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Properties of Tilings by Convex Pentagons

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Abstract. Let us consider an edge-to-edge and strongly balanced tiling of plane by pentagons. A node of valence s (≥3) in an edge-to-edge tiling is a point that is the common vertex of s tiles. Let W_1 be a finite closed disk satisfying the property that the average valence of nodes in W_1 is nearly equal to 10/3. Then, let T denote the union of the set of pentagons meeting the boundary of W_1 but not contained in W_1 and the set of pentagons contained in W_1, and let V_s denote the number of s-valent nodes in T. If the tiling in T is formed of only 3- and k-valent nodes, then $V_3 : V_k \approx 3k - 10 : 1$ where $k \geq 4$. On the other hand, if the tiles in edge-to-edge tiling are congruent convex pentagons, then at least two of the edges (of this congruent convex pentagon) are of equal length.

1. Introduction

Tiling refers to coverage of the plane with polygons (tiles) without gaps or overlapping. Especially a single congruent polygon that tiles the Euclidean plane is called a *prototile* or a *polygonal tile*, and the plane tiling with convex polygons has primarily been studied in an attempt to exhaust all the conditions of prototile. The current state of knowledge in the tilings by congruent polygons can be summarized as follows. Any single triangle and quadrilateral, including concave quadrilaterals, is tileable (i.e., all prototiles), since the sums of the (interior) angles of triangles and quadrilaterals divide evenly. On the other hand, tiling with other convex polygons is not necessarily possible because of constraints on angles and edge-lengths. In the case of convex hexagons, prototiles can be categorized into three types (BOLLOBÁS, 1963; GRÜNBAUM and SHEPHARD, 1987; SUGIMOTO, 1999). For the convex polygons with seven or more edges, no prototiles exist. For the convex pentagons, there are at present 14 types (see Figs. 1 and 2), but it remains unproven whether this is the perfect list of such pentagons (KERSHNER, 1968; GARDNER, 1975; KLARNER, 1981; GRÜNBAUM and SHEPHARD, 1987; SCHATTSCHNEIDER, 1987; WELLS, 1991; SUGIMOTO, 1999; SUGIMOTO and OGAWA, 2000, 2003c, 2005). As shown in Figs. 1 and 2, each of convex pentagonal tiles is defined by some conditions between lengths of edges and

