Letter *Forma*, **21**, 151–158, 2006

Characterization of Attractors in Gumowski-Mira Map Using Fast Lyapunov Indicators

L. M. SAHA[1]*, M. K. DAS[2] and Mridula BUDHRAJA[3]

[1]*Department of Mathematics, Zakir Husain College, University of Delhi, Delhi-110002, India*
[2]*Institute of Informatics and Communication, University of Delhi South Campus, NewDelhi-110021, India*
[3]*Department of Mathematics, Shivaji College, University of Delhi, New Delhi-110027, India*
**E-mail address: lmsaha@sify.com, mrinal@iic.ac.in, nkbudhraja@yahoo.com*

(Received May 18, 2006; Accepted June 30, 2006)

Keywords: Fast Lyapunov Indicator, Gumowski-Mira Map

Abstract. In this Letter we have studied the behavior of the Fast-Lyapunov Indicators for various attractors of the Gumowski-Mira map. It was observed that the Fast-Lyapunov indicators show a typical type of behavior for different kinds of attractors, viz. chaotic, periodic, and bounded attractors. We have shown the possibility of a *transient chaos* in GM-map. The importance of FLI in dynamical system or maps is suggested.

1. Introduction

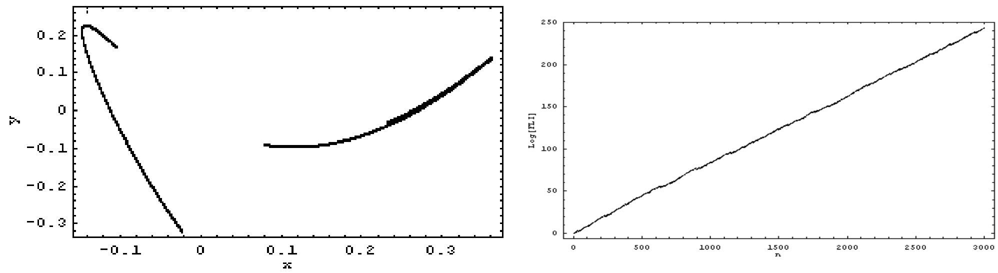
Analyzing chaos has been an important area of research in the field of chaotic dynamics. Scientists have always tried to find various ways which may help us in diagonising chaotic and regular motion of various discrete and continuous dynamical systems. The various methods that have been largely used to characterize chaos are: Poincare section of surface, the Lyapunov exponents, Frequency-Map Analysis, Lyapunov-Characteristic Exponents and recently Fast-Lyapunov Indicators.

Froeschele *et al*. introduced Fast-Lyapunov Indicators (FLIs) in 1997 and it has been found that FLIs can be used to detect chaotic/regular orbits very efficiently. In this Letter, we apply the FLIs to study the various attractors of the Gumowski-Mira map and showed the possibility of observing *transient chaos* over a range of parameters.

2. Definition of Fast-Lyapunov Indicators

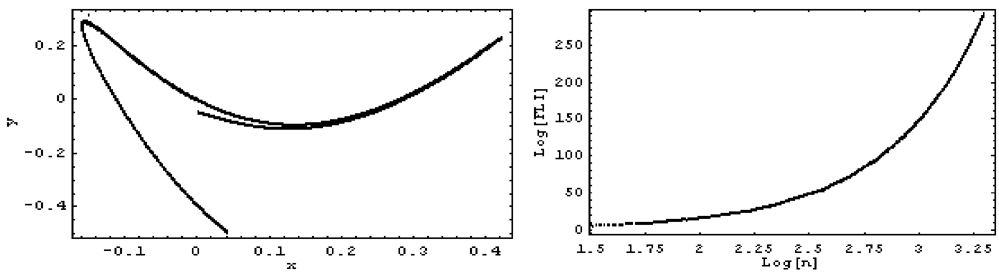
According to Froeschele (1997, 2001), the "Fast-Lyapunov Indicators (FLIs)" are defined as follows:

Starting with an m-dimensional basis $V_m(0) = (v_1(0), v_2(0), ..., v_m(0))$, embedded in an n-dimensional space and with an initial-condition $(x_1(0), x_2(0), ..., x_n(0))$, we take at each iteration the largest among the vectors of the evolving basis.



