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Materializing 3D Quasi-Fuchsian Fractals

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Abstract. This paper reports experiments of materializing recent new discovered mathematical surfaces, 3D quasi-Fuchsian fractals. Three different models in glass, plastic, and metal are created to realize rich mathematical properties including self-similarity, 3-embeddable, simply-connected, and complicated surface consisting of infinite number of cusps. Different models can complementary provide mathematicians and anyone unprofessional better understanding of the mathematical properties of the new discovery.

1. Introduction

A three dimensional sculpture is worth than a thousand words. This 3D analogue of a famous saying was true to mathematicians around the end of nineteenth century in Europe. Many precise surface models in hot research topic had been crafted for argebraic geometry, differential geometry, elliptic functions, and complex functions (SCHILLING, 1904). The existing models that can be seen in mathematical departments (KOHNO, 1996, 1997; MASUDA *et al.*, 2002) give us insights of the rich mathematical properties of the beautiful surfaces with the great reality. Since twentieth century, geometry has been developed to be abstract with advanced concepts introduced by scheme theory and cohomology, the target of geometry had been moved to abstract spaces that are rarely visible in our three dimensional worlds.

In 1985 the saying virtually came back true to mathematicians by HOFFMAN and MEEKS (1985) when they drew beautiful computer graphics of a surface as the result of calculating complicated equations by COSTA (1984). The computer graphics not only reviled the beauty of the surface, but also gave important observations that led a discovery. Thus they proved that the surface is the first example of an embedded complete minimal surface of R^3 in about two centuries after the discovery of the catenoid and helicoids (MEUSNIER, 1785). After that many minimal surfaces of R^3 with various interesting mathematical properties have been proven as embedded with fascinating computer graphics (GRISSE-BRAUCKMANN and WOHLGEMUTH, 1996; KAPOULEAS, 1997).

Using recent evolving 3D printer technologies, precise and artistic sculptures of embedded minimal surfaces had been challenged by mathematicians, computer scientists, artists and their collaborations, DICKSON (1991–2005), Brent Collins (SEQUIN, 1997), GROSSMAN (2000–2005), SEQUIN (1998–2005), Gilbert Riedelbauch (HYDE *et al*., 2003). Depending on their applications, various 3D printer techniques can be used including stereolithography (SLA), selective laser sintering (SLS), fused deposition modeling (FDM), laminated object manufacturing (LOM), inkjet-based three dimensional printing (3DP), laser induced damage image (LIDI) (KODAMA, 1981; BURNS, 1993; TROITSKI, 1999). Most of 3D printer techniques other than LIDI are called rapid prototyping, and they have common processes to grow models by adding and bonding materials layer-by-layer. LIDI etches dots inside optically transparent materials by a pulsed laser beam. 3D printer techniques have also been applied to various topological structures that had required patient hand-crafting such as legacy surfaces of nineteenth centuries, polyhedra, knots, embedded graphs on genus-N surfaces and in space, tessellations on surfaces and in space, 3D version of iterated function systems fractals, 3D slices of quaternion fractals, and chaos attractors (DICKSON, 1991–2005; SEQUIN, 1998–2005; WATANABE and IKEGAMI, 2000; KANNARI, 2000, 2002; KOHNO, 2001; HART, 2002).

In 2002, the saying again virtually came back true in a new area of mathematics by AHARA and ARAKI (2002, 2003) when we drew beautiful computer graphics (Fig. 1) of limit sets of three dimensional Kleinian groups, also referred as 3D quasi-Fuchsian fractals. This computer graphics gave us important observations that led a discovery that the groups of the limit sets are new families of three dimensional quasi-Fuchsian groups. Our computer graphics were the first examples of the limit sets of three-dimensional quasi-Fuchsian groups.

