

Counting the Patterns on Some Symmetric Polyhedrons— Rotations in a Question of Japan Mathematics Contest

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(Received October 11, 2006; Accepted November 17, 2006)

1. Introduction

As an ordinary mathematician working in the university, I usually make research in mathematics. My speciality is complex analysis which is nowadays considered to be a field in traditional mathematics. For me, complex analysis is traditional not only because it is a continuation of the works of great mathematicians like Cauchy¹⁾, Riemann²⁾, Oka³⁾ and Kodaira⁴⁾, but also because it has a pure flavor comparable with greek mathematics and some articles of Platon.

However, at some point of my research life, I happened to get involved in an educational activity which is called “Japan mathematics contest (=JMC)”. JMC is a contest, held each year by a relatively small group of high school teachers and mathematicians, in which the participants compete each other in their mathematical ability.

As well as in the famous Math-Olympiad (=MO), the participants consist of young people under the age of 18 and they are requested to solve mathematical questions. JMC looks like MO in this respect. Actually JMC is sometimes confused with MO even by my colleagues.

Well, JMC is nevertheless distinguishable from MO. Namely the main purpose of JMC is to invoke unexpected mathematical ability of young people. Recognizing excellent talents among them is only secondary.

Accordingly, each participant of JMC is given 330 minutes to answer some of the posed 3 questions by writing papers. It is admitted to look into textbooks of any kind.

As for the questions, we usually choose materials for them from everyday life and try to relate these rudimentary things to substantial mathematical ingredients.

The purpose of the present note is to exhibit one of such questions and give a sketch of its solution. Judging from the answers and responses of the participants, the author believes that it was a successful question.

¹⁾A. L. Cauchy (1789–1857): the founder of complex analysis.

²⁾G. F. B. Riemann (1826–1866): the founder of geometric complex analysis.

³⁾K. Oka (1901–1978): a developer of complex analysis of several variables.

⁴⁾K. Kodaira (1915–1997): a developer of complex analysis on manifolds.

2. A Question of JMC

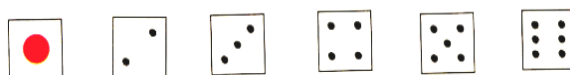
There are two sections in JMC, one for the senior high school students and the other for the junior high school students, although we admit any people under 18 to participate in the senior section. The following question was posed for the junior section in 2004 (cf. [J], The Committee of Japan Mathematics Contest, 2004).

Question (patterns of diagonals on the faces of polyhedrons)

(1) On a tetrahedron consisting of 4 faces of equilateral triangles, 4 lines are drawn, one for each face, into two congruent right triangles. How many such patterns are there? Here one counts the number of patterns by identifying those which coincide each other by some rotation in the space.

(2) Similarly, count the number of patterns made by lines on the faces of a cube, where one of the diagonals is drawn on each face.

(3) How many dices are there in the above mentioned sense? Here the faces of a dice consist of the patterns;



which are arranged in such a way that the sum of the numbers of the spots on the opposite sides are 7.

The idea of making questions (1) and (2) arose from (3) and S. Mukai's⁵⁾ expository talk about the patterns on the cube which are made by coloring the faces. Although questions (3) must have been well known, at least to some of the gamblers, it came into my attention for the first time by K. Namba's⁶⁾ remark. He made it during a short idle break in the laborious work of making entrance examinations. To this remark he added an amazing story of quest for the dices. He told us that from time to time he visited the department stores scattering along the loop railway line in Tokyo to collect different patterns of dices as many as possible. As a result he could complete the collection by finding all the possible patterns!

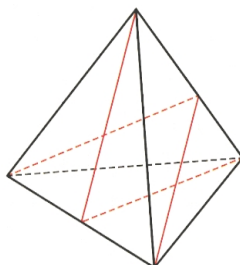
3. Sketch of the Solutions

(1) For each face one can draw a line satisfying the condition in 3 ways. Therefore, putting the identification by rotation aside, the totality of the patterns is counted as $3^4 = 81$.

Now consider the following pattern.

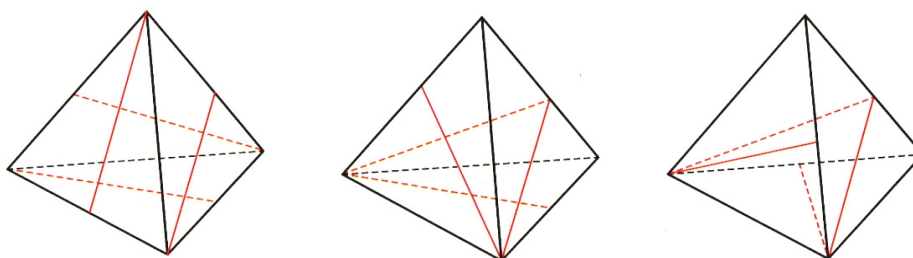
⁵⁾S. Mukai (1953–): professor of the Research Institute for Mathematical Sciences, Kyoto University. (When he gave the above talk, he was a professor of Graduate School of Polymathematics, Nagoya University.)

⁶⁾K. Namba (1939–): professor emeritus of Tokyo University.



Note that it is left invariant by 4 rotations. Here displacing no points is counted as such a rotation (a rotation of magnitude 0 around any axis), and the other rotations are easily seen to be those of magnitude 180° around the lines connecting the mid-points of the mutually opposite edges. Since there exist altogether 12 rotations which leave the tetrahedron invariant, the above pattern is varied by such rotations in $12 \div 4 = 3$ ways.

