

## How Many Facets on Average can a Tile Have in a Tiling?

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**Abstract.** A question, how many faces can have a convex polytope which tiles space by its copies, is long standing and intriguing. Another, more general question, how many faces on average can convex tiles have in a face-to-face tiling, is related to the concept of the complexity of a tiling. In the paper it will be said out about basic results in this field. In particular, it will be shown that in Euclidean 3D-space there are periodic tilings whose all tiles are pairwise combinatorially isomorphic and have arbitrarily large number of faces, and also there are periodic Voronoi tilings whose each tile has faces as many as you like.

### 1. Introduction; Basic Notions

In this paper tilings of space by polytopes will be considered. A *tiling* of space by polytopes is defined as a collection of polytopes placed in space in such a way that no two polytopes overlap and the polytopes of this collection cover the whole space without gaps.

One of the most natural characteristics of the complexity of a tiling is the average number of facets in tiles of the tiling. We deal very often with tilings whose all the tiles have very simple structure. For instance, the so-called Delone tilings for generic point sets consist of simplices only. In crystallography tilings consisting of pairwise congruent tiles appear. These tilings are of special interest. Here we note that one of two questions posed by Hilbert in the 18th of his celebrated problems (titled by: “Building of space from congruent polyhedra”) concerned namely these tilings and their tiles which are called spacefillers\*. The average number of faces per tile in a tiling by a spacefiller is obviously equal to the number of faces of the spacefiller. At the same time, though the most wellknown spacefillers such as the cube or the regular hexagonal prism are simply arranged, not all spacefillers are so simple. For instance, in Euclidean 3D-space there is a spacefiller with 38 faces (see below for more details). A well-known question, arising here, is if there is such a positive number which bounds from above the number of faces in

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\*A *spacefiller* is defined as a polytope which can tile space by its congruent copies.

any spacefiller. This long standing question is one of the most challenging problems in the whole tiling theory in Euclidean space. Besides of a purely mathematical interest, an answer (of any kind) to this question would be very significant, in particular, for the problem of enumeration of spacefillers, for further developing of local theorem on isohedral tilings and regular point sets (DELONE *et al.*, 1976). The question remains still unanswered even for dimension 3. We emphasize that in three-dimensional space with a constant *non-zero* curvature (i.e. in hyperbolic and spherical spaces) the problem has a solution: in both these spaces there are convex spacefillers in which the number of faces exceeds an *á priori* given positive number. Therefore, the fact, that this question remains unsolved only in Euclidean space, makes the problem yet more intriguing.

A tiling is called *isohedral* if its symmetry group operates on the set of all its tiles transitively. A convex polytope which can tile space in isohedral way is also called a *stereotope* (after E. Fedorov and G. Voronoi). Isohedral (and also multihedral) tilings are widely used in crystallography. The symmetry group of a tiling, by definition, consists of all isometries (i.e. of rigid motions of space) which move the tiling into itself. The transitivity property of the symmetry group for an isohedral tiling, mentioned above, means that in the isohedral tiling for any two tiles there is a symmetry of the tiling which moves the first tile into the second one. Therefore, all the tiles in an isohedral tiling are pairwise congruent, i.e. they are spacefillers. Thus, a stereotope is a particular case of a convex spacefiller. It is well-known that in Euclidean plane isohedral tilings can consist of either arbitrary triangles and quadrangles, or pentagons and hexagons of special kind only (DELONE *et al.*, 1978; GRÜNBAUM and SHEPHARD, 1986). Moreover, it is simple to show that if a convex polygon tiles plane it has at most six sides.

It is interesting that in hyperbolic plane situation is quite different. There are isohedral tilings by regular hyperbolic polygons with any number  $n$ ,  $n \geq 3$ , of sides. Indeed, in hyperbolic plane for a fixed integer number  $n \geq 3$  there is the whole continuous family of regular  $n$ -gons. The value of an angle in regular  $n$ -gons is continuously and monotonically decreasing from  $\pi(1 - 2/n)$  to 0 as the sidelength of the polygon grows from 0 up to  $\infty$ . The biggest angle value  $\pi(1 - 2/n)$  corresponds to the angle value in a regular *euclidean* polygon. The smallest value 0 corresponds to the point when vertices of an enlarging regular hyperbolic polygon come out on the absolute of hyperbolic plane. Therefore, given  $n \geq 3$ , for any integer  $m$  such that  $2\pi/m < \pi(1 - 2/n)$  there is such a well-defined sidelength that an angle in a corresponding regular  $n$ -gon is equal to  $2\pi/m$ . This polygon forms an isohedral tiling of hyperbolic plane; in this tiling exactly  $m$   $n$ -gons meet around any vertex.

As to Euclidean space of an arbitrary dimension  $d$ , an upper bound for the number of facets\* in a  $d$ -dimensional stereotope does exist. In our paper (Sec. 2) we reproduce a finding of an upper bound given by DELONE and SANDAKOVA (D-S, 1961). After that, in Sec. 3, we will present a construction of an isohedral tiling to show that the *strict convexity* of a tile is quite essential for existence of an upper bound. Without this requirement the number of neighbors cannot be upper bounded. This fact gives additional interest to the upper bound problem for the number of facets in a strictly convex spacefiller.

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\*A *facet* means a face of dimension  $d - 1$  in a  $d$ -dimensional polytope, if  $d = 3$  a facet is an ordinary 2-dimensional face of a 3-dimensional polytope.

The idea by D-S of finding an upper bound is applicable not only to isohedral tilings. In DOLBILIN *et al.* (1998) the upper bound problem was studied for *multihedral* tilings, i.e. for tilings whose tiles are distributed over  $k$  equivalent classes with respect to the symmetry group (or orbits). Isohedral tilings represent a particular case of a multihedral tiling when the number  $k$  of orbits equals one. According to another terminology, multihedral tilings with  $k$  tile orbits are called *tile- $k$ -transitive tilings*.