In this paper, we report experiments of materializing the 3D quasi-Fuchsian fractals with richer mathematical properties including self-similarity, 3-embeddable, simply-connected, and complicated surface consisting of infinite number of cusps, than ever such as 3D version of iterated function systems fractals and 3D slice of quaternion fractals (HARD *et al*., 1989). Section 2 explains more details about the mathematical background of quasi-Fuchsian fractals, and later sections describe three approaches with different models in glass, plastic, and metal. And we discuss that the saying really comes back true.

2. Quasi-Fuchsian Fractals

In this section we explain details about the mathematical background of quasi-Fuchsian fractals. The two dimensional limit set of quasi-Fuchsian groups and their deformation space are hot topics of Kleinian groups and hyperbolic 3-manifolds. More comprehensive and introductory literature specific for two dimensional cases can be found in MUMFORD *et al*. (2002).

Quasi-Fuchain fractals can be defined as the limit sets of quasi-Fuchsian subgroup of Mob(S^n), the group of Mobius transformations of the n-dimensional space with infinity, $S^n = R^n \cup \{\infty\}$.

A Mobius transformation of S^n is, by definition, a composition of some inversions of S^{n-1}, thus in three dimensional case inversion of spheres or mirror reflections of planes. In almost articles of mathematics, we often consider only orientation preserving Mobius transformations, that is, even number compositions of inversions and reflections. However the condition orientation preserving is required in mathematics only from a technical reason and we do not consider the condition in this paper.

For a subgroup G of $\mathrm{Mob}(S^n)$, let the discontinuity set $\Omega(G)$ be

$$\Omega(G) = \{x \in S^n | \text{the point } x \text{ possesses a neighborhood } U(x) \text{ such that the intersection } U(x) \cap gU(x) \text{ is empty for all but finite elements } g \in G\}.$$

The complement $\Lambda(G) := S^n \backslash \Omega(G)$ is called the limit set of the group G. This definition is clear for mathematicians but not for others. So we give another comprehensive definition here. The following definition is not strictly same, however, it is easier for us to understand the limit set geometrically. Suppose that a group G is generated by some inversions and reflections. Let A be the fundamental domain of the group. That is, the fundamental domain A is a maximal area such that any two points in A cannot be transformed to each other by any elements in the group. In fact the fundamental domain A is given by the intersection of the exterior of generating S^{n-1}. Then we have the discontinuity set as follows.

$$\Omega(G) = \bigcup_{g \in G} gA.$$

And the limit set is the complement of the discontinuity.

A group is a Kleinian group if the limit set is not empty. It is known that the limit set consists either of 0, 1, 2 or infinite amount of points. A Kleinian group is a Fuchsian group if the limit set is a closed round S^{n-1}. The limit sets of two dimensional and three dimensional fuchsian groups are a circle and a sphere respectively. The Kleinian group is called a quasi-Fuchsian group if the limit set is homeomorphic to a closed S^{n-1} and there exists a homeomorphism $f: S^n \to S^n$ that conjugates the Klenian group to a Fuchsian group.

A fundamental invariant of an Kleianian group is the Hausedorff dimension, also referred as a fractal dimension, of the limit set (McMullen, 1999). In two dimensional cases, it is observed that the Hausedorff dimension of the limit sets of quasi-Fuchsian groups are non-integer, thus the limit sets are fractal. Fractal is a word invented by Benoit Mandelbrot to specify the complicated phenomena of shapes with self-similarity.

Images of the limit set can be approximately drawn by iterating inversions and reflections that generates the group starting from an initial point as many times as possible. Any point can be initial point to draw approximated limit set but if the initial point is known as a part of limit set the result image can be precise. Many beautiful computer graphics of them and the condition of inversions and reflections for quasi-Fuchsian groups can be found in Mumford *et al*. (2002).

In Ahara and Araki (2002, 2003), we found that the limit set of quasi-Fuchsian groups are also fractal. Before that, no one knew what the shape of the limit set of three dimensional quasi-Fuchsian groups look like although very few examples of three dimensional quasi-Fuchsian groups were reported (Mednykh *et al*., 2002).

