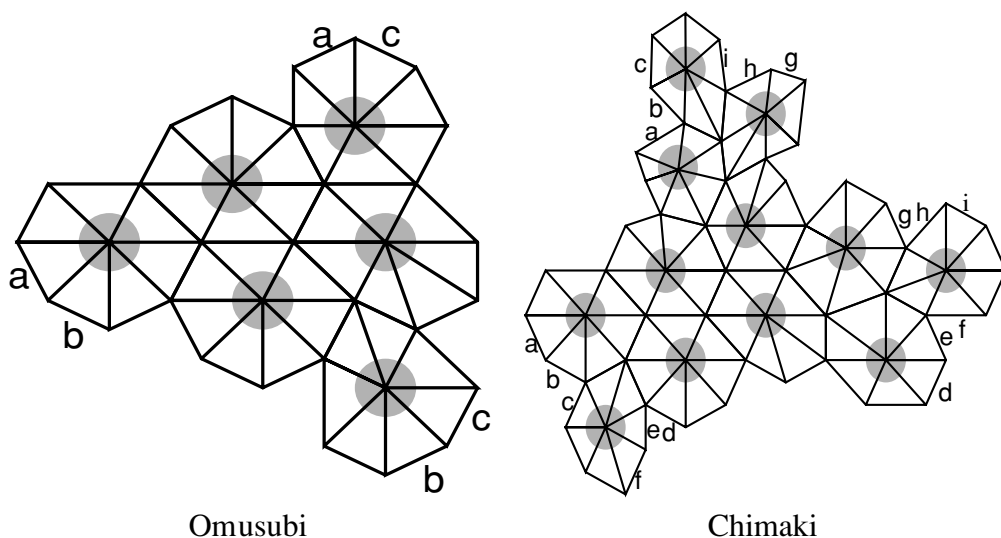


Fig. 10. (6,6,7) hyplanes (developing figures).

In (6,6,7) case, we cannot extend this configuration as in case (7,7,7). But mathematical proof of it has not been known. One considerable reason is as follows: If we consider a $K = -1$ surface with the same symmetry, it must have a singular point (for example, a cusp point) on it. Indeed, the intersection point of the surface and a rotational axis must be singular. This fact is shown from the definition of the Gauss curvature and symmetric action on the tangent spaces. In (7,7,7) case, the shape is so simple that we can extend infinitely. In Omusubi (resp. Chimaki) (6,6,7) case, there are two (resp. four) vertices on a rotational axis. If we remove 1-neighborhood of such vertices, then we may make an infinite polyhedron. (Of course it is not complete.)

4. Geometry of Hyplane

Hyplane is a polyhedral analogue of the hyperbolic plane in R^3 . Therefore we can observe geometry on hyplane. It is well known that the inner angle sum of a hyperbolic triangle is less than π . This is Gauss's theorem on the hyperbolic plane. We have Gauss's theorem on hyplane. Indeed, if a triangle on hyplane surrounds at least one vertex, then the inner angle sum is less than π . We show this theorem easily using a concept *discrete curvature*. For each vertex v , discrete curvature is defined by

$$K(v) := 2\pi - \left(\sum_{a:\text{angle}} a \right).$$

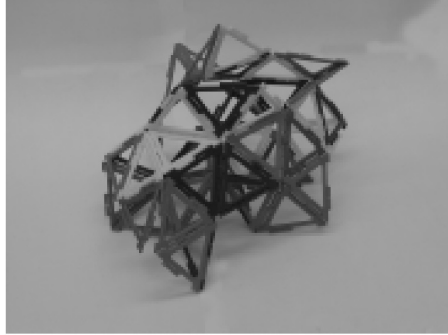


Fig. 11. Polydron.

And we have the following discrete Gauss's theorem.

Theorem Suppose that a triangle on a hyplane surrounds a contractible area A . Then

$$(\text{internal angle sum}) = \pi + \sum_{v \in A} K(v).$$

Here v is a vertex included in the area A .

In hyplane case, the discrete curvature $K(v)$ is constant and

$$K(v) = 2\pi - \frac{abc\pi}{bc + ca + ab} < 0.$$

It follows that for any triangle on a hyplane the angle sum is less than π .