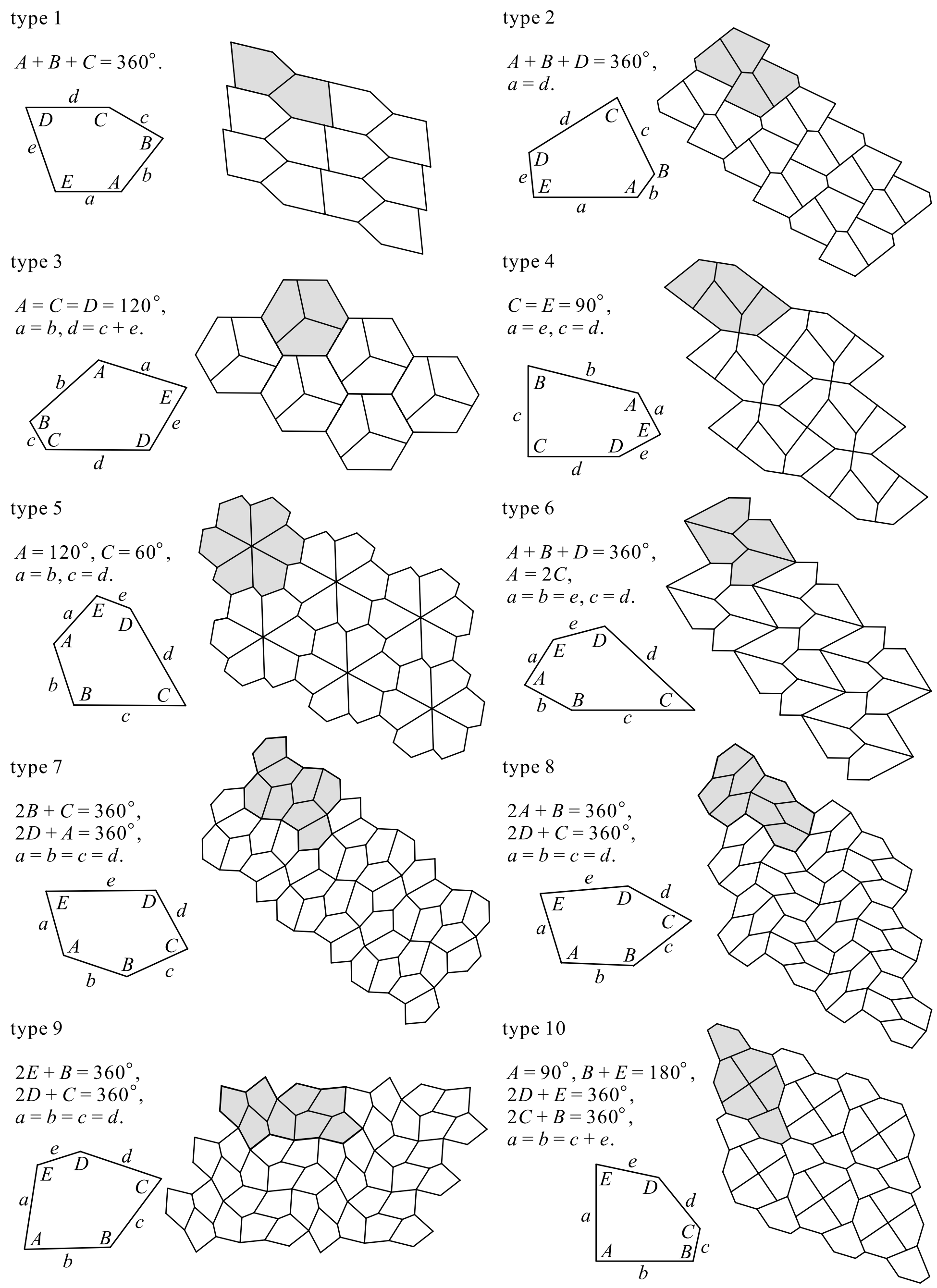

Fig. 1. Convex pentagonal tiles of type 1–10.

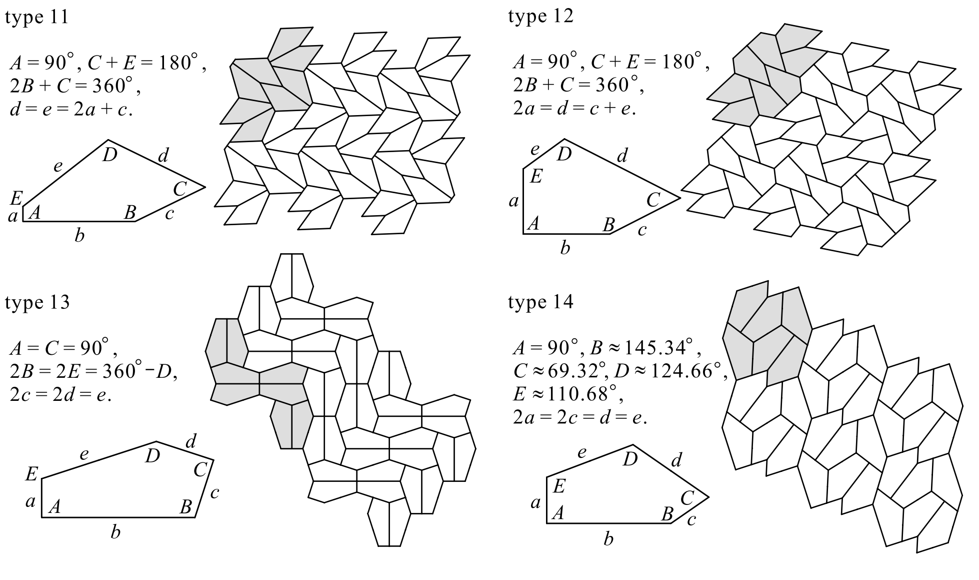

Fig. 2. Convex pentagonal tiles of type 11–14.

magnitudes of angles; but some degrees of freedom remain. (However, only the pentagonal tile of type 14 has one degree of freedom, that of size. For example, the exact value of C in pentagon of type 14 is $\cos^{-1}((3\sqrt{57} - 17)/16) \approx 1.2099$ rad $\approx 69.32°$, and the values of angle B, D, and E can be obtained by C.) Then, unless a convex pentagonal tile is a new prototile, any convex pentagonal tile belongs to one or more of 14 types.

In the study of tiling the plane with congruent convex polygons, the case of pentagon is the only unsolved problem. Then we believe that the problem has yet to be approached from a new point of view. For example, the tilings themselves can be distinguished into two kinds by the connecting method: edge-to-edge tilings and non-edge-to-edge tilings. In edge-to-edge tilings, the vertices and edges of polygons coincide with the vertices and edges of the tiling. In non-edge-to-edge tilings, the vertices of polygons may contact the edges of adjoining polygons, that is, there is no restriction on how adjoining polygons meet. As to the pentagonal tilings, though the edge-to-edge tilings are still pentagonal even in topological point of view, the non-edge-to-edge tilings are not. However, the distinction between these two connecting methods is seldom done in the previous studies of convex pentagonal tiling problem. Our interest lies more in edge-to-edge tiling since it is more essential (OGAWA *et al.*, 2001; SUGIMOTO and OGAWA, 2000, 2003a, 2003c, 2005). Therefore, throughout the report, we will consider only edge-to-edge tiling. Hereafter, as long as cautions are unnecessary, an edge-to-edge tiling is written simply a tiling. On the other hand, so far the classification and exhaustive studies of tilings are not very much noticed to the present. However, for convex pentagonal tiles, it will be impossible to express the necessary and sufficient conditions for identifying prototiles without classifying tilings. Therefore, we should pay attention to the properties of tilings. First, in this report, we show the properties which become important and famous in tiling with polygons (see Proposition in Subsection 2.2). Next, the properties about pentagonal tiling are shown (see Theorem 1 in Subsection 2.3, Theorem 2 in Subsection 2.4, and Theorem 4 in Section 3) and the classification of convex pentagons with equal edges is also shown (see Table 2 in Section 3).

2. Properties of Nodes in Strongly Balanced Tilings

2.1. Definition

A *node* of valence κ in a tiling is a point that is the common vertex of κ polygons (tiles). That is, in this report, hereafter a vertex of a tiling is called a node. Note that the valence κ of a node is at least three.

