Original Paper *Forma*, **21**, 37–48, 2006

World of Scientific Puzzle Art Using Layer Manufacturing

Yasunari WATANABE[1,2,3]*, Yuji IKEGAMI[2,3], Kenji YAMAZAWA[2] and Yoshiko MURAKAMI[1]

[1]*Department of Information, Teikyoheisei University, Ichihara, Chiba 290-0193, Japan*
[2]*Institute of Physical and Chemical Research, Wako, Saitama 351-0198, Japan*
[3]*Interdisciplinary Institute of Science, Technology and Art ISTA, Asaka, Saitama 351-0036, Japan*
E-mail address: watanabe@thu.ac.jp

(Received October 13, 2005; Accepted November 25, 2005)

Keywords: Puzzle Art, Layer Manufacturing, 3D Escher Pattern, 3D Jigsaw Puzzle, Color Matching Puzzle

Abstract. Three types of intelligent puzzle, a three-dimensional Escher puzzle (3DES), a 3D jigsaw puzzle (3DJG) and a color matching cubocatahedron puzzle (CUB), are developed by layer manufacturing. Each puzzle contains a common packing problem. Furthermore, there exist two important mechanisms, the rotation of CUB and the translation of 3DJG. 3DES is obtained on the basis of periodic arrangement of icosahedrons, 3DJG is patterned from a dovetail joint, and CUB is derived from the closest packing of the face-centered cubic (fcc) structure.

1. Introduction

It has been 15 years since layer manufacturing was developed as a mold technique. The advances in layer manufacturing techniques have been remarkable and the field of their application is expanding to not only mechanical engineering but also art. We tried to build three types of geometrical/artistic three-dimensional (3D) model as follows using two types of layer manufacturing system, namely, laser stereolithography (LST) and powder layer manufacturing (PLM). In this paper three types of intelligent puzzle are presented (WATANABE and IKEGAMI, 2002; WATANABE *et al.*, 2004), namely, a color matching puzzle with the shape of a cuboctahedron based on the fcc close-packed structure of metals, the 3D Escher puzzle depending on crystallographic space group, and 3D jigsaw puzzle (3DJG) of which the sliding system is connected to the packing structure of rods. The idea of the 3DJG comes from Japanese carpentry techniques, particularly the use of the dovetail joint. We will show that layer manufacturing plays an important role in the interdisciplinary field of Science and Art.

2. Principle of Layer Manufacturing

The molding system that involves slicing and stacking the 3D CAD data of objects is called rapid prototyping. There are two processing methods to produce a mold, namely,