It is commonly known that 3D-space can be tiled by congruent regular hexagonal prisms. As to a prism with the bigger number of faces, likely, it is not a spacefiller, though this fact is not established yet. However, what is of interest, as shown in Sec. 4, a space can be tiled by “combinatorial” prisms with arbitrary number of faces, i.e. by polytopes, all combinatorially equivalent to a prism with arbitrarily fixed number of faces. All the tiles in this tiling being pairwise combinatorially equivalent (or isomorphic)\* and are not required to be congruent. Such tilings are called *monotypic*. The tiling we will present in Sec. 4 is a tile- $(2n - 4)$ -transitive tiling whose all tiles are combinatorially equivalent to the  $n$ -gonal prism where  $n$  is an arbitrarily large positive integer number. So, in 3D-space there are periodic monotypic tilings whose all tiles have any large number of faces.

One should emphasize that tilings studied in Secs. 2, 3, and 4 are not necessarily Voronoi tilings. The question on the mean number of facets in Voronoi domains of a tiling is of special interest because of numerous applications. In Sec. 5 we will present periodic Delone sets\*\* in 3D-space such that *all* Voronoi domains\*\*\* of the points from the  $X$  have more faces than any *á priori* given number.

Here we mention the following relevant results on Voronoi tilings. In his pioneering work, MEIJERING (1953) proved that in a “typical” random Voronoi tiling in  $\mathbb{E}^3$ , i.e. in Voronoi tilings for Poisson point process, the mean number of faces and vertices in tiles is equal to 15.53... and 27.07..., respectively. In CHRIST *et al.* (1982), it was proved that in the Poisson-Voronoi 4D-cell the mean number of 3-faces is equal to  $340/9$  ( $=37.77...$ ). The expectation of the number of vertices in Poisson-Voronoi cell for any dimension has been derived in MILES (1970).

In TANEMURA (2003, 2005), a large-scale simulation of Poisson-Voronoi cells in dimension 2, 3, 4, and 5 is performed. The computer simulations have well-confirmed the theoretical results and suggested the mean number of facets in the Poisson-Voronoi 5D-cell to be equal to 88.56... .

Voronoi tilings can be used also as a tool for constructing spacefillers with many faces. ENGEL (1981) by constructing a Voronoi stereotope for a special crystallographic group found a stereotope with 38 faces. This number of faces in a 3D-spacefiller still remains

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\*Two polytopes are called *combinatorially equivalent* each other if they have the same numbers of vertices, of edges, of faces, respectively, and if these sets of vertices, sets of edges, sets of faces of both polytopes can be respectively put into one-to-one correspondences keeping all incidences between the vertices, edges, and faces. For example, such two polytopes as a  $n$ -gonal prism and a  $n$ -gonal truncated pyramid are combinatorially equivalent.

\*\*We remember that point set  $X$  in space is called a Delone set if there exist two positive numbers  $r$  and  $R$  such that any open ball of radius  $r$  taken in space contains at most one point from  $X$  and any closed ball of radius  $R$  contains at least one point from  $X$ .

\*\*\*A definition of a Voronoi tiling can be seen in Sec. 5.

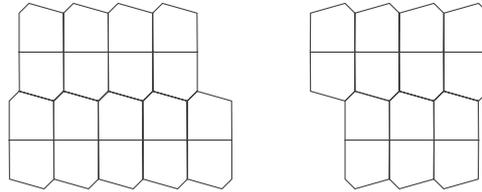


Fig. 1. One stereotope with different isohedral tilings.

unbroken record. The huge gap between Engel's stereotope with 38 faces and D-S's estimate (390 faces), or slight refinement by TARASOV (1997) (378 faces), says rather on rough estimate.

## 2. Parallelotopes and Stereotopes

### 2.1. On the Bieberbach-Schoenflies theorem

Let a stereotope be realized in certain isohedral tiling. Since a fixed stereotope can be realized in different isohedral tilings (see Fig. 1) we take just one of these tilings. By definition of an isohedral tiling  $\mathfrak{T}$  its full group of symmetries  $Sym(\mathfrak{T})$  operates on a set of all the tiles transitively. This group is discrete and has a compact fundamental domain or, by definition, the  $Sym(\mathfrak{T})$  is a *crystallographic group*.

The simplest example of a crystallographic group is a *translational group*  $T$ , i.e. a group generated by  $d$  linearly independent translations, where  $d$  is the dimension of space. If tiling  $\mathfrak{T}$  consists of one translational orbit of tiles\* then the tile is called a *parallelootope*.

SCHOENFLIES (1891) proved that in 3D-space a crystallographic group  $G$  has a translational subgroup  $T$  of a finite index  $h = [G:T]$ . In his XVIII problem D. Hilbert stated a task, in particular, to extend the Schoenflies theorem proved for  $d = 3$  to any dimension  $d > 3$  what was done by BIEBERBACH (1910, 1913).\*\*

Finiteness of the index  $h$  implies that the translational subgroup  $T$  is generated by exactly  $d$  linearly independent vectors. The presence in crystallographic groups of full-dimensional subgroups of pure translations provides finiteness of essentially different

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\*This means that all tiles of the tiling  $\mathfrak{T}$  are being obtained from a certain tile  $P$  by means of translations of the group  $T$ .

\*\*In fact, HILBERT (1901) wrote: ... *The fact of the finiteness of the groups of motions in elliptic space is an immediate consequence of a fundamental theorem of C. Jordan whereby the number of essentially different kinds of finite groups of linear substitutions in  $n$  variables does not surpass a certain finite limit dependent upon  $n$ . The groups of motions with fundamental regions in hyperbolic space have been investigated by Fricke and Klein in the lectures on the theory of automorphic functions and finally FEDOROV (1890), SCHOENFLIES (1891) and lately Rohn have given the proof that there are, in euclidean space, only a finite number of essentially different kinds of groups of motions with a fundamental region. Now, while the results and methods of proof applicable to elliptic and hyperbolic space hold directly for  $n$ -dimensional space also, the generalization of the theorem for euclidean space seems to offer decided difficulties. The investigation of the following question is therefore desirable: Is there in  $n$ -dimensional euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region?*

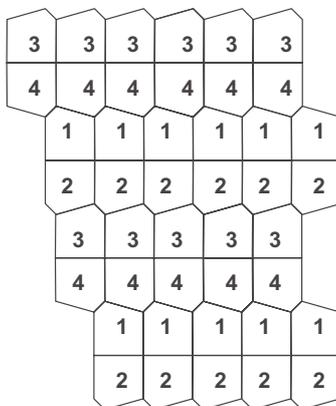


Fig. 2. Isohedral tiling splits into  $h/k$  “lattices”.

crystallographic groups and makes possible to find all of them, at least in low dimensions  $d = 3, 4, 5$ . Due to the theorem of JORDAN (1878, 1880) the index  $h$  can be upper bounded dependently only on dimension  $d$ . Later an upper bound for  $h$  has appeared explicitly in FEIT (1995), FRIEDLAND (1997). In particular, for dimensions  $d = 1, 3, 5$  and for all  $d > 10$  the order  $2^d d!$  of the orthogonal group over the integers (the symmetry group of a  $d$ -dimensional cube) is equal or bigger than the index  $h$  of a translational subgroup in a crystallographic group operating in  $\mathbb{E}^d$ .