(a) **(b)**

Fig. 1. (a) The GM-attractor for $a = -1.1$, $b = -0.2$, $\mu = -1.845$. (b) The behavior of the FLIs indicating chaos.



(a) **(b)**

Fig. 2. (a) The GM-attractor for $a = -1.1$, $b = -0.2$, $\mu = -1.95924896$. (b) The behavior of the FLIs indicating chaos.

Thus, the FLI is defined as:

$$\text{FLI} = \sup, \|v_j\|, \; j = 1, 2, ..., m.$$

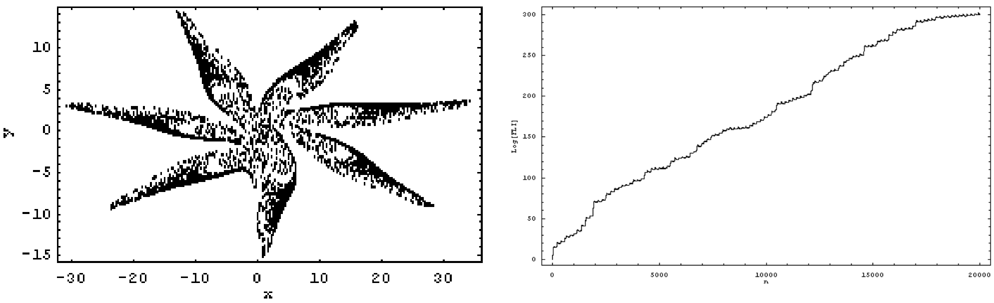
Froeschele has shown that FLIs increase exponentially for chaotic orbits and linearly for regular orbits.

We have applied this definition of FLI to study the various attractors of the Gumowski-Mira map (hereafter referred to as GM-map).The GM-map is defined as:

$$x_{n+1} = y_n + a\left(1 - by_n^2\right)y_n + G\left(x_n\right),$$

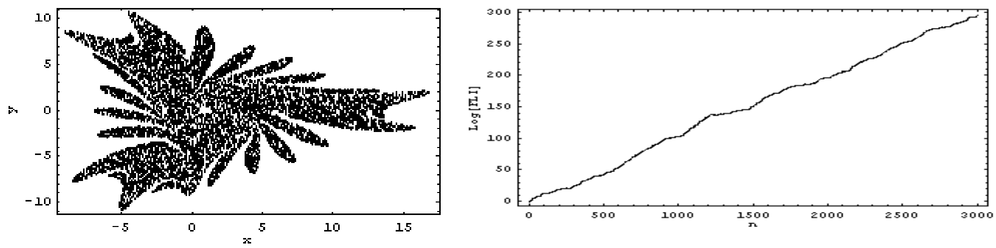
$$y_{n+1} = -x_n + G\left(x_{n+1}\right),$$

where $G(x) = \mu x + [2(1 - \mu)x^2/(1 + x^2)]$; a, b and μ are parameters.



(a) (b)

Fig. 3. (a) The GM-attractor for a = 0.008, b = 0.05, μ = –0.9. (b) The behavior of the FLIs indicating "slow chaos" when compared to earlier cases.



(a) (b)

Fig. 4. (a) The GM-attractor for a = 0.008, b = 0.05, μ = –0.6. (b) The plot of Log(FLI) versus iterations n.

The various patterns in GM-map have been simulated by Otsuba *et al*. (2000) and Ali (2005).

3. Application to the Gumowski-Mira map

The GM-map admits a wide variety of attractors and we get different attractors for very small changes in the parameter μ. In this letter, we intend to characterize various attractors of GM-map, based on the behavior of the FLIs, as: chaotic, periodic, bounded and quasi-chaotic.