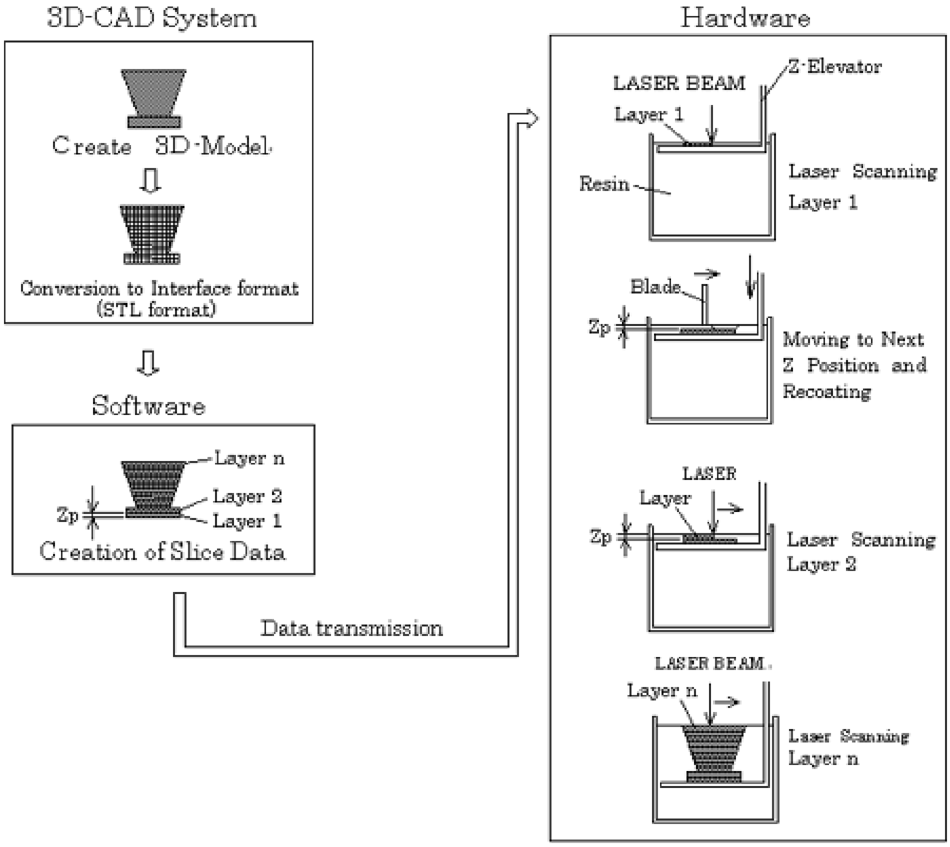

Fig. 1. Process of layer manufacturing.

laser stereolithography (LST) and powder layer manufacturing (PLM). Liquid resin hardened using a laser beam for the former or print assembly using a binder solution for the latter is used as a building material of the objects. The basic process of modeling in LST for generating the 3DJG is as follows.

(1) Data of the 3D model is generated using the CAD system and transformed into standard STL(stereolithography) data which is represented by a triangular patch.

(2) STL data is processed to generate data of many cross sections, which are sliced according to the stacking pitch. The scanning speed of the laser beam and the positions of objects are set up and input in the molding machine.

(3) The scanning laser beam is irradiated along the slice data of objects, the irradiated region of liquid resin hardens to become a thin film and a cross-section pattern identical to the slice data is obtained.

(4) The elevator stage on which hardening objects are mounted moves downwards by the amount of slice pitch.

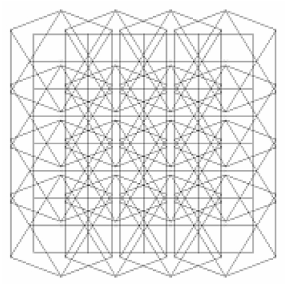

Fig. 2. Wire frame of icosahedron assembly.

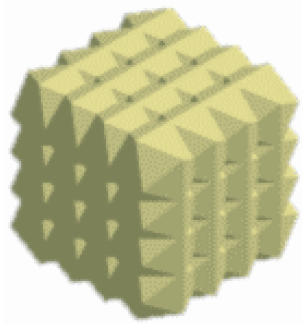

Fig. 3. Shading of icosahedron assembly.

(5) By recoating a liquid surface of resin, a new thin liquid layer is generated automatically on the hardened resin layer.

(6) Processes (3), (4) and (5) are iterated until objects are produced. As the amount of slice pitch is less than that of the depth of hardened resin, a new layer adheres strongly on the previous hardened layer.

After all the processes are finished the elevator is pulled up and the object taken out. The process of PLM of the 3D Escher pattern is essentially the same as that of STL. Calcium sulfate powder is used instead of liquid resin in STL. Solidification in PLM is carried out using binder solution instead of a laser beam in STL. One of the advantages of PLM is the coloring of objects.

3. 3D Escher Patterns (3DES)

3.1. Construction process of 3DES

Three-dimensional Escher patterns (3DES) are generated using a PLM machine (Color-Machine COSMART-1). Two types of solid model are presented, whose original data are designed using CADPAC2 software. In the preparation of the model, building an icosahedron is considered as a basic unit of modeling, which is an isotropic regular polyhedron that has the maximum facet in the five regular polyhedrons. Several inscribed cubes in the icosahedrons can be arranged such that they form a cubic lattice decorated by a set of icosahedrons. The assembled icosahedrons are formed to be a 3D periodic structure with an overlapped region of icosahedrons inside and icosahedron facets outside of the assembly (icosahedron assembly shown in Figs. 2 and 3). The model construction of 3D Escher pattern begins with the deformation of the unit cell of the cubic lattice by iterative arithmetic operation of the CAD system. The exterior of icosahedrons is available for creating the body of animals that has mirror symmetry.

It is important for the creator of 3DES to imagine what motif is the best by viewing an assembly of icosahedrons. That is, the problem is what motif can be determined by the contour of a solid. It is certain that the problem depends on the artistic sense of the creator. After several trials to produce a characteristic image, an elephant image comes across the

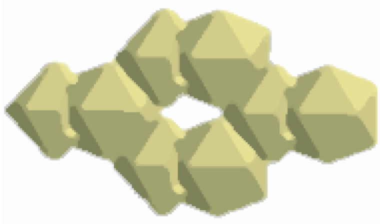

Fig. 4. Elephant set built using icosahedron pair.