2.2. *The upper bound for the number of facets in a parallelotope and a stereotope*

DELONE and SANDAKOVA (D-S) (1961) found that a  $d$ -dimensional stereotope has the number  $f_d$  of facets upper bounded as follows:

$$f_d \leq 2(2^d - 1) + (h - 1)2^d. \tag{1}$$

Here in (1)  $h = [Sym(\mathfrak{T}) : T]$  is the index of translational subgroup  $T$  in symmetry group  $Sym(\mathfrak{T})$  of isohedral tiling  $\mathfrak{T}$  of space with the given stereotope.

Take a tile  $P$  in the isohedral tiling  $\mathfrak{T}$  and let  $Stab(P)$  be the stabilizer of  $P$  in  $Sym(\mathfrak{T})$ , i.e. a subgroup of all the symmetries of the  $\mathfrak{T}$  that leave the  $P$  invariant. Let also  $k$  be the order of  $Stab(P)$ . Then the tiling  $\mathfrak{T}$  splits into  $h/k$  lattices of tiles and inequality (1) can be rewritten as follows

$$f_d \leq 2(2^d - 1) + (h/k - 1)2^d. \tag{2}$$

It is easy to prove that condition  $h/k = 1$  corresponds to the case when we deal with a tiling of space by parallelotopes. On the other side, MINKOWSKI (1897) obtained a non-refinable upper bound for the number of  $d$ -dimensional parallelotope’s facets

$$f_d \leq 2(2^d - 1).$$

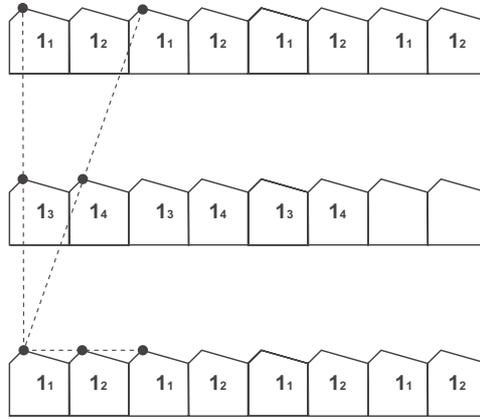


Fig. 3. A lattice  $\mathfrak{T}_1$  splits into  $2^d$  classes:  $d = 2, 2^d = 4$ .

Thus, the famous upper bound for a parallelotope by MINKOWSKI can be thought as a particular case of the D-S estimate when  $h/k = 1$ .

**Proof** of (2). Indeed, the tiling  $\mathfrak{T}$  is isohedral if  $\mathfrak{T}$  consists of one orbit  $Sym(\mathfrak{T}) \cdot P$ . This tile orbit splits into  $h/k$  “tile lattices”, each consisting of  $T \cdot P_i, 1 \leq i \leq h/k$  (see Fig. 2). In other words, each tile lattice consists of parallel copies of certain tile  $P_i \in \mathfrak{T}$  which is moved into other tiles of  $\mathfrak{T}_i$  by translations of subgroup  $T$ .

Without loss of generality, we may take tile  $P_1 \in \mathfrak{T}_1$  and estimate the number of its facets. First of all, we estimate from above number  $f'$  of facets shared by  $P_1$  with tiles from the same lattice  $\mathfrak{T}_1$ . We will show that

$$f' \leq 2(2^d - 1). \tag{3}$$

Following Minkowski we break lattice of tiles  $\mathfrak{T}_1$  into  $2^d$  modulo 2 classes. In order to do this, we consider a point lattice  $\Lambda$ . It consists, by definition, of all points having in some basis integer coordinates:

$$\Lambda := \{ \mathbf{x} = (x_1, x_2, \dots, x_d), | x_i \in \mathbb{Z} \}.$$

Two points  $\mathbf{x}$  and  $\mathbf{x}'$  are said to be *equivalent modulo 2* if  $x_i - x'_i \equiv 0 \pmod{2}$ . It is obvious that the set of all the modulo 2 equivalence classes can be put into one-to-one correspondence with the set of  $d$ -element (0-1)-sequences. Therefore there are sharply  $2^d$  equivalence classes (Fig. 3).

It is important that, besides this coordinate definition of the partition of  $\Lambda$  into  $2^d$  classes, there is a coordinate-free, purely geometric description of this partition: two points  $\mathbf{x}$  and  $\mathbf{x}'$  of lattice  $\Lambda$  belong to the same modulo 2 class if and only if the midpoint of segment  $[\mathbf{x}, \mathbf{x}']$  (i.e. a point  $(\mathbf{x} + \mathbf{x}')/2$ ) also belongs to the lattice  $\Lambda$ .

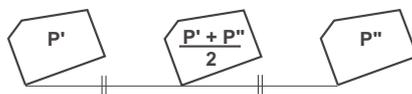


Fig. 4. Minkowski half-sum.

Two tiles from one class cannot share a facet because inbetween there is one more tile from another class from the lattice  $\Lambda$ . Therefore, the tile  $P_1$  has no facets shared with tiles from the same  $P_1$ 's class. But in the same lattice there are  $2^d - 1$  more classes. It is not hard to make sure that in each class there cannot be more than two tiles sharing a facet with  $P_1$ . This implies that the number  $f'$  of facets shared by  $P_1$  and tiles from the same lattice  $\mathfrak{T}_1$  does not exceed  $2(2^d - 1)$ . Thus, (3) is proved.