Similarly, each of the patterns, which remain invariant by the 180 degree rotation around a line connecting two opposite edges,



changes in $12 \div 2 = 6$ ways.

It is easy to see that there exist only these 4 illustrated patterns (and their equivalents obtained by rotations) which are left invariant by some nontrivial rotation.

Therefore, any of the other patterns varies by any of the nontrivial rotations.

As a result we have an equation

$$3 \times 1 + 6 \times 3 + 12 \times \square = 81.$$

Here we have in the box \square the number of mutually inequivalent patterns each of which varies in 12 ways under rotation.

Since $3 \times 1 + 6 \times 3 + 12 \times 5 = 81$, we know that the totality of inequivalent patterns under rotation is $1 + 3 + 5 = 9$.

(2) Similarly as in the case of tetrahedron, by counting the patterns at first without considering the effect of rotations, we obtain the number $2^6 = 64$.

Since the number of rotations moving the cube to itself is 24, as one can count it by finding the axes and angles of rotations, one can classify the diagonal patterns on the cube

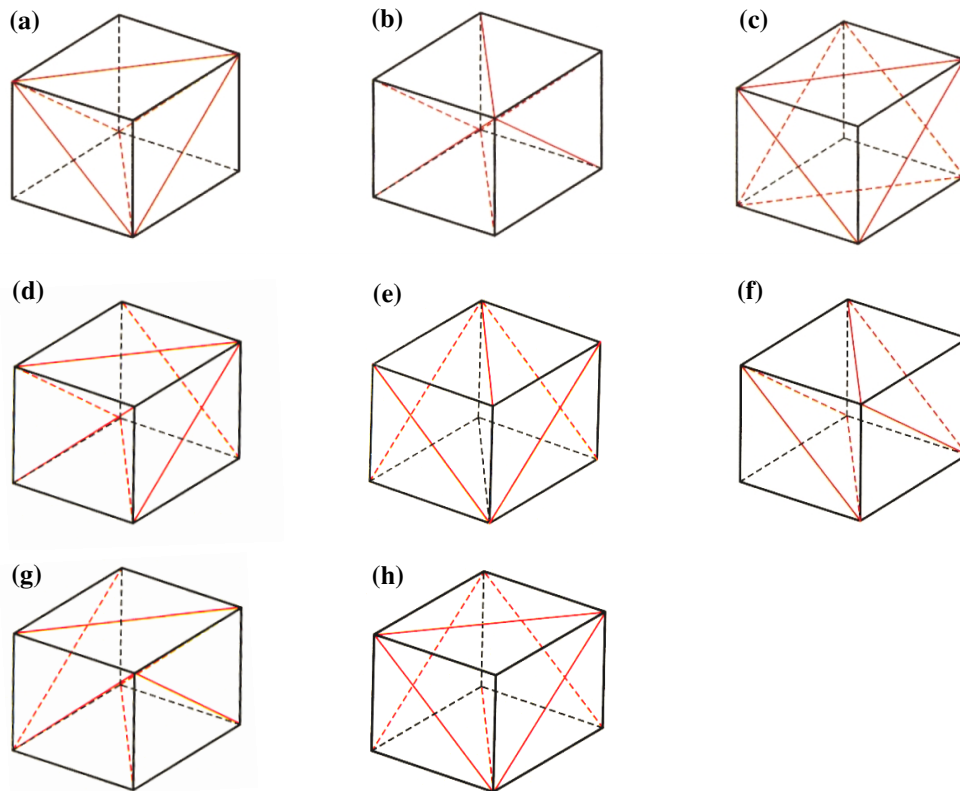


Fig. 1. Diagonal patterns on the cube.

by listing up the patterns according to a priori possible orders 24, 12, 8, 6, 4, 3, 2, 1 of rotational symmetry. Here the order means the number of rotations fixing the pattern.

As a pattern of order 12 one may take (Fig. 1(a)) which varies in $24 \div 12 = 2$ ways. Similarly the two patterns (b) and (c) (Fig. 1) vary in $24 \div 6 = 4$ ways, the three patterns (d), (e) and (f) (Fig. 1) in $24 \div 4 = 6$ ways, the pattern (g) (Fig. 1) in $24 \div 2 = 12$ ways, and the pattern (h) (Fig. 1), being totally asymmetric, varies in 24 ways.

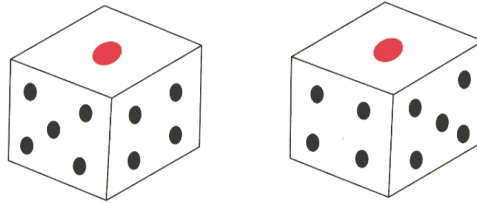
Since

$$2 \times 1 + 4 \times 2 + 6 \times 3 + 12 \times 1 + 24 \times 1 = 64,$$

the above list exhausts all the possibility.

Hence the total number of inequivalent diagonal patterns on the cube is 8.

(3) The three faces which are invariant under the rotation of 90° are arranged in two ways as



These two patterns are inequivalent to each other under any rotation.

On the other hand, there are two possibilities for each face in the opposite side of these patterns, since the patterns of the faces “2”, “3” and “6” change by 90 degree rotation. Therefore the number of inequivalent patterns as a dice is $2 \times 2^3 = 16$.

REFERENCE

The Committee of Japan Mathematics Contest (2004) Japan Mathematics Contest 2004—Solutions and Comments (in Japanese).