3. Glass Model with LIDI Technique

Laser induced damage image (LIDI) technique is a kind of 3D pointillist technique inside glass. Pulsed laser beam fractures a glass at focal point without damaging the surface. The shapes are drawn as monochrome images with white dots on transparent canvas of a glass block. LIDI also known as Laser Crystal is getting popular in market as paperweights or novelty goods for cartoon characters, animals, and vehicles.

The advantage of LIDI is that any types of shape including unstable and delicate structures can be drawn because the shapes are floating inside glasses. Using LIDI, the image of complicated surface with a multitude of sharp cusps can be observed by anyone without worries. If such surfaces were exposed for observers to be tangible, the delicate structures of the surface could be damaged by little impacts by observers, or the observers could be hurt by the dangerous surface with small sharp needles.

For creating precise glass models of delicate structure of 3D quasi-Fuchsian fractals, the diameter of dots inside glass should be as short as possible. The diameter of dots depends on the wavelength of laser of LIDI systems. In market, most LIDI images are etched with Nd:YAG laser of 1064 nm wavelength. Recent LIDI systems use a non-linear optical crystal, KTP that convert the wavelength of Nd:YAG laser into 532 nm.

The disadvantage of LIDI is that connected structures like lines and surfaces cannot be drawn in strict sense because the images consist of discrete dots separating with one another. Etching dots overcrowded can cause crashes inside the glass.

For creating precise glass models of 3D quasi-Fuchsian fractals with simply connected property, the number of dots should be as much as possible and the distance between dots should be as short as possible. The minimum distance between dots depends on the laser radiation energy and material properties. In market, most LIDI images in $50 \times 50 \times 70$ mm size of glass block are etched with between 10,000 to 60,0000 dots depending on their prices because less number of dots costs lower price.

In our experiments, nearly 200,000 dots in $50 \times 50 \times 70$ mm glass block are used to realize the complicated surfaces with 532 nm laser. As shown in Fig. 2, Computer graphics in dot representation can be near exactly reproduced inside the glass block with sharp cusp surface.

4. Metal and Plastic Model with Rapid Prototyping Technique

Rapid prototyping techniques have been developed in manufacturing for ease of making tangible mock-ups in design phase directly from CAD (Computer Aided Design) data sources. Rapid prototyping grows models by adding and bonding materials layer-by-layer. Getting higher precision and use of more various materials, rapid prototyping technologies have been evolved for master model creations of mold casting, direct manufacturing of molds, and finally direct manufacturing of products. Direct manufacturing with rapid prototyping is expanding its area to biology and energy applications (LEE *et al.*, 2000).

The advantage of rapid prototyping is that it is good for representing connected structures like surfaces as tangible solid models. The most rapid prototyping systems require solid modeling representations (REQUICHA, 1977) to CAD data sources. The solid modeling representations ensure that all surfaces meet properly and that the object is geometrically correct. Observers can easily understand the simply connected property by touching around the entire surface and checking no holes on the surface with their hands.

For creating precise rapid prototyping models of simply connected surface of 3D quasi-Fuchsian fractals, the pitch between layers should be as short as possible. The minimum pitch between layers depends on the rapid prototyping systems, their material properties and shape of the model.

The disadvantage of rapid prototyping is the difficulty to construct unstable and delicate structures as solid models. The layered materials can be broken with little pressures if CAD data source provide sharp needle like shape that diameter is as narrows as the pitch.

For creating delicate structures of 3D quasi-Fuchsian fractals with a multitude of sharp cusps, materials should be carefully chosen. Available materials depend on rapid prototyping systems. Some recent rapid prototyping systems provide direct manufacturing in metal by layering metal powders. Most of rapid prototyping in market use resins to create master models for mold casting. The master models in resin are easy to break, however, metal cast models can be hard enough and plastic cast model can be soft enough not to be broken by pressures the observers.