Given a tiling, the *first corona* of a tile is the set of all tiles that have a common boundary point with that tile (including the original tile itself).

Two tiles are called *adjacent* if they have an edge in common, and then each is called an *adjacent* of the other. Therefore, in an edge-to-edge tiling by polygons, the number of adjacents of a polygon is equal to the number of edges of that. Then, a polygon with h edges (and therefore h vertices) will be called a h-gon.

A tiling by polygons is called *normal* if there are positive numbers r and R such that any polygon contains a certain disk of radius r and is contained in a certain disk of radius R.

Given a normal tiling $\Im$ by polygons, let W be a closed disk of radius ρ (>0) on the plane. Then, let F_1 and F_2 denote the set of the polygons contained in W and the set of polygons meeting the boundary of W but not contained in W, respectively. Here, define $F := F_1 \cup F_2$ (i.e., F is the set of polygons generated by W). We denote by $P(F)$ the number of polygons in F. In addition, let $E(F)$ and $N(F)$ denote the number of edges and nodes in F, respectively. The tiling $\Im$ is *balanced* if it is normal and satisfies the following condition: the limits

$$\lim_{\rho\to\infty}\frac{N(F)}{P(F)} \quad \text{and} \quad \lim_{\rho\to\infty}\frac{E(F)}{P(F)} \tag{1}$$

exist and are finite (GRÜNBAUM and SHEPHARD, 1987).

Now, let $P_h(F)$ and $N_\kappa(F)$ be the number of polygons with h adjacents in F and the number of κ-valent nodes in F, respectively. The tiling $\Im$ is *strongly balanced* if it is normal and satisfies the following condition: all the limits

$$\lim_{\rho\to\infty}\frac{P_h(F)}{P(F)} \quad \text{and} \quad \lim_{\rho\to\infty}\frac{N_\kappa(F)}{P(F)} \tag{2}$$

exist (GRÜNBAUM and SHEPHARD, 1987).

Note that, in the following Subsection 2.2, the strongly balanced condition is not needed.

2.2. Average valence of nodes in balanced tilings by polygons with h edges

Given a normal tiling $\Im$ of plane by polygons and let $P(F_2)$ denote the number of polygons in F_2. Since $\Im$ is normal, then (see GRÜNBAUM and SHEPHARD, 1987; BAGINA, 2004):

$$\lim_{\rho\to\infty} \frac{P(F_2)}{P(F)} = 0. \tag{3}$$

Hereafter, let $N(W)$ be the number of nodes of the tiling contained in W.

Lemma 1. *Given a normal tiling $\Im$, if the radius ρ of the disk W tends to infinity then*

$$\lim_{\rho\to\infty} \frac{N(W)}{N(F)} = 1. \tag{4}$$

Proof. First, we consider the upper bound for the number of edges in a polygon in $\Im$. Let p_0 be any polygon. We suppose that m polygons contact p_0. So, the first corona of p_0 in tiling $\Im$ is contained in some disk of radius $3R$. Since any polygon of $\Im$ contains a disk of radius r and these disks do not overlap, we have

$$(m+1)\pi r^2 < \pi(3R)^2 \quad \text{or} \quad m < 9\frac{R^2}{r^2} - 1.$$

Therefore the upper bound for the number of edges in a polygon is $9(R^2/r^2) - 1$.

Thus, for the number $N(F) - N(W)$ we have:

$$N(F) - N(W) < 9\frac{R^2}{r^2} \cdot P(F_2). \tag{5}$$

Next, we consider the upper bound of valence of nodes in $\Im$. We take a node G of valence κ, draw a circle of radius $2R$ centered at a node G. It contains fully the star of node G in the tiling. Since this star contains κ polygons, we get

$$\kappa \cdot \pi r^2 < \pi(2R)^2.$$

Hence for any valence κ then:

$$\kappa < 4\frac{R^2}{r^2}. \tag{6}$$

Now we will prove that

$$N(F) > \frac{3r^2}{4R^2} \cdot P(F). \tag{7}$$

Let u_i denote the number of vertices in the i-th polygon in F ($i = 1, \ldots, P(F)$). Since $u_i \geq 3$ for any $i \in \{1, \ldots, P(F)\}$, then

$$\sum_{i=1}^{P(F)} u_i \geq 3 \cdot P(F).$$

On the other hand, from (6) and $N(F)$ (the number of nodes of the tiling in F), at most $4(R^2/r^2) \cdot N(F)$ vertices of polygons exist in F. Therefore, for the number of vertices of polygons from F, we have

$$4\frac{R^2}{r^2} \cdot N(F) > \sum_{i=1}^{P(F)} u_i \geq 3 \cdot P(F).$$

Therefore, we obtain the inequality (7).

Given R and r, from (5) and (7), we have

$$\frac{N(F) - N(W)}{N(F)} = \left(1 - \frac{N(W)}{N(F)}\right) < \frac{P(F_2)}{P(F)} \cdot \frac{9R^2}{r^2} \cdot \frac{4R^2}{3r^2} = 12\frac{R^4}{r^4}\frac{P(F_2)}{P(F)}. \tag{8}$$

Thus, if $\rho \to \infty$, the relation (4) is derived from (3) and (8). □

Now, let $K(W)$ and $K(F)$ be the sum of valences of $N(W)$ nodes of the tiling contained in W and the sum of valences of $N(F)$ nodes of the tiling in F, respectively;

$$K(W) := \sum_{j=1}^{N(W)} \kappa_j \quad \text{and} \quad K(F) := \sum_{j=1}^{N(F)} \kappa_j$$

where κ_j is the valence of j-th nodes of the tiling in F ($j = 1, 2, \ldots, N(W), \ldots, N(F)$).