3.1. Chaotic attractors

For $a = -1.1$, $b = -0.2$, $\mu = -1.845$ and with initial conditions $x = 0.1$, $y = 0.1$, we get the attractor shown in Fig. 1(a). The behavior of the FLIs is shown in Fig. 1(b). The exponential increase in FLIs indicates that the attractor is a chaotic attractor. It may be noted that the ordinate, Log{FLI}, in Fig. 1(b) is taken with base 10.

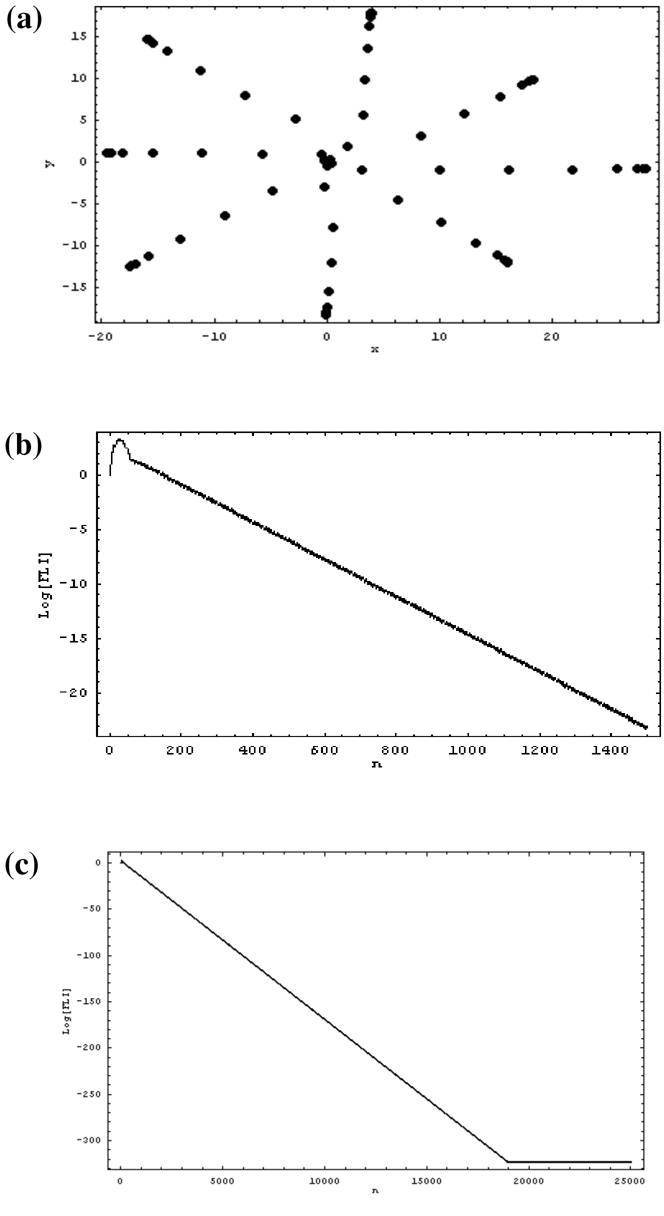


Fig. 5. (a) Periodic orbit for $a = 0.008$, $b = 0.05$, $\mu = -0.71$. (b) The FLIs start decreasing after an initial increase to about 10^5. (c) The FLIs approach zero for periodic motion.