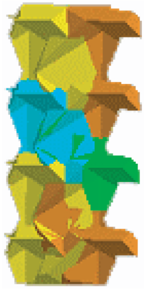

Fig. 5. Complementary set obtained from icosahedron's assembly.

Fig. 6. Elephant.

Fig. 7. Frilled lizard.

creator's mind as a motif. Cutting processes to produce the first image of the elephant are as follows.

First, select the two adjacent icosahedrons and cut them into the trunk, ears and back of the elephant. The space occupied by the solid elephant is called solid space (Fig. 4). Second, take out a column of complementary space (remainder set) of solid space (elephant set) from the whole space (icosahedrons set) as shown in Fig. 5. As the column has a periodic structure, it is easy to find a unit cell within it. Third, cut the unit cell of the remainder into another form of a second motif maintaining the periodicity of the column. The third step is very difficult because the new solid obtained here does not always resemble an animal. Furthermore, the cutting operation causes deformation of the solid elephant. The cutting effect on the elephant form is confirmed by the inversion of the remainder space to the solid one. The only way to obtain reasonable forms in both spaces is to iterate operations of first $\rightarrow$ second $\rightarrow$ third $\rightarrow$ first $\rightarrow$ second $\rightarrow$ third $\rightarrow \cdots$ until a solid model of the animal pattern appears in both spaces. Thus, two animal patterns, an elephant and a frilled lizard, are obtained as shown in Figs. 6 and 7. The packing of 3DES is shown in Fig. 8. The solid model produced by PLM is presented in Fig. 9, in which the packing of the elephant and frilled lizard is shown.

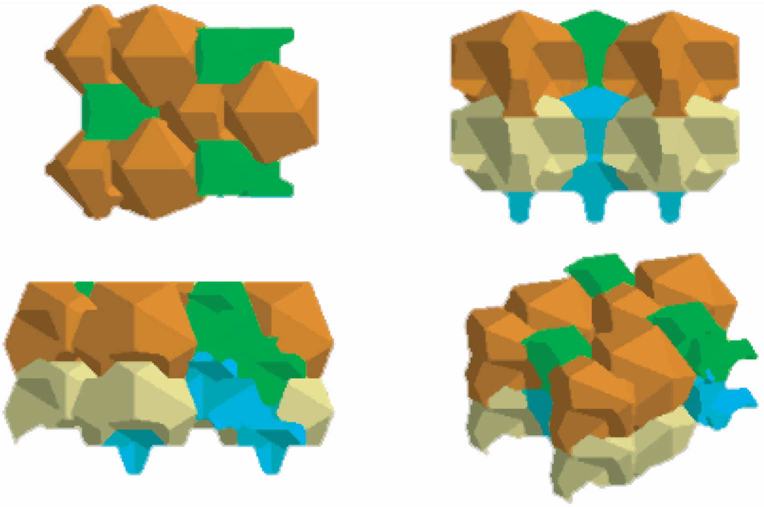

Fig. 8. Packing projection: Plan (upper left), Front view (upper right), Side view (lower left) and Isometric drawing (lower right).

Fig. 9. Packing solid model of two characters produced by PLM.

3.2. Crystallographic data

Crystallographic data such as lattice constant and number of objects in 3DES in the unit cell can be determined easily by the conventional method of crystallography. However, in 3DES there are no atomic parameters as in the crystal structure. Molecules in the crystal correspond to objects in 3DES. Let us compare the difference between crystal structure of molecule and 3DES. Although a unit cell is divided into a molecular space and a vacant

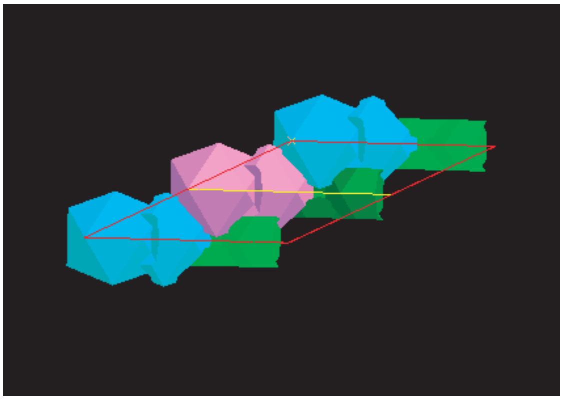

Fig. 10. Arrangement of two objects in unit cell.