Lemma 2. *Given a normal tiling $\mathfrak{J}$, if the radius ρ of the disk W tends to infinity then*

$$\lim_{\rho \to \infty} \frac{K(W)}{K(F)} = 1. \tag{9}$$

Proof. From (6) and the fact that the valence of a node is at least three, we have

$$4\frac{R^2}{r^2} > \kappa_j \geq 3.$$

Therefore:

$$4\frac{R^2}{r^2}\cdot N(F) > K(F) \geq 3\cdot N(F)$$

and

$$4\frac{R^2}{r^2}\big(N(F)-N(W)\big) > \big(K(F)-K(W)\big) \geq 3\cdot\big(N(F)-N(W)\big).$$

Hence, we have

$$\frac{4\frac{R^2}{r^2}\big(N(F)-N(W)\big)}{3\cdot N(F)} > \frac{K(F)-K(W)}{K(F)} > \frac{3\cdot\big(N(F)-N(W)\big)}{4\frac{R^2}{r^2}\cdot N(F)}$$

or

$$\frac{4R^2}{3r^2}\left(1-\frac{N(W)}{N(F)}\right) > \left(1-\frac{K(W)}{K(F)}\right) > \frac{3r^2}{4R^2}\left(1-\frac{N(W)}{N(F)}\right). \tag{10}$$

Given R and r, if $\rho \to \infty$, the relation (9) follows from (10) and Lemma 1. □

From the statement 3.3.5 ("*a normal tiling in which every tile has the same number of adjacents is balanced*") in GRÜNBAUM and SHEPHARD (1987) p. 133, a normal tiling by h-gons ($h \geq 3$) is balanced. (Note that, in this report, we consider the edge-to-edge tiling.) Hereafter, in this Subsection, we will consider a balanced tiling $\Im_h$ of plane by h-gons. Then, we derive the limits $\lim_{\rho\to\infty}\big(E(F)/P(F)\big) = h/2$, $\lim_{\rho\to\infty}\big(N(F)/P(F)\big) = (h/2) - 1$, and $\lim_{\rho\to\infty}\big(P_h(F)/P(F)\big) = 1$ (i.e., the values of limits in (1) and in one of (2) are able to be obtained). The tiling $\Im_h$ satisfies Lemmas 1 and 2.

Here, denote $\bar{\kappa} := \lim_{\rho\to\infty}\big(K(F)/N(F)\big)$, and it is called the average valence of nodes in $\Im_h$.

Proposition. *Given a balanced tiling $\Im_h$ by h-gons ($h \geq 3$), the $\bar{\kappa}$ is found as follows:*

$$\bar{\kappa} = \frac{2h}{h-2}. \tag{11}$$

Proof. Fix a disk W with large radius ρ in $\Im_h$. Then the sum of angles of $P(F)$ h-gons in a set F of h-gons generated by W and the sum of angles of $P(F) - P(F_2)$ h-gons, of the tiling contained in W, are $(h-2)\pi \cdot P(F)$ and $(h-2)\pi \cdot (P(F) - P(F_2))$, respectively. Since the total sum of angles meeting at the $N(W)$ nodes, of the tiling contained in W, is $2\pi \cdot N(W)$, we have

$$(h-2)\pi \cdot P(F) > 2\pi \cdot N(W) > (h-2)\pi \cdot \left(P(F) - P(F_2)\right)$$

or

$$1 > \frac{2 \cdot N(W)}{(h-2) \cdot P(F)} > \left(1 - \frac{P(F_2)}{P(F)}\right).$$

Therefore, due to (3), as $\rho \to \infty$,

$$\frac{2 \cdot N(W)}{(h-2) \cdot P(F)} \to 1. \tag{12}$$

On the other hand, the total number of edges for $P(F)$ h-gons in F and the total number of edges for $P(F) - P(F_2)$ h-gons, of the tiling contained in W, are $h \cdot P(F)$ and $h \cdot (P(F) - P(F_2))$, respectively. Let $\bar{\kappa}_W$ denote the average valence of nodes contained in W; $\bar{\kappa}_W := K(W)/N(W)$. Then we have

$$h \cdot P(F) \geq \bar{\kappa}_W \cdot N(W) \geq h \cdot \left(P(F) - P(F_2)\right)$$

or

$$1 \geq \frac{\bar{\kappa}_W \cdot N(W)}{h \cdot P(F)} \geq \left(1 - \frac{P(F_2)}{P(F)}\right).$$

Therefore, as $\rho \to \infty$, from (3), we have

$$\frac{\bar{\kappa}_W \cdot N(W)}{h \cdot P(F)} \to 1. \tag{13}$$

From Lemmas 1 and 2 the limits

$$\bar{\kappa} = \lim_{\rho \to \infty} \frac{K(F)}{N(F)} = \lim_{\rho \to \infty} \frac{K(W)}{N(W)} = \lim_{\rho \to \infty} \frac{\bar{\kappa}_W \cdot N(W)}{N(W)} = \lim_{\rho \to \infty} \bar{\kappa}_W \tag{14}$$

exists and is finite.