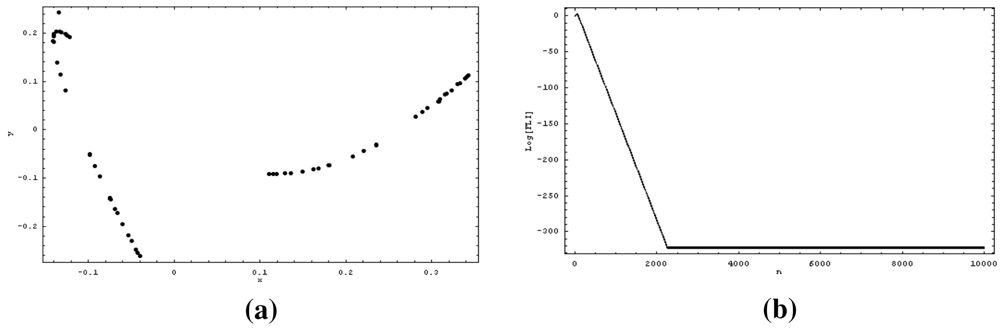


Fig. 6. (a) A periodic attractor for $a = -1.1$. (b) The FLIs approach zero very fastly. $b = -0.2$, $\mu = -1.8$.

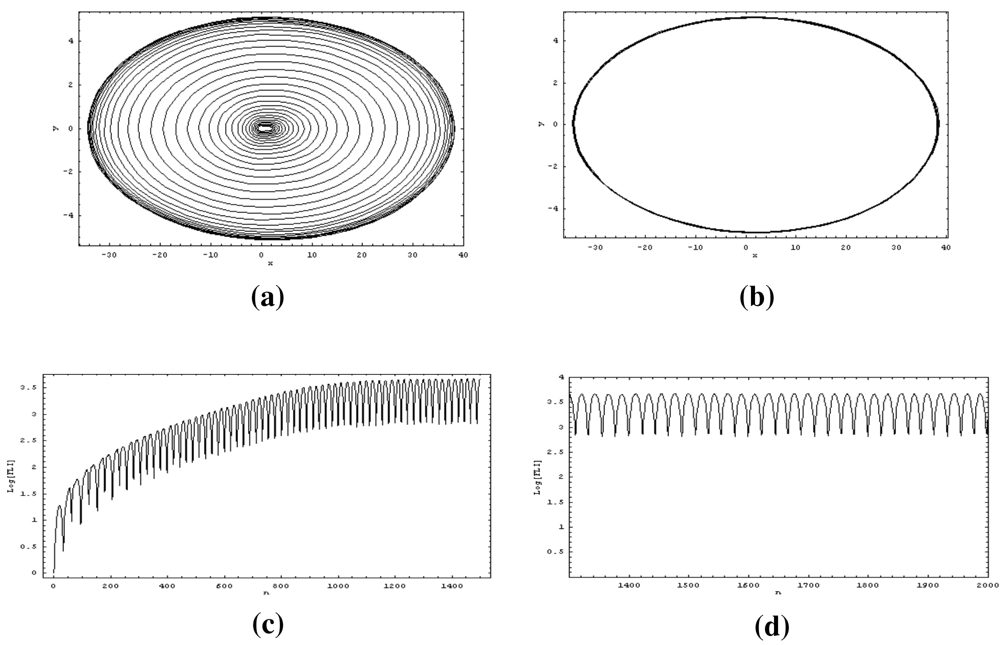
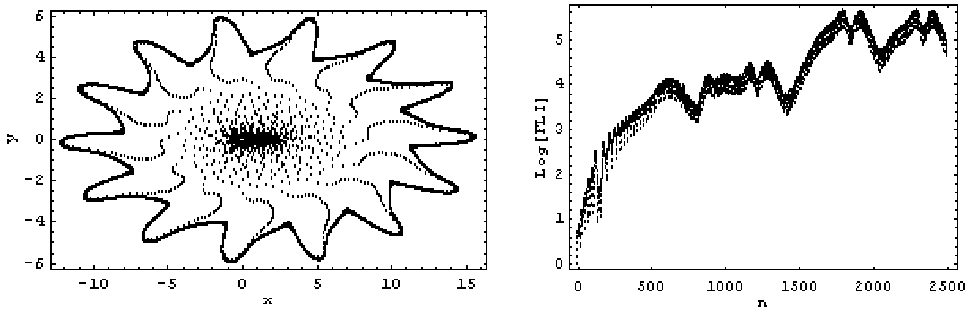