Fig. 11. Single-layer 3D-jigsaw puzzle.

Fig. 12. Three-layer cubic jigsaw puzzle.

space in the crystal structure, it is divided into objects (animals) only in 3DES. That is, the contours of facets surrounding an object form boundary facets between adjacent objects. An essential difference between crystal structure and 3DES is whether there is a space in the structure. The coordinates of vertices of the facets surrounding the object represent the positions of objects. The number of facets surrounding the elephant is 82 and frilled lizard is 60 respectively. The crystallographic data is as follows:

Crystal System: monoclinic, Space group: P*m*.

Lattice constants: $a = 9.43$ cm, $b = 2.36$ cm, $c = 10.54$ cm, $\beta = 153.43°$.

Number of objects in unit cell: $Z = 4$.

The arrangement of objects in the unit cell is shown in Fig. 10.

Fig. 13. Sliding orbits between upper and middle layers.

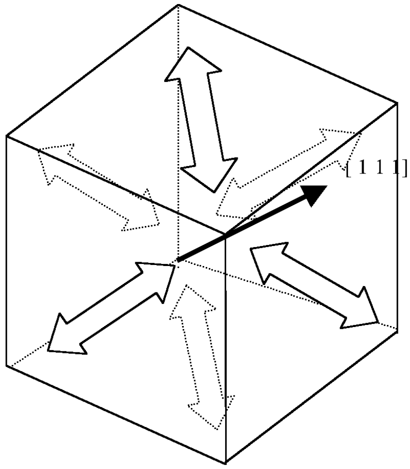

Fig. 14. Sliding direction on each facet of cubic jigsaw puzzle.

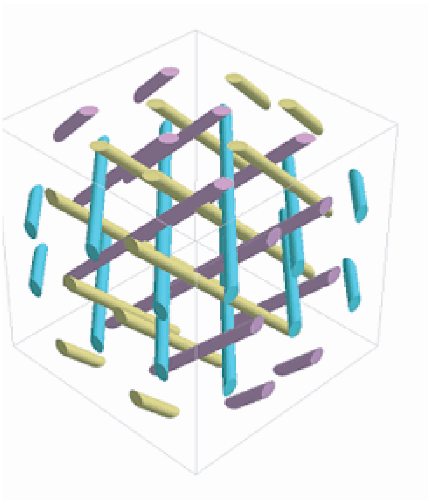

Fig. 15. Orbits around [111] screw axis in cubic jigsaw puzzle shown in Fig. 12.

Fig. 16. Solid 3 × 3 × 3 model produced by STL.

4. 3DJG

The jigsaw puzzle is one of the most popular puzzles. However, a 3DJG has not been developed to date as far as the authors are aware. Some solid jigsaw puzzles such as a glove or the Eiffel Tower are produced but they are a surface-covering-type 2D jigsaw puzzle essentially.

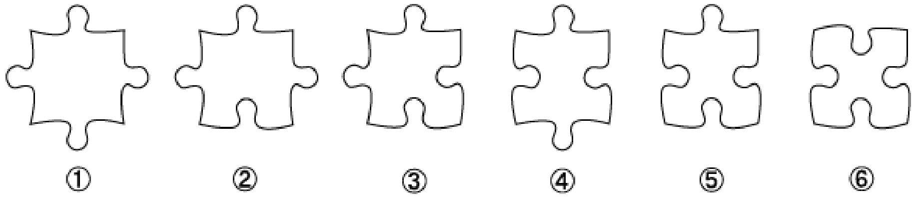

Fig. 17. Joint patterns of 2D jigsaw piece.

The authors developed first a 3 × 3 single-layer jigsaw puzzle as shown in Fig. 11. It seems that each piece cannot be slid upwards, downwards, frontwards or backwards. However, each piece can be separated into nine pieces.