Table 1. Average valence $\overline{\kappa}$ of nodes in balanced tiling by h-gons.

h-gon	$\overline{\kappa}$ (average valence of nodes)
Triangles (h = 3)	6
Quadrilaterals (h = 4)	4
Pentagons (h = 5)	10/3
Hexagons (h = 6)	3

Thus, the relation (11) follows from (12), (13), and (14). □

From Proposition we have the average valences for the cases of triangles (h = 3), of quadrilaterals (h = 4), of pentagons (h = 5), and of hexagons (h = 6) as in Table 1. These results for h = 3, 4, and 6 are well known. (Note that, in order to prove other theorems in this report, we proved the Proposition exactly according to the definition of this research.) For $h = 5$, the average valence is $10/3 \approx 3.33 \cdots$. Since the average valence is not an integer then there must be nodes with valences smaller than $3.33 \cdots$. But the smallest valence at node is three. So in any normal tiling of plane by pentagon there must be nodes with valence 3. In addition, for the same reason (the average number is not an integer), there are no tilings with all nodes of the same valence. If Eq. (11) is applied to the convex polygons with seven or more edges (i.e., $h \geq 7$), their tilings are impossible, since the average valence of nodes is smaller than 3 when $h \geq 7$. Therefore, in the case of convex polygons with seven or more edges, we can understand that no balanced tiling exists.

2.3. Rates of 3- and k-valent nodes in strongly balanced tilings by pentagons

The following arguments are relevant to strongly balanced tilings by pentagons. Then, every strongly balanced tiling is necessarily balanced.

Let $\Im_5$ be a strongly balanced tiling of plane by pentagons. Hereafter, we denote N_s ($s \geq 3$) as the number of s-valent nodes in F of pentagons generated by a closed disk W on a pentagonal tiling $\Im_5$.

Lemma 3. *Given a pentagonal tiling $\Im_5$, if $\Im_5$ is formed of only 3- and k-valent nodes, then*

$$\lim_{\rho\to\infty} \frac{N_3}{N_k} = 3k - 10 \tag{15}$$

where $k \geq 4$.

Proof. When the radius ρ of the disk W tends to infinity (i.e., $\rho \to \infty$), the limit $\lim_{\rho\to\infty}\left(N_3/N_k\right)$ exists since $\Im_5$ is strongly balanced.

The total number of nodes in F is $N_3 + N_k$ and the sum of valences of nodes in F is $3N_3 + k \cdot N_k$. Therefore, from Proposition, we have

$$\lim_{\rho\to\infty} \frac{3N_3 + k \cdot N_k}{N_3 + N_k} = \lim_{\rho\to\infty} \frac{3\dfrac{N_3}{N_k} + k}{\dfrac{N_3}{N_k} + 1} = \frac{10}{3}. \tag{16}$$

Hence, from (16), we obtain (15). □

Here, we consider a closed disk W_1 of finite radius ρ_1 ($R << \rho_1 < \infty$) on the strongly balanced pentagonal tiling $\Im_5$. In addition, we assume that W_1 satisfies the property that the average valence of nodes in W_1 is nearly equal to 10/3. Then, let T denote the finite set of pentagons generated by W_1. Therefore, the each number of pentagons and nodes in T is finite. Note that the set T needs to be finite since we discuss about the number of nodes in a tiling such as Theorem 1. Hereafter, we denote V_s ($s \geq 3$) as the number of s-valent nodes in T.

Theorem 1. *If the tiling in T is formed of only 3- and k-valent nodes, then*

$$V_3 : V_k \approx 3k - 10 : 1 \tag{17}$$

where $k \geq 4$.

Proof. It is clear from Lemma 3. □

Corollary 1. *If the tiling in T is formed of only 3- and 4-valent nodes, then*

$$V_3 : V_4 \approx 2 : 1. \tag{18}$$

Corollary 2. *If the tiling in T is formed of only 3- and 6-valent nodes, then*

$$V_3 : V_6 \approx 8 : 1. \tag{19}$$

Proof. When $k = 4$ and 6 in (17) of Theorem 1, (18) and (19) are obtained, respectively. □

Now, Corollaries 1 and 2 can be applied to tilings with congruent convex pentagons. If a tiling is edge-to-edge, it satisfies Theorem 1. As a result, for convex pentagonal tiles in Figs. 1 and 2, we find that tilings of type 4, 6, 7, 8, and 9 satisfy (18), and tiling of type 5 satisfies (19) (SUGIMOTO, 1999; SUGIMOTO and OGAWA, 2003c, 2004). Note that, although the tilings of type 1 and 2 in the list of 14 types are generally non-edge-to-edge (see Fig. 1), tilings by convex pentagonal tiles which belong to type 1 and 2 can be edge-to-edge in special cases. For example, a convex pentagonal tile with "$A + B + C = 360°$, $a = d$" belongs to type 1 according to the present classification scheme has an edge-to-edge tiling (see Fig. 3(a)). Similarly, a convex pentagonal tile satisfying the condition "$A + B + D = 360°$, $a = d$, $c = e$" is of type 2, and has an edge-to-edge tiling (see Fig. 3(b)). Then, the tilings by these two pentagons satisfy (18) together. Like these examples, when the tilings by congruent convex pentagonal tiles which belong to type 1 or 2 are edge-to-edge, in the range which we know, the tilings satisfy (18).

