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Perfect Polyhedral Kaleidoscopes

Caspar SCHWABE

Kurashiki University of Science and the Arts,
2640 Nishinoura, Tsurajima-cho, Kurashiki, Okayama 712-8505, Japan
E-mail address: schwabe@arts.kusa.ac.jp

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Abstract. There are a lot of confusions in the geometric literature in the use of the terminology about polyhedral kaleidoscopes. This work tries to clarify this situation by presenting two new charts with an exact list of all the perfect polyhedral kaleidoscopes in three-dimensional space.

In this article we distinguish three basic kinds of kaleidoscopes (see Fig. 1).

1. The classic polygonal kaleidoscopes with infinite reflections
2. The polyhedral corner kaleidoscopes with finite reflections
3. The polyhedral closed box kaleidoscopes with infinite reflections

The word ‘perfect’ means that the reflected pattern has no break along the mirror axis. In order to obtain consistent mirror images, the angle between two mirrors should be 360° divided into an even number, or according to Brewster’s angle restriction law (1817): 180 divided into *n*.

For example if the angle between the mirrors is 120° or 72°, an inconsistency of images appear.

In the case of even numbers, there is no such clash: the images superpose on one another.

1. The classic polygonal kaleidoscopes with infinite reflections (see chart 2)

There are only 4 types:

(1) rectangle or square (90°, 90°, 90°, 90°)
(2) right-angled isosceles triangle (90°, 45°, 45°)
(3) equilateral triangle (60°, 60°, 60°)
(4) half of an equilateral triangle (90°, 60°, 30°)

The above is a complete list of all the possible polygonal kaleidoscope reflecting an infinite flat plane. Please note that the forms of (2) and (4) are the same as the two triangular ruler used in schools as well as the greek’s ‘stocheia’, to whom they thought as the ‘atoms’ of the universe.

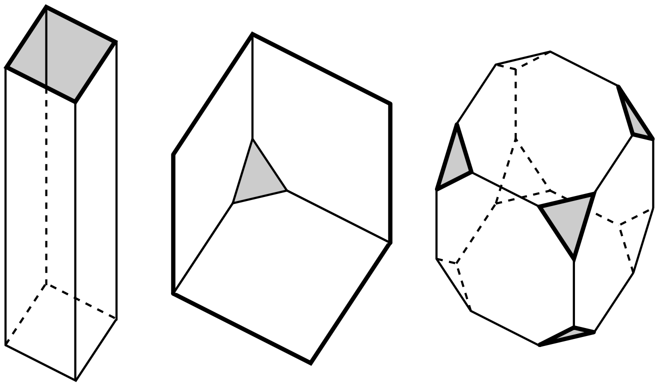

Fig. 1. The three basic kinds of kaleidoscopes.

For readers who are interested in the history of kaleidoscopes, please refer to Denes Nagy's "*Ars Scientifica*: Old and New, East and West, Mirrors and Kaleidoscopes" (1994).

Physicist Sir David Brewster (1817) first coined the term "kaleidoscope" (see references).

It is based on Greek expressions, *kalos* (beautiful), *eidos* (form), and *skopein* (to see).

Brewster called this sort of kaleidoscope a 'polycentric kaleidoscope', because it presents not just a rosette, but (theoretically) infinitely many of them, forming a wallpaper-type pattern.

2. The polyhedral corner kaleidoscopes with finite reflections (see chart 2)

There are only 4 types:

(1) 8-fold octahedron or cat-eye and endless variations on the top right side of the chart

(2) 24-fold hexakis tetrahedron

The exact definitions to the irrational number of angles are:

$$\cos^{-1}\left(1/\sqrt{3}\right) \cong 54.73°$$

$$\cos^{-1}(1/3) \cong 70.53°$$

(3) 48-fold hexakis octahedron

The exact definitions to the irrational number of angles are:

$$\cos^{-1}\left(2/\sqrt{6}\right) \cong 35.26°$$

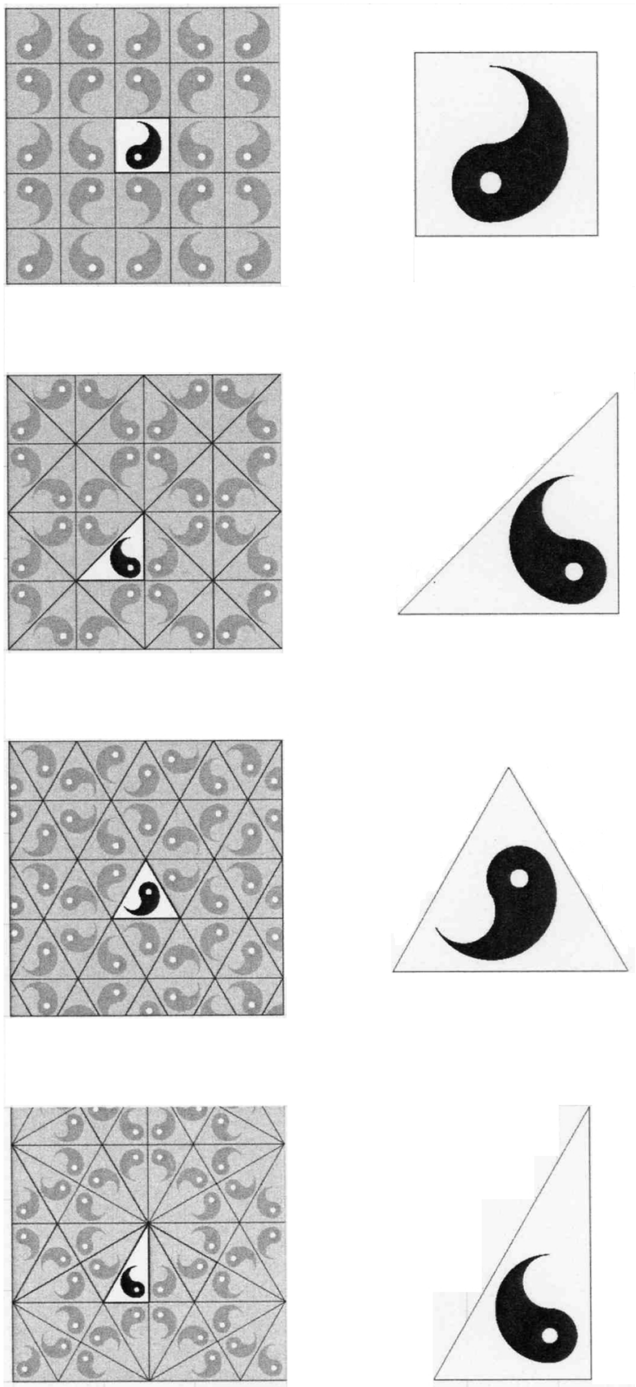

Chart 1. The four classic polygonal kaleidoscopes (infinite reflection).