Now we prove that for  $i, 2 \leq i \leq h/k$ , if the  $P_1$  shares  $f''$  facets with tiles from tile lattice  $\mathfrak{T}_i$  then one has

$$f'' \leq 2^d. \tag{4}$$

Assume for some  $i$  tile  $P_1$  has more than  $2^d$  neighbors at facets from lattice  $\mathfrak{T}_i$ . This implies that in  $\mathfrak{T}_i$  there is such a modulo 2 class that contains at least two tiles  $P'$  and  $P''$ . Therefore, inbetween there is one more tile from  $\mathfrak{T}_i$ . This tile can be expressed as the Minkowski half-sum  $(P' + P'')/2$  of tiles  $P'$  and  $P''$ \* (Fig. 4) is also a tile from the  $\mathfrak{T}_i$ .

Let  $F'$  and  $F''$  be facets shared by  $P_1$  with  $P'$  and  $P''$ , respectively. Take in these facets arbitrary relatively interior points  $A' \in \text{Relint}(F')$ \*\* and  $B'' \in \text{Relint}(F'')$  (Fig. 5). Since  $P'$  and  $P''$  are supposed to belong to tile lattice  $\mathfrak{T}_i$  there is translation  $\mathbf{t} \in T \subseteq \text{Sym}(\mathfrak{T})$  such that  $P' + \mathbf{t} = P''$ . Facet  $F'$  and point  $A' \in \text{Relint}(F')$  have respective images  $F' + \mathbf{t}$  (a facet of  $P''$ ) and  $A'' \in \text{Relint}(F' + \mathbf{t})$ . Quite analogously, facet  $F''$  and point  $B'' \in \text{Relint}(F'')$  have respective preimages  $F'' - \mathbf{t}$  (a facet of  $P'$ ) and  $B' \in \text{Relint}(F'' - \mathbf{t})$  (Fig. 5). From this it follows that quadrangle  $A'B'B''A''$  is a parallelogram.

Let us take midpoints  $C'$  and  $C''$  of sides  $A'B'$  and  $A''B''$ , respectively. These points  $C'$  and  $C''$  belong to the interiors  $\text{Int}(P')$  and  $\text{Int}(P'')$  of the  $d$ -polytopes  $P'$  and  $P''$ , respectively. Their midpoint  $C := (C' + C'')/2$  belongs to the interior of tile  $P''' = (P' + P'')/2 \in \mathfrak{T}_i, i \neq 1$ . On the other hand, midpoint  $(A' + B'')/2$  belongs to the interior of tile  $P_1 \notin \mathfrak{T}_i$ . Points  $(C' + C'')/2$  and  $(A' + B'')/2$  coincide because they both are the centropoint of parallelogram  $A'B'B''A''$ . Bearing in mind that  $(C' + C'')/2 \in \text{Int}(P''')$  and  $(A' + B'')/2 \in \text{Int}(P_1)$  we get tiles  $P_1$  and  $P'''$  to have an interior point in common. However, tiles  $P_1$  and  $P'''$  under  $i \neq 1$  belong to different lattices and consequently cannot overlap. The contradiction proves (4). Inequalities (3) and (4) imply (2).

\*The Minkowski half-sum of two sets  $A$  and  $B$  in a vector space is a particular case of the Minkowski (or vector) sum and is defined as follows:  $(A + B)/2 := \{(\mathbf{a} + \mathbf{b})/2 \mid \mathbf{a} \in A, \mathbf{b} \in B\}$ .

\*\*Given  $k$ -face  $F$  of a convex  $d$ -polytope  $P$ , as a convex  $k$ -dimensional polytope,  $F$  consists of the  $(k - 1)$ -dimensional boundary (the union of all faces of the  $P$  of lower dimensions belonging to  $F$ ) and the  $k$ -dimensional interior of  $F$  what is denoted by  $\text{Relint}(F)$ . Since we find ourselves in  $d$ -dimensional space the interior of a  $d$ -dimensional polytope consists of all points of the polytope which are contained within the polytope along with their some neighborhoods. The interior of a  $d$ -dimensional polytope  $P$  in  $d$ -space is denoted  $\text{Int}(P)$ .

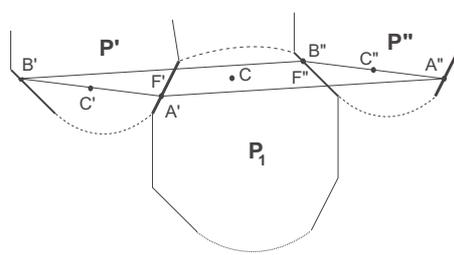


Fig. 5. Proof of (2).

### 3. Non-strict Convex Tilings

It is surprising that the strict convexity property turns out to be very essential in the question of existence of an upper bound. The situation completely changes if we cancel the strict convexity property: we face the situation where no upper bound exists.

We will demonstrate a simple example of an isohedral tiling in which any tile (a space-filler) is non-strictly convex and surrounded by an arbitrarily large number of congruent tiles. We take this construction from ZAMORZAEV (1965) though, probably, it might be known earlier.

There is a simple procedure to obtain a non-strictly convex polytope: we can take a convex polytope, subdivide some of its facets (that are  $(d-1)$ -dimensional polytopes) into a number of convex  $(d-1)$ -dimensional polytopes, and name these smaller  $(d-1)$ -polytopes as *facets* of a “new” polytope.

Now we take a rectangular three-dimensional  $(1 \times 1 \times m)$ -parallelepiped  $P$  and subdivide long faces as depicted on Fig. 6. The two opposite faces of size  $1 \times m$  are subdivided into  $m$  unit squares. The two other long faces are subdivided in one square and one rectangle of size  $1 \times (m-1)$  each. We underline that these nets on the last two faces are required to be placed centrosymmetrically with respect to the centropoint of the  $P$ . By means of this additional subdivision of faces the polytope  $P$  turns out a non-strictly convex polytope which will be also denoted  $P$ .