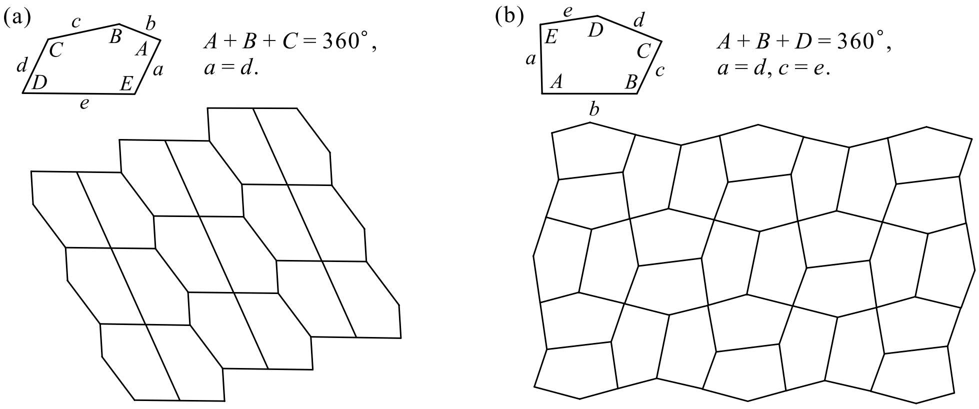

Fig. 3. Tilings by congruent convex pentagons. (a) Convex pentagonal tiles belong to type 1. (b) Convex pentagonal tiles belong to type 2.

Among 14 convex pentagonal tilings in Figs. 1 and 2, the tiling which satisfies (17) is not found in the case of $k = 5$ or $k \geq 7$.

2.4. Remarks

For the tilings (especially periodic tilings) by congruent convex pentagons, the properties $\bar{\kappa} = 10/3$ and (17) should be realized not only globally but also locally. That is, the properties should be included in the fundamental regions of their periodic tilings.

Here, let us investigate the properties of tilings by congruent convex pentagons from the point of our analysis in previous subsections. First, the tilings of type 4, 6, 7, 8, and 9 (see Fig. 1) satisfy (18). They have three kinds of nodes. Especially, the tilings of type 6, 7, 8, and 9 are formed of two kinds of 3-valent nodes and one kind of 4-valent node (SUGIMOTO, 1999; SUGIMOTO and OGAWA, 2003a, 2003b, 2004). For example, the tiling of type 6 in Fig. 1 is formed of three kinds of nodes satisfying $A + B + D = 360°$, $2E + A = 360°$, and $2C + B + D = 360°$. In the **three** (= 2 + 1) kinds of nodes, each of the five vertices A, B, C, D, and E of pentagon appears twice; i.e., the total number of vertices is **10** ($= 5 \times 2 = 3 \times 2 + 4 \times 1$). As a result, we see the relations **10/3** and (**two** kinds of 3-valent nodes) : (**one** kind of 4-valent node) = **2** : **1**. In addition, for the pentagonal tilings that have two-kinds of 3-valent nodes and one-kind of 4-valent node, the even number of pentagonal tiles are necessary in order to form the fundamental region since each of the five vertices of pentagon is used twice in the three kinds of nodes. On the other hand, the tiling of type 4 in Fig. 1 is formed of three kinds of nodes satisfying $A + B + D = 360°$, $4C = 360°$, and $4E = 360°$. That is, there are one-kind of 3-valent nodes and two-kind of 4-valent nodes in the tiling. When there are four vertices C and four vertices E (i.e., $4C = 360°$ and $4E = 360°$) in the tiling, the number of vertices A, B, and D also need to be four, respectively. Therefore, **one** "$4C = 360°$" and **one** "$4E = 360°$" have to correspond to **four** "$A + B + D = 360°$". Hence, (total four 3-valent nodes) : (two kinds of 4-valent nodes) = **4** : (**1** + **1**) = 2 : 1, and **six** (= 4 + 1 + 1) nodes are constructed at **20** ($= 5 \times 4 = 3 \times 4 + 4 \times$

$1 + 4 \times 1$) vertices of pentagons; namely **20/6** = 10/3. Thus, for the tiling of types 4 in Fig. 1, the fundamental region is formed of the four pentagonal tiles. Next, the tiling of type 5 satisfies (19) and is formed of three kinds of nodes satisfying $B + D + E = 360°$, $3A = 360°$, and $6C = 360°$ (see Fig. 1). In order to form the fundamental region of the tiling of type 5, six pentagonal tiles are necessary. Therefore, in the combinations by using the node conditions $B + D + E = 360°$, $3A = 360°$, and $6C = 360°$, each of the five vertices A, B, C, D, and E of pentagon has to appear six times. Thus, **six** "$B + D + E = 360°$" and **two** "$3A = 360°$" have to correspond to **one** "$6C = 360°$"; i.e., (total eight 3-valent nodes) : (one kind of 6-valent node) = (**6** + **2**) : **1** = 8 : 1. Furthermore, since **nine** (= 6 + 2 + 1) nodes are constructed at **30** ($= 5 \times 6 = 3 \times 6 + 3 \times 2 + 6 \times 1$) vertices of pentagons, we see the relation **30/9** = 10/3.

Now, based on the properties that are locally realized, we consider the cases that the tiling in T is formed of the nodes of three or more kinds of valence. Note that the 3-valent nodes in T exist necessarily.