For metal mold casting, we use lost wax investment casting. Lost wax investment casting method was developed by Italian monk some 900 years ago to craft large statues in metal. It is widely used for jewels and complex metal parts in manufacturing today. The master model is sculpted in wax then coated with plaster. The plaster is used as a mold by melted out the wax, and molten metal is cast into the mold. The mold is broken away after the metal is cooled and hardended. Today, the master models can be created in a wax like resin with inkjet-based three-dimensional printer directly from CAD data source.

In our experiments of metal models, three wax master models ($15 \times 15 \times 15$ mm) are created in pitch of 0.072 mm. Only one cast model in capper is survived as shown in Fig. 3. One master model in a wax like resin is crashed during plaster coating process. One cast model in aluminum is hardended on the way to the edge of cusps because the specific gravity of aluminum is light.

For plastic mold casting, we make electroformed molds. Electroformed molds are molds made by electroplating metals on the master model. The molds are reused for mass production. The master model is melted out and a molten plastic is layered on the surface of the mold. Before hardended, the plastic is pulled out of the mold without breaking the mold. Today, master models can be created in resin such as polyethylene with a selective laser sintering technique directly from CAD data source.

In our experiments of plastic models, a hundred plastic models are created from a pair of master models ($100 \times 100 \times 100$ mm) in pitch of 0.15 mm. As shown in Fig. 4, the cusps are configured not to be sharp by adjusting parameters of quasi-Fuchsian fractals (AHARA and ARAKI, 2002) because plastic models must be pulled out of the mold at creation process. A pair of plastic model is connected into a wholly simply connected surface. The seam of the pair corresponds the plane of mirror reflection.

5. Discussion

In previous sections we mentioned pros and cons of three different models, and applied these methods to 3D quasi-Fuchsian fractals. As expected, simply connected property can be represented with tangible solid models in metal or plastic. And complicated surface with infinite of sharp cusps can be approximately represented with models in glass or metal.

We found more advantages of these materialized models than we had expected through demonstrations of these models to many persons including mathematicians and someone unprofessional. One advantage is that see-though surface of glass model provides insight of the shape of exterior of the surface. The other is that plastic model is good for understanding self-similarity property by painting and disconnecting the surfaces. Such material specific advantages could not be found if we only observed virtual sculptures in computer graphics.

Observing the exterior shape of the limit set is important for mathematician because the exterior shape and interior shape divided by the limit set of quasi-Fuchsian groups are different in general. By changing some parameters of the fractals (AHARA and ARAKI, 2002), these shapes can be deformed. The interior shape can be characterized as a multitude of convex cusps, and the convex cusps are approximately represented in metal models. However, the exterior shapes characterized as concave cusps are difficult to observe in metal models.

Glass models can approximately represent both a multitude of convex and concave cusps at once because the surfaces are not so densely drawn with a multitude of dots. In glass models, deep concave cusps can also be seen as the sharp convex cusp from other side of the surface. Observer can also find that two deep concaves cusps dig from different area are nearly contacting at the edge of the cusps in see-through surface of glass models.

Self-similarity is a basic property of fractals but is rather difficult property that requires more careful observations than other properties. Observers may be able to find the property by paying attention to repeated sequences of sharp cusps in glass and metal models. Plastic models, even though without such sharp cusps, can provide much better understanding of self-similarity than other material because of its cheaper cost.

Plastic models are cheap enough to paint some colors on the surface. Observers can paint the same color to similar parts on plastic model. During painting, observers can find that all the parts will be drawn in single color and that a bigger part consists of a multitude of small copies of the part.

Plastic model are cheap enough to cut in peaces. Using a cutter, observers can easily disconnect the plastic model along the lines of reflection sphere on the surface. Any cuts along a line of reflect sphere on the surface corresponds to 2D version of quasi-Fuchsian fractals. Observer can check the similarity in terms of Mobius transformations by calculating the equality between cross ratios of different cuts.