Other geometrical problem on hyplane is whether there exists an immersed compact hyplane. This is a hyplane version of Hilbert's theorem. Hilbert shows that there does not exist an immersed complete $K = -1$ surface in R^3 . We know that there are complete but not compact configurations in $(7,7,7)$ case. So the remaining problem is whether there exists an immersed compact hyplane. For example, the following is the simplest open problem:

Problem (existence of regular 28 faced polyhedron)

Using 28 regular triangles, construct a closed $(7,7,7)$ -hyplane. The whole figure is homeomorphic to a closed surface of genus 2.

We get the number 28 of faces from calculation of the Euler number. We know an example of developing figure of the regular 28 faced polyhedron. That is the developing figure of Chimaki $(7,7,7)$ hyplane in Fig. 6.

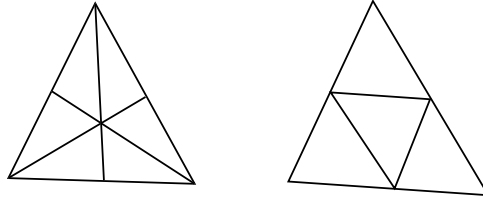


Fig. 12. Subdivision of a triangle.

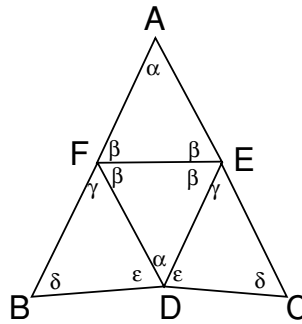


Fig. 13. Subdivision of (6,6,7) hyplane (calculation).

To investigate this open problem, a toy “Polydron” is very useful. This toy consists of faces of regular polygons and we can construct them up easily. See Fig. 11.

5. Subdivision

Hyplane is an analogue of $K = -1$ surface in R^3 , but it is a polyhedron after all. If we want to make smoother configuration, we need to consider subdivision of hyplane. Subdivision is a configuration such that we divide each face into some small pieces. There are two standard ways to divide a triangle into small triangles. See Fig. 12. Here we adopt the right one and consider a subdivision of (6,6,7) hyplane.

There is no unique way to make a subdivision. Here is one way but it is not the only one. First we divide an isosceles triangle into 4 parts. See Fig. 13. Vertices D, E, F are added. Recalling the condition (H4), we know that the angle sum at D, E, F are more than 2π , so the angle $\angle AFB, \angle BDC, \angle CEA$ are greater than π . There are three mathematical demands to determine the shape of subdivision.

- (D1) The angle sum of each vertex is constant.
- (D2) There are 7 faces together at A , there are 6 faces together at B, C .
- (D3) The figure is symmetrical with respect to a vertical line.

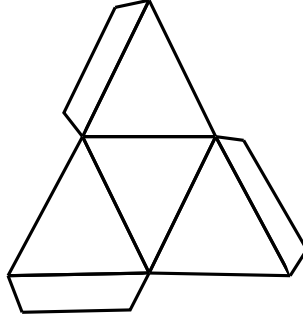


Fig. 14. Subdivision of (6,6,7) hyplane (precise figure).

These demands do not determine the shapes of all triangles. So we add one more condition that $\triangle AEF$ and $\triangle DEF$ are congruent to each other. And we make a series of equations for Fig. 13.

Then we have

$$\begin{cases} 7\alpha = 4\beta + 2\gamma = 6\delta = 4\varepsilon + 2\partial \\ \alpha + 2\beta = \pi \\ \gamma + \delta + \varepsilon = \pi. \end{cases}$$

It is easy to solve these equations and we get

$$\begin{cases} \alpha = 24\pi / 83 \\ \beta = 59\pi / 166 \\ \gamma = 25\pi / 83 \\ \delta = 28\pi / 83 \\ \varepsilon = 30\pi / 83. \end{cases}$$

In this case, the discrete curvature $K(v)$ is

$$K(v) = 2\pi - 7\alpha = -\frac{2\pi}{83}.$$

For a (6,6,7) hyplane, $K(v) = -18^\circ = \pi/10$ and $-2\pi/83$ is closer to zero. The following Fig. 14 is the precise construction of the faces.