We will parquet space with copies of the polytope  $P$ . We put the first squared beam  $P_1$  into the “horizontal” position  $0 \leq x \leq m, 0 \leq y \leq 1, -1 \leq z \leq 0$  (see Fig. 7). The second beam  $P_2$  is put into the “vertical” position  $0 \leq x \leq 1, 1 \leq y \leq m+1, -1 \leq z \leq 0$ . Which isometry moves the  $P_1$  into the  $P_2$ ? It is clear that under a gliding reflection  $\tau$  at plane with equation  $x-y=1/2$  (one should remember that we are in space, not in plane)  $P_1$  moves into  $P_2$ . This isometry is the pure reflection followed by translation by vector  $(1/2, 1/2, 0)$ .

Now we desiminate this pair of the  $P_1$  and  $P_2$  by translational group  $T$  into the whole tiling of space by congruent parallelepipeds. On the first stage by applying all translations, spanned by the two generating shifts  $\mathbf{t}_1 = (1, 1, 0)$  and  $\mathbf{t}_2 = (2m, 0, 0)$ , to the pair  $P_1, P_2$  we get a parketed layer  $L$  (Fig. 7). Then we shift this layer into another parallel layer and put it on the  $L$  from above by means of the third, linearly independent translation  $\mathbf{t}_3 = (m, 0, 1)$ . Under this translation each “horizontal” parallelepiped of the first layer  $L$  moves up and is

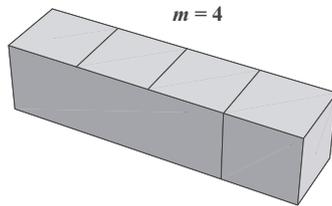


Fig. 6. Parallelepiped  $P$  of size  $1 \times 1 \times m$ .

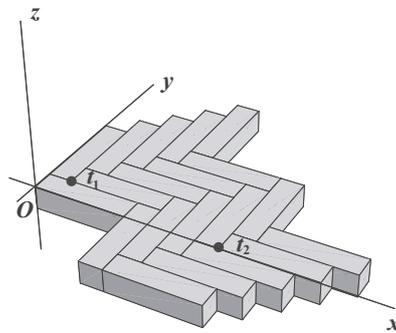


Fig. 7. Parketted layer  $L$ .

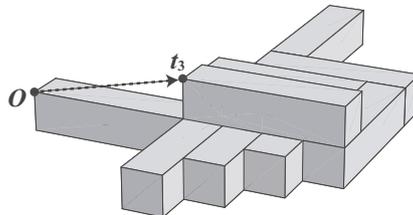


Fig. 8. Shift of the layer on  $L$  by translation  $\mathbf{t}_3 = (m, 0, 1)$ .

put upon exactly  $m$  “vertical” parallelepipeds from the layer  $L$ . It is easy to see that the “new” parallelepiped shares with each of the  $m$  “vertical” parallelepipeds from the first layer a common small square subsurface (Fig. 8). Exactly in the same way, each “vertical” parallelepiped from the first layer translated by vector  $\mathbf{t}_3$  turns out in the next layer  $L + \mathbf{t}_3$ . This image of a vertical parallelepiped, being in the second layer lies exactly on  $m$  horizontal parallelepipeds from the layer  $L$  and also has a common small square face with each of them. By repeat of applying this translation  $\mathbf{t}_3$  to layer  $L$  over and over again we get a tiling  $\mathfrak{T}$  of space by parallelepipeds.

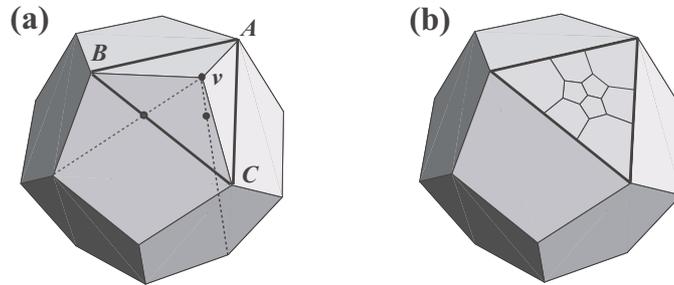


Fig. 9. Projection of a polyhedron from a vertex onto simplicial section.

The  $\mathfrak{T}$  is an isohedral face-to-face tiling. The number of faces in a (non-strictly convex) stereotope  $P'$  is equal to  $2(m + 3)$ . We see that, in contrast to the D-S case, in isohedral face-to-face tilings with non-strictly convex stereotopes a tile can have an arbitrarily large number of facets and, respectively, the same number of neighbors along them.

One should underline that the face-to-face property of tiling has been attained in the last case for the sake of artificial subdivision of the boundary of polytope  $P$ . On the other hand, namely due to this subdivision the parallelepiped  $P$  turned from a convex polytope into a non-strictly convex polytope. If we did not make subdivision of the boundary we would keep strict convexity of  $P$  but lose its face-to-face property.

#### 4. Monotypic Strictly Convex Tiles with Many Faces

The aim of the section is to present a face-to-face monotypic tiling in  $E^3$  whose all tiles are strictly convex and have arbitrarily many faces. We remind, by definition of a monotypic tiling, all tiles in the tiling have the same combinatorial type. In our case this type will be the combinatorial type of the  $n$ -gonal prism where  $n \geq 3$ . Additionally, this tiling will be a multihedral, or more concretely, tile- $(2n - 4)$ -transitive tiling, that is, it will consist of precisely  $2n - 4$  orbits of tiles with respect to its symmetry group.