6. Concluding Remarks

We tried three different approaches to materialize our recent discovery in mathematics, 3D quasi-Fuchsian Fractals and found that different models can complementary provide better understanding of the mathematical properties of the new discovery than expected. When we drew the first image of 3D quasi-Fuchsian Fractals, we though that a three dimensional computer graphic is worth than a thousand words. Now we can say that a three dimensional sculpture is worth than a thousand computer graphics.

As future work if cost permitted, more challenges can be done for much better understanding of the mathematical properties. Complicated surface consisting of infinite number of cusps could be more approximated if more amount of much sharper cusps were materialized by recent direct manufacturing in lower pitch. Observer could use magnifying glasses to see more detail structure of the surface in the same way to observe natural

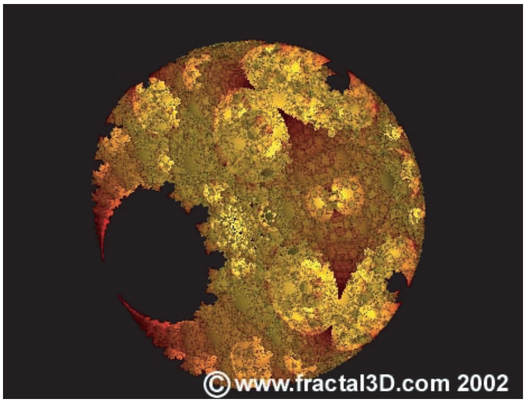

Fig. 1. Computer graphics of limit sets of three dimensional quasi-Fuchsian groups.

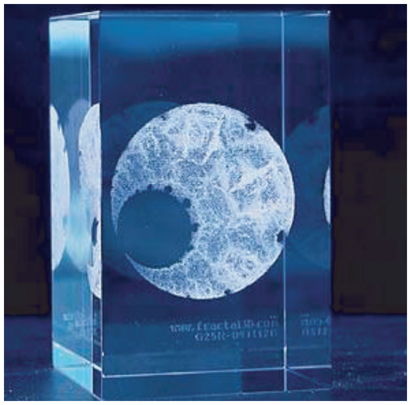

Fig. 2. Glass model of 3D quasi-Fuchsian fractal with Laser Induced Damage Image.

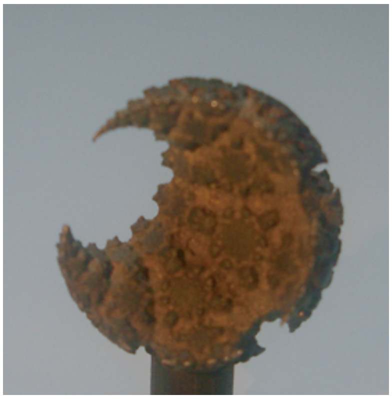

Fig. 3. Cupper model of 3D quasi-Fuchsian fractal with inkjet based 3D printer and lost wax investment casting.

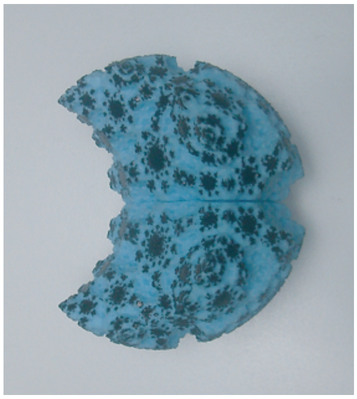

Fig. 4. Plastic model of 3D quasi-Fuchsian fractal with selective laser sintering technique and electroformed molds.

minerals and insects. Self-similarity could be more understandable if large model as life size of our body were created. Observers could experience zooming to the surface only by walking ahead to the large object and they can find similar patterns from both far and near points of view on the surface.

We are currently working on more 3D limit sets of Kleinian groups and their deformation spaces (ARAKI and ITO, 2003) with more interesting mathematical properties. We may need to try different approaches to materializing new properties such as loxodromic spirals and degenerated structures.

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