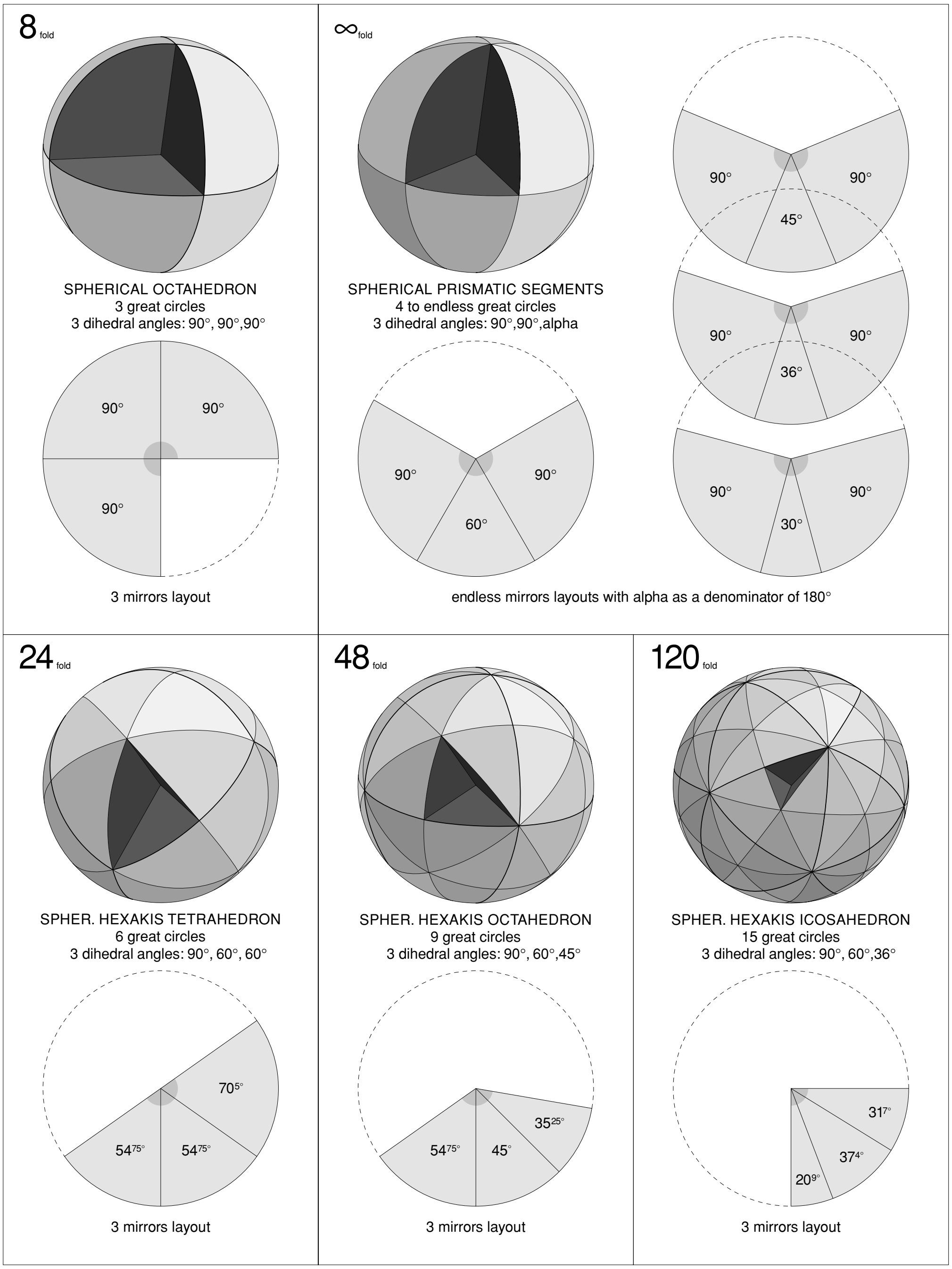

Chart 2. Great circles polyhedral corner kaleidoscopes (finite reflexion).