By stacking the single-layer jigsaw puzzle shown in Fig. 11 it can be extended to a perfect 3D puzzle. For example, consider a new 3D cubic jigsaw puzzle which has 3 × 3 × 3 = 27 pieces, that is, a three-layer structure as shown in Fig. 12. It seems that every piece cannot be slid along the *x*, *y*, and *z* directions as well as in the single-layer case. How can each piece slide and be taken apart? The problem is solved by considering the geometry of the orbit on which the pieces of the puzzle slide. The orbits take parallel positions around the screw axes of the body-diagonal direction or cubic [111] axis. Figure 13 shows the orbit structure alone taken out from 3DJG. Since the orbits go through the center of the edge of the jigsaw piece that slides along the diagonal directions, namely, the [110], [011] and [101] axes, six stacking layers of the orbits are formed in the 3DJG structure as shown in Fig. 13. The sliding system of layers is shown schematically in Fig. 14 using arrow notation representing the sliding direction of the stacking layers. Thus, the upper layer and middle layer can slide together in the same diagonal direction as shown in Fig. 15. The solid model of a 3 × 3 × 3 object produced by LST is shown in Fig. 16.

In order that every piece slide easily it is necessary for orbits not to interact with each other. In the 2D jigsaw puzzle, there are six joint patterns (Fig. 17) that are combinations of concave and convex dovetail joints. In the 3D case, the joint pattern is determined necessarily by the conditions that each piece can slide along the orbit without obstruction. It is elucidated that the convex and concave orbits have to be arranged alternately at regular intervals to avoide obstructing each other. The joint pattern of the jigsaw piece appearing on the surface of the 3DJG is restricted to one pattern, which corresponds to ④ in Fig. 17. The surface pattern changes according to the sliding direction of each layer. If the sliding direction of the middle layer is perpendicular to the upper layer, another pattern of jigsaw piece will appear on the surface but in this case the sliding orbits interrupt each other. If every layer can slide in the same direction, it is possible to build a jigsaw puzzle with any number of layers.

3DJG is an application of the dovetail joint used in Japanese traditional carpentry. It is very interesting that the arrangement of sliding orbits in the structure of the 3DJG corresponds to a rod structure model having three axes (O’KEEFE *et al*., 2001; TESHIMA *et al*., 2002) and that there exists compound corresponding to the three-axis structure of the rod model (O’KEEFE and ANDERSON, 1977).

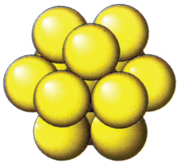

Fig. 18.

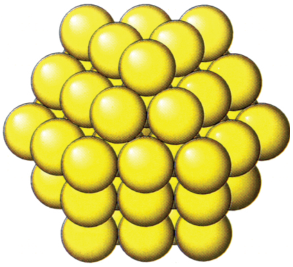

Fig. 19.

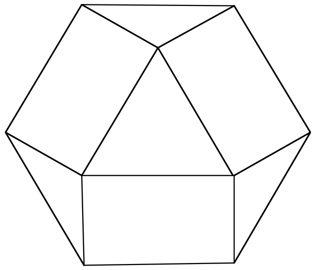

Fig. 20.

Fig. 18. Smallest three-layer cluster of FCCP (K1).
Fig. 19. Five-layer cluster of FCCP (K2).
Fig. 20. Cuboctahedron.

5. Color Matching Puzzle—RainbowCube

5.1. FCCP structure

Problems regarding the densest packing of spheres have been investigated by many authors. One of the closest-packed structures with very little space between spheres of uniform size is called the face-centered close-packed structure (FCCP). FCCP is obtained by stacking a honeycomb structure of the densest-packed spheres on a plane. The smallest cluster unit of FCCP is built by the following steps. First, consider a regular hexagonal sheet in which six spheres are arranged around a center sphere. Second, a regular triangle of three spheres is arranged on the upper side of the hexagonal sheet and another on the lower side. An outside view of the smallest cluster unit (K1) and its multilayer structure (K2) are shown in Figs. 18 and 19, respectively. Because the upper triangle and lower one are related by the center of inversion, the stacking structure takes the ABCABC··· period. If the upper and lower triangles have the same direction, the stacking period is ABABAB··· and it is called the hexagonal close-packed structure (HCP). The shape of cluster K1 is cuboctahedron (CUB) (Fig. 20), which is one of the Archimedes polyhedrons and has three 4-fold axes and four 3-fold axes, that is, facets of three squares and four regular triangles as shown in Fig. 20. The sheet of spheres stacks perpendicular to the triangle facet. Stacking layers are formed along [111], [1-11], [11-1] and [1-1-1], and their inversion directions crystallographically. The crystal structures of metals in nature such as Au, Ag, Cu and Al have the FCCP structure. It is possible to explain the physical and mechanical properties of metals such as malleability and flexibility using this cluster model. Since the atomic sheet can rotate or slide between layers, metals with FCCP have more flexible, ductile properties and are easier to process than materials with a HCP structure, in which atomic layers are stacked along a single rotational axis.