The construction is based on a nice idea of SCHULTE (1984). We will demonstrate this idea by means of a way which one can pave plane by pentagons in. Let us take a convex polytope with pentagonal faces only and with at least one simplicial vertex  $v$ , for example, a regular dodecahedron. We draw a plane through all the three end points, say  $A, B, C$ , of edges coming out vertex  $v$ . This plane cuts off the pyramid with the apex  $v$  (Fig. 9a). The faces of the “truncated” dodecahedron are of three kinds: a sole “new” triangular face, three cut “old” faces, and nine uncut faces of the dodecahedron. Now we project from the  $v$  all the uncut faces of the “truncated” dodecahedron on triangle  $ABC$ . The projection of an uncut pentagonal face is a convex pentagon in triangle  $ABC$  (Fig. 9b). A common edge of two *uncut* faces is projected onto a common edge of the two corresponding pentagons in triangle  $ABC$ . If an edge of an *uncut* face is shared with some *cut* face then the edge is projected into a side of  $ABC$ . In our case of dodecahedron each side of triangle  $ABC$  is

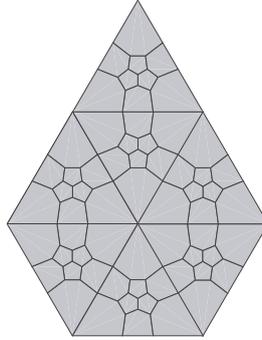


Fig. 10. Monotypic tiling with pentagons.

divided by three segments which all are the projections of edges of three different uncut faces. So, on this stage we get an edge-to-edge subdivision of triangle  $ABC$  with 9 (=12 – 3) convex pentagons which are the projections of all the uncut pentagonal faces of the dodecahedron.

Now note that triangle  $ABC$  is regular with angles  $\pi/3$ . Therefore, three reflections in all its sides generate a crystallographic group of a special kind, a so-called Coxeter group  $G$ . By applying isometries of  $G$  to the triangle  $ABC$  paved already by pentagons we get a tiling of plane by pentagons (Fig. 10). This tiling is edge-to-edge. Indeed, inside of these triangles the edge-to-edge property is inherited from the original triangle  $ABC$ . As for an edge of a pentagon which lies on a side of a triangle, this edge under reflection at the triangle's side remains fixed and at the same time becomes an edge of a symmetrical pentagon which lies in a neighboring triangle. Thus, the face-to-face property of the tiling also holds on the common boundary of the triangles.

Now we are to construct a face-to-face tiling of space  $\mathbb{E}^3$  by polytopes combinatorially equivalent to a  $n$ -gonal prism. Take the standard unit 3D-sphere  $\mathbb{S}^3 \subset \mathbb{E}^4$

$$\mathbb{S}^3 := \{ \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \} \tag{5}$$

and consider two circles in  $\mathbb{S}^3$

$$C_1 := \{ \mathbf{x} \in \mathbb{S}^3 | x_3 = x_4 = 0, x_1^2 + x_2^2 = 1 \}$$

and

$$C_2 := \{ \mathbf{x} \in \mathbb{S}^3 | x_1 = x_2 = 0, x_3^2 + x_4^2 = 1 \}.$$

Put on each of these circles vertices of a regular  $n$ -gon

$$X_1 := \left\{ \mathbf{x}_i = \left( \cos \frac{2\pi i}{n}, \sin \frac{2\pi i}{n}, 0, 0 \right), 0 \leq i \leq n-1 \right\}, \tag{6}$$

$$X_2 := \left\{ \mathbf{x}_j' = \left( 0, 0, \cos \frac{2\pi j}{n}, \sin \frac{2\pi j}{n} \right), 0 \leq j \leq n-1 \right\}. \quad (7)$$

We draw through each point  $\mathbf{x}$  of  $X := X_1 \cup X_2$  hyperplane  $H_{\mathbf{x}}$  tangent to  $\mathbb{S}^3$ . It is obvious that the intersection of closed halfspaces  $H_{\mathbf{x}}^+$  is a 4-dimensional convex circumscribed polytope  $P(X)$  which is arranged as follows.

First, we will show this polytope to be an *isohedron*, i.e. such a polytope whose symmetry group operates transitively on the set of its facets. Consider a group of isometries of space  $\mathbb{E}^4$  generated by two elements  $g_n$  and  $\tau$ , where  $g_n$  is a rotation  $2\pi/n$  of space  $\mathbb{E}^4$  around fixed plane  $x_3 = x_4 = 0$  and the isometry  $\tau$  commutes axis  $Ox_1$  with  $Ox_3$  and axis  $Ox_2$  with  $Ox_4$ :

$$G = \langle g_n, \tau \rangle. \quad (8)$$

The rotation  $g_n$  is presented by matrix

$$\begin{pmatrix} \cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} & 0 & 0 \\ -\sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The  $\tau$  is represented by matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is clear that  $X = G \cdot \mathbf{x}_0$ , where in notations of (6)  $\mathbf{x}_0 = (1, 0, 0, 0)$ . Indeed, the cyclic group  $\langle g_n \rangle$  turns the point  $\mathbf{x}_0$  into all the points of  $X_1$  and rotation  $\tau$  sends  $X_1$  into  $X_2$  and, vice versa, the set  $X_2$  into  $X_1$ . It follows that group  $G$  transitively operates on the set of tangent hyperplanes at all the points of  $X$ . Consequently, polytope  $P(X)$  is an isohedron. All the facets are pairwise congruent 3-dimensional polytopes. We show that they are regular  $n$ -gonal prisms.

Take hyperplane  $x_1 = 1$  touching  $\mathbb{S}^3$  at point  $\mathbf{x}_0 \in X$  and two point groups:

$$\mathbf{v}_j = \left( 1, \tan \frac{\pi}{n}, \frac{1}{\cos \frac{\pi}{n}} \cos \frac{\pi(2j+1)}{n}, \frac{1}{\cos \frac{\pi}{n}} \sin \frac{\pi(2j+1)}{n} \right), 0 \leq j \leq n-1, \quad (9)$$

and

$$\mathbf{v}_j' = \left( 1, -\tan \frac{\pi}{n}, \frac{1}{\cos \frac{\pi}{n}} \cos \frac{\pi(2j+1)}{n}, \frac{1}{\cos \frac{\pi}{n}} \sin \frac{\pi(2j+1)}{n} \right), \quad 0 \leq j \leq n-1. \quad (10)$$

These points form vertices of a regular prism displaced in the hyperplane  $x_1 = 1$ . Really, points (9) and (10) form vertices of two regular  $n$ -gons to be in two parallel planes, respectively. These polygons are the bases of the prism. Aside edges of the prism are formed by segments  $[\mathbf{v}_j, \mathbf{v}_j']$  pairwise linking the corresponding vertices of the bases. Simple calculations show that

$$|\mathbf{v}_j, \mathbf{v}_j'| = |\mathbf{v}_j, \mathbf{v}_{j+1}|,$$

i.e. the side faces of the prism are squares. Denote this prism by  $P_1$ . The axis of prism  $P_1$  is a line given by three equations:  $x_1 = 1, x_2 = x_3 = 0$ .