Using this figure, we can make a Chimaki type (6,6,7) subdivisional hyplane as in Fig. 15. It looks like a tetra-pot on seashore.

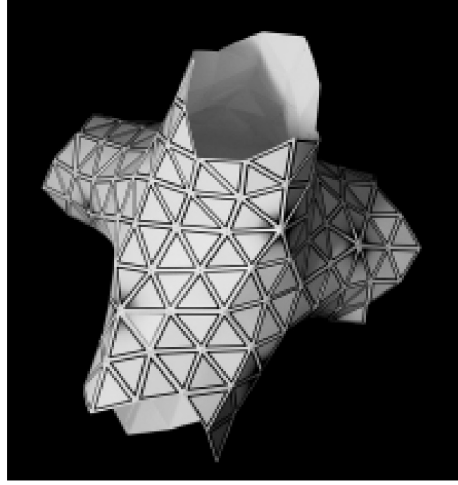


Fig. 15. Subdivision of (6,6,7) hyplane (chimaki type).

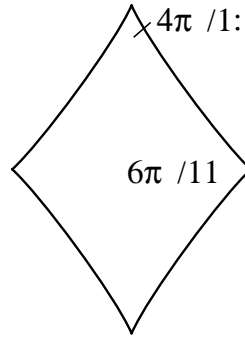


Fig. 16. A hyperbolic rhombus.

6. Irregular Tiling and Hyplane

Tiling pattern of (a, b, c) hyplane is the same as that of $(2\pi/a, 2\pi/b, 2\pi/c)$ hyperbolic triangles. We can understand this feature naturally because a $K = -1$ surface is an isometric immersion of the hyperbolic plane to R^3 . From this viewpoint, for an irregular tiling pattern of the hyperbolic plane by triangles of the same figure, we can construct a hyplane-like polyhedron. Here is an example. In Fig. 16, there is a hyperbolic rhombus with angles $4\pi/11$, $6\pi/11$. We can make an irregular tiling on the hyperbolic plane. To make 2π by summing up $4\pi/11$ and $6\pi/11$, there are two following ways:

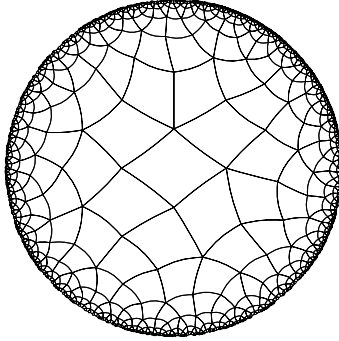


Fig. 17. Acyclic tiling of the hyperbolic rhombi.

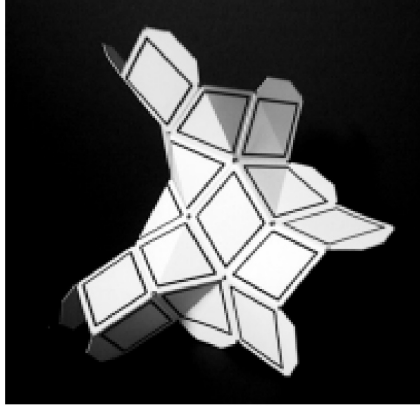


Fig. 18. Acycliv hyplane of the rhombus.

$$\frac{4\pi}{11} + \frac{6\pi}{11} + \frac{6\pi}{11} + \frac{6\pi}{11} + \frac{4\pi}{11} + \frac{4\pi}{11} + \frac{4\pi}{11} + \frac{4\pi}{11} + \frac{6\pi}{11} = 2\pi.$$

This fact suggests that a tiling of this rhombus may be irregular. In Fig. 17 there is an example of tiling. This tiling is presented first in AHARA (2004) and we have few mathematical results.

On the other hand, we can construct a hyplane-like polyhedron from this tiling. First we divide a rhombus into two isosceles triangles. After simple calculations we have a triangle with angles $2\pi/5$, $3\pi/10$, $3\pi/10$, and we can make configurations by using this triangle. In Fig. 18 we see an example. In Fig. 19, we make a subdivision of this tiling.