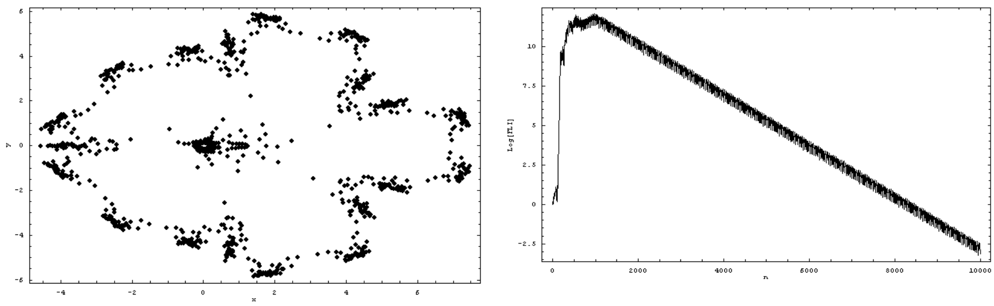
Fig. 7. (a) The evolution of GM-map as a spiral for $a = 0.008$, $b = 0.05$ and $\mu = 0.99$. (b) The boundary of its final settlement. (c) The evoluton of FLIs corresponding to (a). (d) The evoluton of FLIs corresponding to (b).

Similarly, for $a = -1.1$, $b = -0.2$, $\mu = -1.95924896$ and with initial conditions $x = 0.1$, $y = 0.1$, we get the attractor shown in Fig. 2(a) and the corresponding behavior of the FLIs is shown in Fig. 2(b), again indicating chaos. Observe that the FLIs increases exponentially to 10^{300} in about 3500 iterations.



(a) **(b)**

Fig. 8. (a) The GM-attractor for a = 0.008, b = 0.05 and μ = 0.9. (b) The evoluton of FLIs.



(a) **(b)**

Fig. 9. (a) The GM-attractor for a = 0.008, b = 0.05 and μ = 0.2. (b) The evoluton of FLIs.

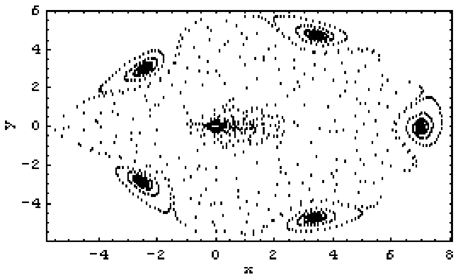
For a = 0.008, b = 0.05, μ = –0.9, and initial conditions x = 0.1, y = 0.1, the attractor is shown in Fig. 3(a) and the corresponding plot of the FLIs is shown in Fig. 3(b). Here again the motion is chaotic but this is "slow-chaos" when compared to chaos shown in the above figures.

For a = 0.008, b = 0.05, μ = –0.6, x = 0.1, y = 0.1, the attractor is shown in Fig. 4(a) and the plot of Log(FLI) versus iterations n is shown in Fig. 4(b).

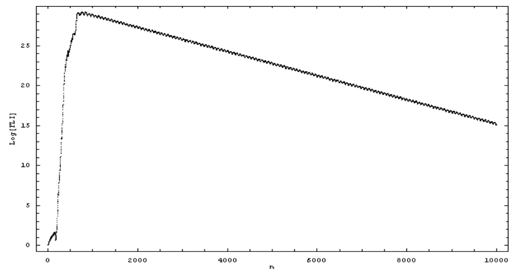
3.2. Periodic Attractors

For a = 0.008, b = 0.05, μ = –0.71, and initial conditions x = 0.1, y = 0.1, we get the phase-plot as shown in Fig. 5(a). The behavior of the FLIs is shown in Fig. 5(b) and Fig. 5(c).