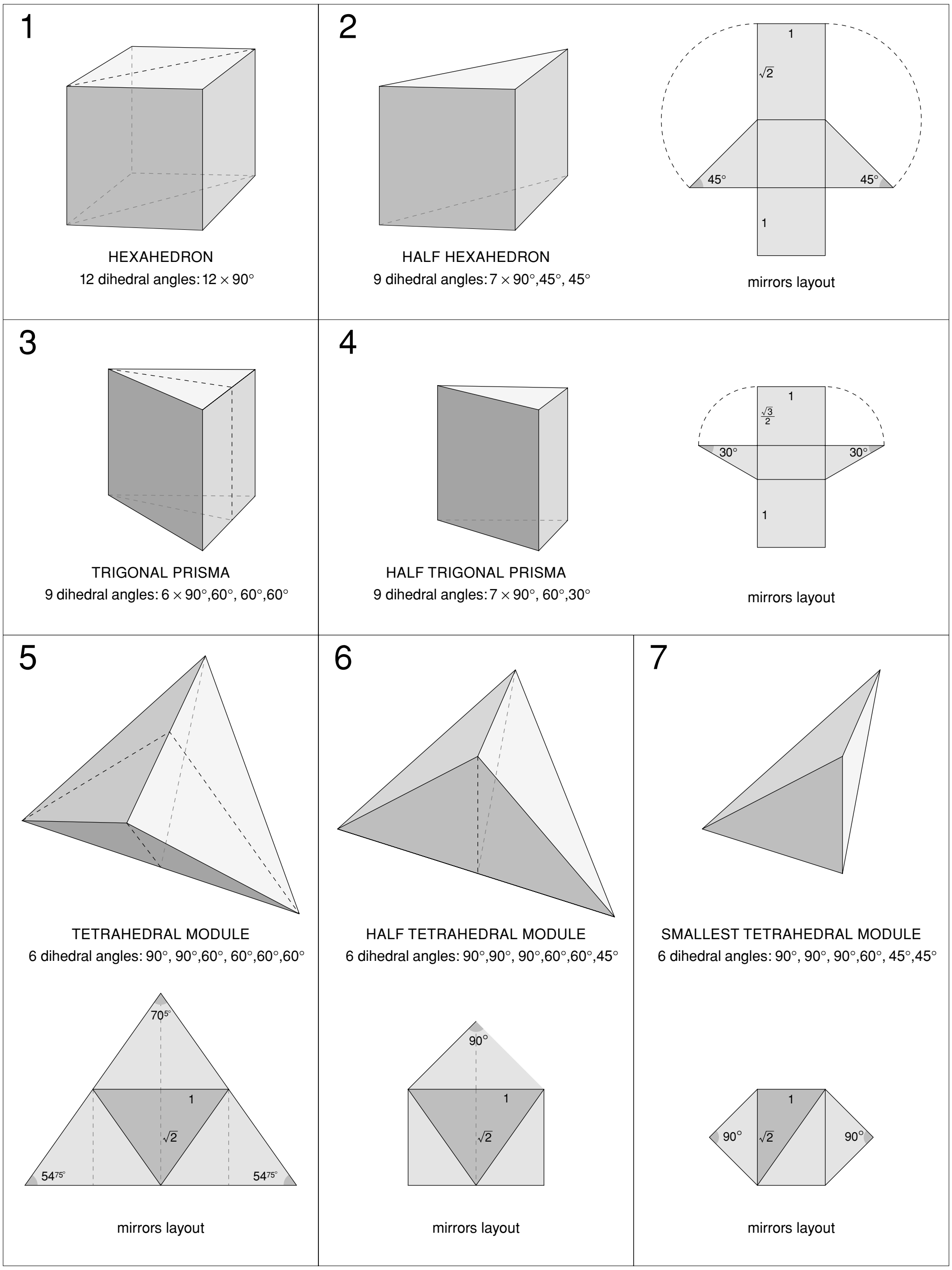

Chart 3. Perfect polyhedral closed box kaleidoscopes (infinite reflexion). After Shubnikov/Koptsik (1972).

(4) 120-fold hexakis icosahedron (the most fundamental region of a sphere)

The exact definitions to the irrational number of angles are:

$$\cos^{-1}\left[\left(1+\sqrt{5}\right)/2\sqrt{3}\right] \cong 20.91°$$

$$\cos^{-1}\left[\sqrt{\frac{1}{3}\left(1+\frac{2}{\sqrt{5}}\right)}\right] \cong 37.37°$$

$$\cos^{-1}\left[\sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{5}}\right)}\right] \cong 31.72°$$

These sorts of kaleidoscopes were investigated by late H. S. M. Coxeter (1974) and he has given them the name ‘polyhedral kaleidoscopes’, because they reflect polyhedra.

They can be constructed with 3, 6, 9 and 15 great circles, similar to Fuller’s (1982) geodesic domes.

From the first type, the classical right angled corner scope (8-fold) or cat-eye, there are also endless variations (12-, 16-, 20-fold, etc.) as long as the dihedral angle is a denominator of 180°.

A very special position within the perfect polyhedral kaleidoscopes holds the 120-fold icosahedral kaleidoscope with its unique 10-fold axis. The layout of the three mirrors measure 90°.

3. The polyhedral closed box kaleidoscopes with infinite reflections (see chart 3)

There are 7 types:

(1) Cube, (similar to (1) of the polygonal kaleidoscopes)
(2) Half cube (similar to (2) of the polygonal kaleidoscopes)
(3) Trigonal prisma (similar to (3) of the polygonal kaleidoscopes)
(4) Half trigonal prisma (similar to (4) of the polygonal kaleidoscopes)
(5) Tetrahedral module (similar to the 24-fold polyhedral corner kaleidoscopes)
(6) Half tetrahedral module (similar to the 48-fold polyhedral corner kaleidoscopes)
(7) Most fundamental tetrahedral module (similar to Fuller’s Quanta B module)

The shapes of the closed box kaleidoscopes are actually space filling blocks as well. With these kaleidoscopes the mirrors form a closed space and reflect an infinite spatial structure. Note: there are no icosahedral symmetries possible in spatial lattices. (5-fold axis)

There are two ways to see the inside: to go inside the mirrors, or to peep in through holes.

No. (5), (6), (7) have been extensively researched by Buckminster Fuller. He has given them the name of Quanta A module and Quanta B module and regarded them (just as the ‘stocheia’ of the old greeks) as the building blocks of the universe.

The charts in this abstracts are a preprint from the symmetry chapter of a new book called "GEOMETRIC ART" which will be on sale from May 2006, published by Kousakusha in Tokyo.

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