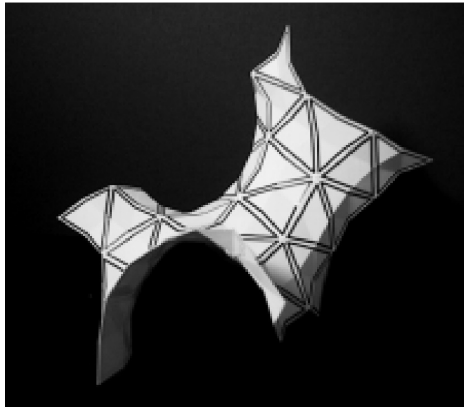


Fig. 19. Acycliv hyplane of the rhombus(subdivisional figure).

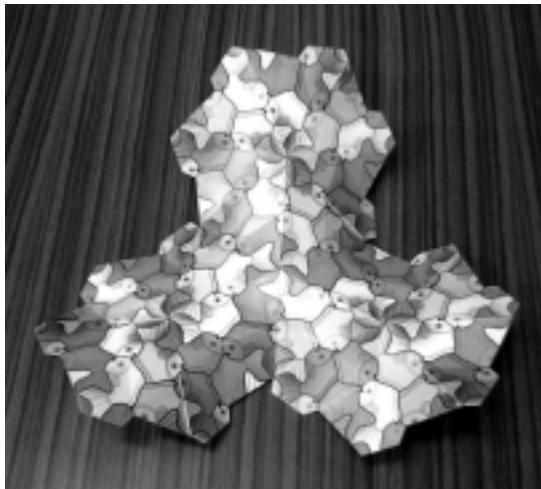


Fig. 20. A tessellation of goldfish on hyplane.

7. Applications

Here we present three applications of hyplane.

First we consider an application to three-dimensional art. In hyplane all faces are triangles and are congruent to each other. For example we can make a tessellation design art on hyplane. See Fig. 20. This is a sample work by the author. On $(6,6,7)$ subdivisional hyplane, tessellation design of gold fishes are drawn. We see that at some vertices there are 6 caudal fins together but there are vertices where 7 caudal fins meeting together.

Second we consider an educational toy of hyplane. There are many kinds of toys by which we can make polyhedra easily. Polydron (POLYDRON) is a good example. So if there are pieces of triangles with 63,63,54 degrees in Polydron, we can construct hyplane very easily.

The author held a workshop on hyplane for junior high school students in 2003. Here he prepared developing figures and participants made paper crafts of hyplane. This workshop is very good as education of geometry.

Last, we consider application to architecture. Hyplane consists of triangles and it may be good for structure of a building. But unfortunately there is no offer to use hyplane to construct a building until now.

8. Conclusion

We present a new type of shape of polyhedron through hyplane. We know lots of polyhedra, but most of polyhedra are based on a sphere and they arise from spherical (or elliptic) geometry. Hyplane is the only polyhedron arising from the hyperbolic geometry and it looks like a horse saddle everywhere. Hyplane is descended from the idea of hyperbolic geometry whose history is long.

REFERENCES

- AHARA, K. (1999) Hyplane, software, <http://www.math.meiji.ac.jp/~ahara/>
 AHARA, K. (2004) Hyplane (Japanese column), Sugaku-Seminor, Vol. 4–12.
 GRAY, A. (1998) *Modern Differential Geometry of Curves and Surfaces with Mathematica*, 2nd ed., CH/CRC Studies Adv., CRC.
 GPS PROJECT (1998) GPS Project, <http://www.math.kobe-u.ac.jp/~kobayasi/GPS/GPSIndex.html>
 INOGUCHI, J., KOBAYASHI, S.-P. and MATSUURA, N. (2005) Differential geometry of surfaces and soliton equations, Rikkyo Lecture Note No. 8 (in Japanese).
 POLYDRON (Polydron), <http://www.polydron.co.uk/>
 WEEKS, J. R. (2001) *The Shape of Spaces*, 2nd ed., Marcel Dekker Inc.