5.2. Mechanism of color-matching puzzle of CUB

The idea of manufacturing K1 was first proposed by WATANABE (1982) and its

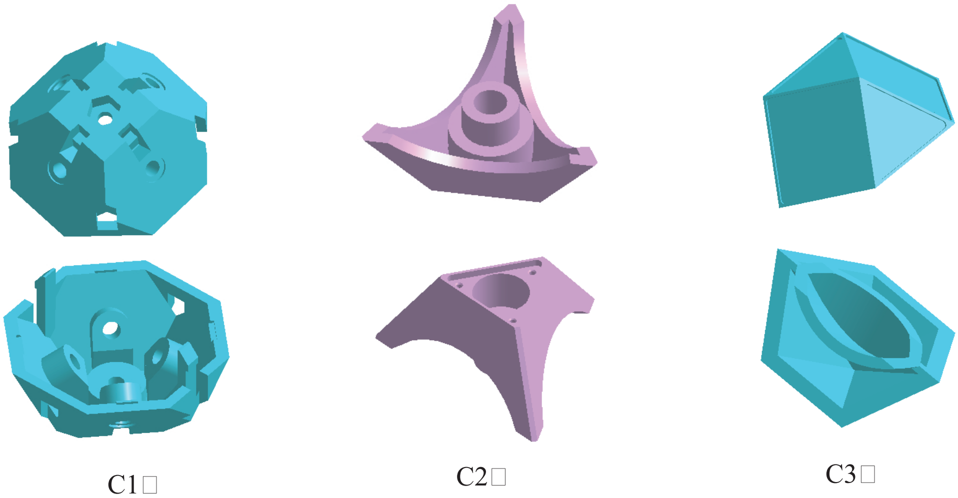

Fig. 21. Component parts of “RainbowCube”, Center unit (left), Rotation center unit (middle), Moving unit (right).

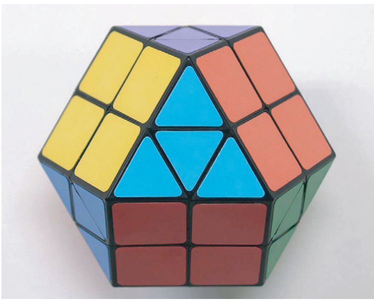

Fig. 22. Outside view of RainbowCube.

prototype was manufactured using a cutting machine. The K1 model (Fig. 18) can be built using small atomic balls for crystal structure modeling kit. However, it is impossible to rotate the atomic sheet around any 3-fold rotation axis using the ball model. To solve the problems authors developed a new mechanical model of K1, which can rotate every atomic sheet. The physical properties of metals with the FCCP structure can be explained to students using the new K1 model. By dividing a CUB into three layers normal to every 3-fold axis, it is decomposed into three types of part, namely, one octahedron (C1: Center unit of new model, Fig. 21), eight truncated equilateral triangle pyramids with 3-fold rotational symmetry (C2: Rotation center unit, Fig. 21) and twelve hexahedrons with mirror symmetry (C3: Moving unit, Fig. 21), which are composed of two trapezoids on the inside surface and

Table 1. Comparison of structures of Rainbow Cube and Rubic Cube.

Properties of structure	RainbowCube	RubicCube
Shape	Cuboctahedron	Cube
Number of facets	3 (square)*, 4(regular triangle)*	6 (square)
Number of rotation axes	4	3
Symmetry of rotation axes	3-fold	4-fold
Number of matching colors	14 (7*)	6
Number of layers	3, 5	2, 3, 4, 5
Combination number	Nearly 48×10^7	Nearly 43×10^{17}

*Independent region with inversion center.

two squares and two regular triangles on the outside surface, respectively. C1 is positioned in the center of the new KI. The rotation axis is attached on the 3-fold axis of C1. The 3-fold rotation axis of each C2 fits that of C1. The circular arc orbit is cut out of the base edge of the trapezoid of C2. The trapezoid of the inside surface of C3 has a convex orbit, which fits every concave circular arc of C2. Thus, one C2 and three C3 connect together and rotate around the 3-fold axis as a united body. Assembling all the parts of K1, the CUB model is obtained as shown in Fig. 22