Why this prism is a facet of the isohedron  $P(X)$ ? To show this it suffices to check that the intersection of tangent hyperplane  $x_1 = 1$  and polytope  $P(X)$  is exactly the prism  $P_1$ . There are two groups of these tangent hyperplanes of polytope  $P(X)$ . Hyperplanes of the first group touch the sphere  $\mathbb{S}^3$  at points of  $X_1$  hyperplanes of the second group are tangent hyperplanes at points of  $X_2$ . Here are their equations.

$$H_{\mathbf{x}_i}, \mathbf{x}_i \in X_1: x_1 \cos \frac{2\pi i}{n} + x_2 \sin \frac{2\pi i}{n} = 1, \quad 0 \leq i \leq n-1, \quad (11)$$

and

$$H_{\mathbf{x}_j'}, \mathbf{x}_j' \in X_2: x_3 \cos \frac{2\pi j}{n} + x_4 \sin \frac{2\pi j}{n} = 1, \quad 0 \leq j \leq n-1. \quad (12)$$

By putting the coordinates of  $\mathbf{v}_j$  from (9) into Eqs. (11) and (12), one can check that for a fixed  $j, 0 \leq j \leq n-1$ , a vertex  $\mathbf{v}_j \in P_1$

- (i) belongs exactly to four hyperplanes  $H_{\mathbf{x}_0}, H_{\mathbf{x}_1}, H_{\mathbf{x}_j},$  and  $H_{\mathbf{x}_{j+1}}$  and
- (ii) lies in the interiors of half-spaces for all the rest tangent hyperplanes, i.e.  $\mathbf{v}_j \in \text{Int}(H_{\mathbf{x}}^+), \mathbf{x} \in X \setminus \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_j', \mathbf{x}_{j+1}'\}$ .

The same can be said on the prism's vertices  $\mathbf{v}_j'$ . Namely, vertex  $\mathbf{v}_j', 0 \leq j \leq n-1$

- (i') belongs exactly to four hyperplanes  $H_{\mathbf{x}_0}, H_{\mathbf{x}_{n-1}}, H_{\mathbf{x}_j'},$  and  $H_{\mathbf{x}_{j+1}'}$  and
- (ii') lies in the interior of all the rest half-spaces, i.e.  $\mathbf{v}_j' \in \text{Int}(H_{\mathbf{x}}^+), \mathbf{x} \in X \setminus \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_j, \mathbf{x}_{j+1}'\}$ .

Therefore, according to (i) and (i') exactly two hyperplanes from (11)  $H_{\mathbf{x}_1}$  and  $H_{\mathbf{x}_{n-1}}$ , and all the  $n$  hyperplanes from (12) cut out off hyperplane  $H_{\mathbf{x}_0}$  a regular prism  $P_1$  with  $n+2$  faces. According to (ii) and (ii') all other  $n-2$  hyperplanes leave all the vertices of  $P_1$  on the same side as polytope  $P(X)$ . Therefore, the intersection of all halfspaces, determined

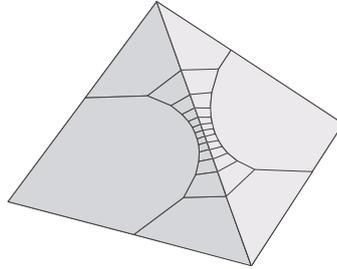


Fig. 11. Combinatorics of net's track on the boundary of simplex  $ABCD$  (case  $n = 15$ ).

by the last  $n - 2$  hyperplanes, with the hyperplane  $x_1 = 1$  is a convex 3-dimensional polytope which contains the prism  $P_1$ . Therefore the prism  $P_1$  is the maximal part of  $P$  which is contained in hyperplane  $x_1 = 1$ . Therefore, prism  $P_1$  is exactly a facet (or 3-face) of 4-dimensional polytope  $P(X)$  (GRÜNBAUM, 1967).

So, the polytope  $P(X)$  is confined by  $2n$  (as many as vertices in  $X$ ) facets, each of them is a regular  $n$ -gonal prism. Moreover, this polytope is *simple*, i.e. at each vertex as minimal as possible for a given dimension number of edges meet (in particular, in dimension 4 exactly 4 edges meet). Therefore we can apply to this polytope the construction described above. Draw in  $\mathbb{E}^4$  a hyperplane coming through the 4 endpoints of edges meeting at a vertex, say  $O$ . The hyperplane crosses four facets of  $P(X)$  along simplex, say  $ABCD$ . All the rest  $2n - 4$  prismatic facets of  $P$  are projected in one-to-one way from the center  $O$  onto simplex  $ABCD$ . Combinatorial structure of this tessellation of  $ABCD$  by convex prisms\* is inherited from the  $P(X)$ . Combinatorics of a net consisting of vertices, edges, and faces of images of  $n$ -gonal prisms on the boundary of simplex  $ABCD$  is depicted on Fig. 11. Simplex  $ABCD$ , saying in general, does not tile space. But since any simplex can be affinely mapped onto any other simplex, by an appropriate affine transformation we can map this simplex into a so-called Coxeter simplex.

Remember, by definition, a *Coxeter polytope*, in particular, a *Coxeter simplex* is such a polytope that has all dihedral angles equal to  $\pi/m_{ij}$ , where  $m_{ij}$  are integer numbers  $\geq 2$ . It is well-known that reflections in facial planes supporting a Coxeter polytope generate a discrete group for which the given polytope is a fundamental cell. So, a Coxeter polytope and its copies under the mentioned reflections and their compositions tile the whole space in isohedral way.