If the tiling in T is formed of only 3, 4, and 5-valent nodes, then

$$V_3 : V_4 : V_5 \approx (2x + 5y) : x : y, \tag{20}$$

where $x = 1, 2, 3, \ldots$ and $y = 1, 2, 3, \ldots$. We will prove this relation later. The total number of pentagonal vertices on nodes in T is $(3V_3 + 4V_4 + 5V_5)$, and the total number of nodes in T is $(V_3 + V_4 + V_5)$. Since T is the finite set of pentagons generated by W_1 and the average valence of node in W_1 is nearly equal to 10/3, $(3V_3 + 4V_4 + 5V_5)/(V_3 + V_4 + V_5) \approx 10/3$. Therefore, V_3 is nearly equal to $2x + 5y$ for $V_4 = x$ and $V_5 = y$. Here, we consider the concrete character of tiling that satisfies (20). For example, if it is possible that the pentagonal tiling satisfies $V_3 : V_4 : V_5 \approx 12 : 1 : 2$, the pentagons of number of the multiple of 10 are necessary in order to form the fundamental region. It is because that, when each of the five vertices A, B, C, D, and E of pentagon appear ten times, **15** (= 12 + 1 + 2) nodes are constructed at **50** ($= 5 \times 10 = 3 \times 12 + 4 \times 1 + 5 \times 2$) vertices of pentagons (i.e., **50/15** = 10/3).

From similar consideration, we can obtain the following relations.

If the tiling in T is formed of only 3-, 4-, and 6-valent nodes, then

$$V_3 : V_4 : V_6 \approx (2x + 8y) : x : y, \tag{21}$$

where $x = 1, 2, 3, \ldots$ and $y = 1, 2, 3, \ldots$.

If the tiling in T is formed of only 3-, 5-, and 6-valent nodes, then

$$V_3 : V_5 : V_6 \approx (5x + 8y) : x : y, \tag{22}$$

where $x = 1, 2, 3, \ldots$ and $y = 1, 2, 3, \ldots$.

If the tiling in T is formed of only 3-, 4-, 5-, and 6-valent nodes, then

$$V_3 : V_4 : V_5 : V_6 \approx (2x + 5y + 8z) : x : y : z, \tag{23}$$

where $x = 1, 2, 3, \ldots$, $y = 1, 2, 3, \ldots$, and $z = 1, 2, 3, \ldots$.

From the above consideration, we find the following Theorem 2.

Theorem 2. *If the valence number of all nodes in T is finite, then*

$$V_3 \approx \sum_{k \geq 4} (3k - 10) \cdot V_k. \tag{24}$$

Proof. The total number of pentagonal vertices on nodes in T is $3V_3 + \sum_{k \geq 4} k \cdot V_k$, and the total number of nodes in T is $V_3 + \sum_{k \geq 4} V_k$. Since the average valence of node in T is nearly equal to 10/3,

$$\frac{3V_3 + \sum_{k \geq 4} k \cdot V_k}{V_3 + \sum_{k \geq 4} V_k} \approx \frac{10}{3}. \tag{25}$$

Hence, from (25), we obtain (24). □

Besides the relations (20)–(23), it is possible to search for similar relations. However, in the pentagonal tilings of 14 types in Figs. 1 and 2, we see that there are only tilings which satisfy (18) or (19).

3. Classification of Convex Pentagons with Equal Edges and Properties of Their Tiling

Pentagons can be categorized by the number of equal-length edges and their positions, from figures with five unequal edges to those with five equal edges. Here, the edge-lengths are designated symbolically in anticlockwise order, with identical symbols for edges of identical lengths, and descriptions of congruent shapes with different starting points or mirror-reflections are excluded. Beginning with equilateral pentagons, followed by those with four equal-length edges, etc., there are a total of 12 unique combinations (see Table 2) (SUGIMOTO, 1999; SUGIMOTO and OGAWA, 2000, 2003a, 2003c, 2005). For example, the combination [**11111**] in Table 2 is the pentagon with all the identical edge-lengths; i.e., it is equilateral pentagon. On other hand, the combinations [**11122**] and [**11212**] in Table 2 are the pentagons that have the edge-lengths of two kinds, but the arrangements of five edge-lengths are different.

Here, we summarize the study of tilings by congruent convex pentagons with combination [**11111**] (i.e., equilateral convex pentagons) (HIRSCHHORN and HUNT, 1985; SCHATTSCHNEIDER, 1987; BAGINA, 2004). Hirschhorn and Hunt gave the following theorem.

Theorem 3 (HIRSCHHORN and HUNT, 1985). *An equilateral convex pentagon tiles the plane if and only if it has two angles adding to 180°, or it is the unique equilateral convex pentagon with angles A, B, C, D, E satisfying 2B + C = 2D + A = 2E + A + C = 360° (A ≈ 89.26°, B ≈ 144.56°, C ≈ 70.88°, D ≈ 135.37°, E ≈ 99.93°).*

Table 2. Twelve combination of edges of pentagon.