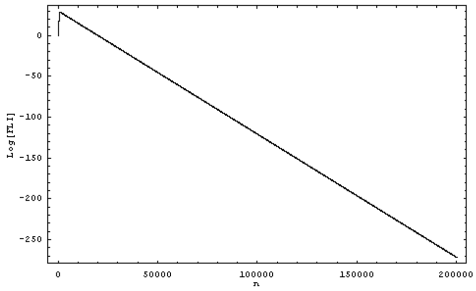
Similar plot of FLIs is observed for the periodic orbit shown in Fig. 6(a) with initial conditions $x = y = 0.1$ and parameter values a = –1.1, b = –0.2, μ = –1.8. The run of FLI with n is shown in Fig. 6(b).



(a)



(b)



(c)

Fig. 10. (a) The phase plot of GM-attractor for a = 0.008, b = 0.05 and μ = 0.2909. (b) The evoluton of FLIs. (c) Long term evolution of FLIs.

3.3. Bounded attractors

For $a = 0.008$, $b = 0.05$, $\mu = 0.99$ and initial conditions $x = y = 0.1$, the attractors evolves as a spiral as shown in Fig. 7(a) and finally settles on the boundary shown in Fig. 7(b).

The evolution of the FLIs corresponding to Figs. 7(a) and 7(b) are shown respectively in Fig. 7(c) and Fig. 7(d).

Similarly, for the attractor shown in Fig. 8(a), with a = 0.008, b = 0.05, μ = 0.9, and initial conditions $x = y = 0.1$, the FLIs are shown in Fig. 8(b). It may be noted that as in the above case, the FLIs first increase and then oscillate between fixed values.

3.4. Transition from chaos to regular attractors

In Fig. 9(a), we have shown an attractor with parameter values a = 0.008, b = 0.05, μ = 0.2, and with initial conditions x = 0.1, y = 0.1. The behavior of the FLIs is shown in Fig. 9(b). It is observed that after an initial increase to about in first 500 iterations, the FLIs start decreasing. The increase in the FLIs first indicates chaos, but later on the FLIs indicate periodic motion. We therefore find the initial evolution to chaotic domain is of transient nature. In the present case the long term behavior of the system in phase plane is regular/

periodic as indicated by the decrease of Log{FLI}. In GM-map, we therefore find that after an initial transient chaotic behavior, the orbit may settle down onto a periodic attractor depending on the values of the parameter a, b and μ.

Similar case is found for the phase-plot shown in Fig. 10(a) with parameter values $a = 0.008$, $b = 0.05$, $\mu = 0.2909$ and with initial conditions $x = 0.1$, $y = 0.1$. The FLIs are shown in Fig. 10(b) and Fig. 10(c).

4. Conclusion

We note that the FLIs are quiet efficient in analyzing various types of motions or orbits in GM-map. It is observed that sufficiently large number of iterations should be carried out before we label an attractor as being of certain type, as the initial 1000–2000 iterations may misguide us. The computation of FLI enable us to study the motion over a large time scale as the computation of FLI is not very time consuming and convergence/divergence of FLI is much faster than the Lyapunov characteristic index (LCI). The fact that a chaotic system may ultimately evolves to a periodic system can be studied using FLI suggests the study of other dynamical systems or maps using FLIs.

REFERENCES

ALI, M. (2005) Chaos, predictability and controllability in nonlinear systems, Ph.D. Thesis, submitted to Delhi University, 91–112.

FROESCHELE, C., GONCZI, R. and LEGA, E. (1997) The fast Lyapunov indicator: A simple tool to detect weak chaos, application to the structure of the main asteroidal belt, *Planet. Space Sci.*, **45**, 881–886.

LEGA, E. and FROESCHLE, C. (2001) On the relationship between fast Lyapunov indicator and periodic orbits for symplectic mappings, *Celestial Mechanics and Dynamical Astronomy*, **81**, 129–147.

OTSUBO, K., WASHIDA, M., ITOH, T., KATSUURA, K. and HAYASHI, M. (2000) Computer simulation on the Gumowski-Mira transformation, *Forma*, **15**, 121–126.