Thus, the cluster model of the FCCP crystal structure (K1) can be replaced by CUB, where C1 (octahedron) and C3s (hexahedrons) correspond to the center atom and its twelve surrounding atoms, respectively. C2 is interpreted as a void in FCCP but this part plays an important role in rotating the atomic sheet perpendicular to the 3-fold axis. CUB is expected to be used as a teaching model for crystallography, that is, an ideal model to explain the physical and mechanical properties of metals. As for the HPC model, the same ideal polyhedron model as CUB can be built, in which the stacking layer is formed only along the c axis. If a single crystal of FCCP metals could be obtained and the physical properties could be observed, the CUB model would be helpful in understanding the anisotropy of the mechanical properties.

Another interesting application of CUB is a color matching puzzle. The CUB structure is similar to the well-known color matching puzzle "Rubic's cube", while its shape is that of a cube, its structure corresponds to a simple cubic crystal system having three rotation axes with 4-fold symmetry. However, CUB is cuboctahedron in shape, its structure corresponding to an FCCP crystal system having four rotation axes with 3-fold symmetry. The comparison of the structures of the two puzzles is summarized in Table 1. Because of the inversion center in CUB with fourteen facets, there are seven independent facet, namely, four triangles and three squares. The CUB original pattern is designed such that each facet is given the same color and cold colors are assigned to triangles and warm colors to squares. Such a color arrangement creates a sharp contrast between adjacent facets. CUB is named "RainbowCube" after the seven facet colors.

In this puzzle, the challenge is to find the shortest way to revert the dispersed color pattern by arbitrary rotations back to the original pattern. We can enjoy RainbowCube not only as a mathematical puzzle and artistic *objet*, but can also use it as a teaching tool for crystallography and physics. RainbowCube is shown in Fig. 22, which is produced using molds of component pieces fabricated by LST.

6. Concluding Remarks

Here, we presented three types of mathematical and geometrical solid puzzle. Some manual operation is necessary to solve these solid puzzles. Rotation orbits for RainbowCube and translation ones for 3DJG are present, whereas the 3DES is solved by only manual operation. While the former two puzzles are designed so that the problem is solved by decisive operation, the latter one has no mechanism to solve the problem. The 3DES is based on crystallographic operation (rotation and translation). Consequently, the problem can be solved by manual operation implicitly considering a crystallographic algorithm by trial and error.

Every solid puzzle presented here is composed of manifold parts with a complex shape. Layer manufacturing using the CAD/CAM system is the most suitable and useful technique for creating new solid puzzles. Although they all have polyhedral-type components, more beautiful and elegant solid puzzles are expected in the future.

This work is partly supported by the Hayao Nakayama Foundation for Science & Technology and Culture.

The construction of the 3DES was quoted from a graduation thesis of Teikyohesei University "A development of 3D Escher pattern" by Yumiko Tajima, Sato Takahashi and Yoshiko Murakami.

REFERENCES

O'KEEFE, M. and ANDERSON, S. (1977) Tanabe Rod packing and crystal chemistry, *Acta Cryst*, **A33**, 914–923.

O'KEEFE, M., PLEEVERT, J., TESHIMA, Y., WATANABE, Y. and OGAWA, T. (2001) The invariant Cubic Rod (cylinder) packings: Symmetries and coordinates, *Acta Cryst*, **A57**, 110–111.

TESHIMA, Y., WATANABE, Y. and OGAWA, T. (2002) A new structure of cylinder packing, *Lecture Note in Computer Science (Springer)*, LNCS-2098, pp. 351–361.

WATANABE, T., IKEGAMI, Y., YAMAZAWA, K. and MURAKAMI, Y. (2004) The world of scientific puzzle art using layer manufacturing, *Proceeding of Sixth Interdisciplinary Symmetry Congress and Exhibition of ISIS-Symmetry*, pp. 270–273.

WATANABE, Y. (1982) *Cube Puzzle in Nature*, Rikagakukenkyusyo News, Vol 72, No. 6, 6 pp. (in Japanese).

WATANABE, Y. and IKEGAMI, Y. (2002) The world of scientific art using layer manufacturing method, *Proceedings of 23rd Rapid Prototype Symposium*, pp. 105–107 (in Japanese).