Edges	Combination	Example
Equilateral	[**11111**]	$a = b = c = d = e$
Two kinds	[**11112**]	$a = b = c = d \neq e$
	[**11122**]	$a = b = c \neq d = e$
	[**11212**]	$a = b = d \neq c = e$
Three kinds	[**11123**]	$a = b = c, d \neq e, d \neq a \neq e$
	[**11213**]	$a = b = d \neq c \neq e, a \neq e$
	[**11223**]	$e \neq a = b \neq c = d \neq e$
	[**11232**]	$d \neq a = b \neq c = e \neq d$
	[**12123**]	$e \neq a = c \neq b = d \neq e$
Four kinds	[**11234**]	$a = b \neq c \neq d \neq e, c \neq e, a \neq d, a \neq e$
	[**12134**]	$a = c, b \neq d \neq e, b \neq e, a \neq b, a \neq d, a \neq e$
Five kinds	[**12345**]	$a \neq b, a \neq c, a \neq d, a \neq e, b \neq c, b \neq d, b \neq e, c \neq d, c \neq e, d \neq e$

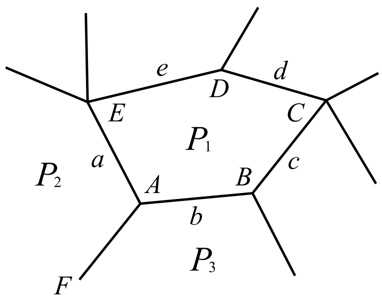

Fig. 4. Pentagons P_1, P_2, and P_3.

Therefore, the tiles of tilings by congruent convex pentagons with combination [**11111**] belong to type 1, 2, or 7.

On the other hand, we give the following Theorem 4.

Theorem 4. *If the tiles in tiling are congruent convex pentagons, then at least two of the edges (of this congruent convex pentagon) are of equal length.*

Proof. Given a tiling $\mathfrak{I}_c$ by convex pentagons, let P_1 be the pentagon of edges $EA = a$, $AB = b$, $BC = c$, $CD = d$, and $DE = e$ (see Fig. 4). Since the nodes of valence 3 are surely necessary in the pentagonal tiling, we suppose that the vertex A of P_1 exists on a 3-valent node. Then, let P_2 and P_3 be the pentagons which share edges EA and AB, respectively. In addition, we denote AF as the common edge of P_2 and P_3.

If all pentagons in $\mathfrak{I}_c$ are congruent, the edge AF in P_2 is equal to b or e, since P_1 and

P_2 are adjacent with their common edge EA. Similarly, if all pentagons in $\Im_c$ are congruent, the edge AF in P_2 is equal to a or c, since P_1 and P_2 have their common edge AB. Therefore, any 3-valent node of tiling by congruent convex pentagons with combination [**12345**] can not exist. Thus, the tiling by congruent convex pentagons with combination [**12345**] is impossible.

On the other hand, the tiling by congruent convex pentagon with combination [**12134**] exists. For example, see the tiling by congruent convex pentagon with "$A + B + C = 360°$, $a = d$" in Fig. 3(a). □

4. Conclusion

In the convex pentagonal tiling problem, it is important to consider properties of tilings and tiles both. First, when a tiling is edge-to-edge and strongly balanced, we observe the number and kinds of nodes in tiling and gave Lemma 3 and Theorem 1 (see Subsection 2.3). Next, for tiles, we find that the pentagons can be classified into 12 kinds by the number of equal-length edges and their positions (see Table 2) (SUGIMOTO, 1999; SUGIMOTO and OGAWA, 2000, 2003a, 2003c, 2005). Then, we see that the tiling by congruent convex pentagon is impossible when all edges of convex pentagon are of different length (see Theorem 4) (SUGIMOTO and OGAWA, 2003c). On the other hand, it is known that the equilateral convex pentagons which can tile the plane have to belong to type 1, 2, or 7 in the present list. Among 12 cases of pentagons of Table 2, the two cases were solved. But, the investigations about other 10 cases have not been completely finished yet.

The properties of nodes in pentagonal tiling have not been hardly noticed or formally discussed until now. That is, before this report, there were no previous reports describing the concrete nodes' properties (Lemma 3, Theorem 1 and 2, and Corollary 1 and 2). In addition, the relations between the combinations of pentagonal edges and their pentagonal tilings had not been discussed in as much detail. Theorem 4 also had not been shown formally until now. Hence, for the first time, we have shown and proven their properties explicitly.

The convex pentagonal tilings problem has yet to be fully approached scientifically. The solution to this problem requires a systematic approach. Thus, although our results may be elementary, we assert that the properties in this report are important in order to consider the convex pentagonal tiling problem.

We actually tackled the convex pentagonal tiling problem from the properties achieved in this report. Specifically, for the tilings by pentagons with combination [**11112**] which are formed of two kinds of 3-valent nodes (including the case when the two kinds are identical) and one kind of 4-valent node (i.e., the tilings satisfy (18)), we investigated in SUGIMOTO and OGAWA (2003a, 2003b, 2004, 2005). The results of SUGIMOTO and OGAWA (2003a, 2003b, 2004, 2005) are summarized as follows. According to the present classification (see Figs. 1 and 2), the convex pentagonal tiles which are directly yielded by our investigation belong to one or more of type 1, 2, 4, 7, 8, or 9. Therefore, as mentioned in the introduction, a new prototile was not found in SUGIMOTO and OGAWA (2003a, 2003b, 2004, 2005). However, our investigation on the basis of the properties shown in this report should be applicable to create a perfect list of convex pentagonal tiles. Incidentally, we found some new convex pentagonal tilings in SUGIMOTO and OGAWA (2004).

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