In space all the Coxeter simplices have been enumerated. As a possible sort of a Coxeter simplex in 3D-space one can take the following simplex. Divide a cube's face, which is a square, into eight congruent triangles by drawing two diagonals and two segments linking the midpoints of opposite sides. Construct a pyramid over such a triangle as over the base with the apex in the cube's centropoint. The obtained tetrahedron will be a Coxeter simplex. It constitutes the  $1/48$  part of the cube. This Coxeter simplex has three

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\*In fact, here we talk only on polytopes of combinatorial type of a prism because such affine properties of a prism, as parallelity of its bases or of its side edges, under central projection are being lost.

dihedral angles equal to  $\pi/2$ , two angles  $\pi/4$ , and one dihedral angle  $\pi/3$ . Remember that the original simplex  $ABCD$  was filled up by projections of  $2n - 4$  prisms with  $n + 2$  faces each. Therefore the Coxeter simplex is filled up by the same number of affine images of those combinatorial prisms.

By means of a Coxeter group generated by this Coxeter simplex, the simplex itself along with  $n$ -gonal prisms filling it, is spread over space and generate a multihedral (tile- $(2n - 4)$ -transitive)) face-to-face tiling of space by convex polytopes all combinatorially equivalent to a  $n$ -gonal prism.

### 5. On Some Voronoi Tilings in 3D-Space

Now we are going to present explicitly in 3D-space Delone sets such that the corresponding Voronoi tessellations have all domains with as many faces as you want.

First, we should remember what a Voronoi tiling and a Delone tiling for a given Delone set are. Let  $X \subset \mathbb{E}^3$  be a Delone set (a definition of Delone set has been given earlier) and  $\mathbf{x} \in X$ . A *Voronoi domain*  $V_{\mathbf{x}}$  of the point  $\mathbf{x}$  in the set  $X$  is a set of all the points of space  $\mathbf{y} \in \mathbb{E}^3$  such that  $|\mathbf{y}, \mathbf{x}| \leq |\mathbf{y}, \mathbf{x}'|$  for any other point  $\mathbf{x}' \in X$ .

Due to discreteness of Delone set  $X$ , a Voronoi domain is a polytope and, moreover, it is a convex polytope. Since the distance between any two points of  $X$  not lesser than  $2r$  this polytope cannot be too small: it must contain inside a ball of radius  $r$ . Due to the  $R$ -property of a Delone set this Voronoi domain cannot be too large: it must be contained in a certain ball with radius  $R$ .

Given a Delone set  $X$ , the full collection of Voronoi domains  $\{V_{\mathbf{x}} | \mathbf{x} \in X\}$  for all the points of  $X$  forms a tiling of the whole space  $\mathbb{E}^3$  which is called a *Voronoi tiling*  $\text{Vor}(X)$  (or a *Voronoi diagram*) for a given point set  $X$ .

Given Delone set  $X$ , a polytope  $D$  is called a *Delone polytope* if

- (1) its vertices belong to the set  $X$ ;
- (2) the polytope is inscribed into a certain sphere;
- (3) besides the vertices of  $D$  there are no points of  $X$  anymore either inside the circumsphere or on it itself.

The family of all possible Delone tiles forms a face-to-face tiling of space, which is called a *Delone tiling*  $\text{Del}(X)$ .

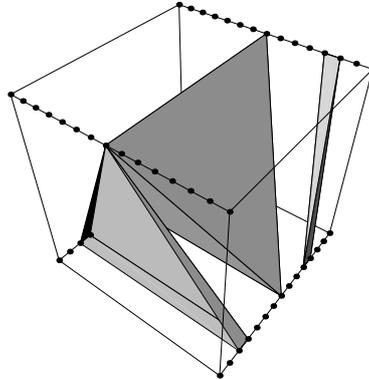
It is not hard to see that tiling  $\text{Del}(X)$  is dual to tiling  $\text{Vor}(X)$ . Moreover, this duality has metrical character. Namely, faces and edges of Voronoi tiles are *perpendicular* to corresponding edges and faces, respectively, of Delone tiles. It follows, in particular, the number of faces in a Voronoi tile is equal to the number of edges of a Delone tiling meeting at the vertex corresponding to the Voronoi tile.

Now, before starting our construction, one should note that the element of this construction is used as a sample of a set of  $N$  points located on two skew lines and generating  $O(N^2)$  Delone tetrahedra for this set (OKABE *et al.*, 2000).

Cubic point lattice  $\mathbb{Z}^3 \subset \mathbb{E}^3$  of integer points  $(x, y, z)$  can split into two parts

$$\mathbb{Z}^3 = \mathbb{Z}_0^3 \cup \mathbb{Z}_1^3,$$

where the first sublattice  $\mathbb{Z}_0^3$  consists of points with even  $z$ -coordinates and the points of

Fig. 12. Three types of Delone polytope (case  $m = 7$ ).

the second sublattice  $\mathbb{Z}_1^3$  have odd  $z$ -coordinates.

Given an integer number  $m \geq 1$  and two point sets:

$$\Lambda_0 := \bigcup_{i=1}^{2m} \left\{ \mathbb{Z}_0^3 + \left( \frac{i}{2m}, 0, 0 \right) \right\}$$

and

$$\Lambda_1 := \bigcup_{j=1}^{2m} \left\{ \mathbb{Z}_1^3 + \left( 0, \frac{j}{2m}, 0 \right) \right\},$$

we consider their union

$$X := \Lambda_0 \cup \Lambda_1.$$

**Theorem.** *The restriction of  $\text{Del}(X)$  of the whole space  $\mathbb{E}^3$  to the cube  $K = [0, 1]^3$  is a tessellation of the cube consisting of  $4m^2 + 1$  simplices and  $4m$  pyramids with rectangular bases. The tessellation of the  $K$  completely determines the whole tiling by means of the group  $\text{Sym}(X)$  (see Fig. 12).*

**Proof.** First of all, we note that set  $X$  consists of  $m + 1$  orbits with respect to  $\text{Sym}(X)$  which is a crystallographic group from the cubic family of spatial groups.

The fundamental domain of the group  $\text{Sym}(X)$  is the  $1/8$  part of the cube  $K$ . Therefore, if we find all Delone tiles entering the  $K$  then we will determine the Delone tiling of the whole space. Introduce notations for points:  $\mathbf{x}_i = (i/2m, 0, 0)$ ,  $\mathbf{x}'_i = (i/2m, 1, 0)$ ,  $\mathbf{y}_j = (0, j/2m, 1)$ ,  $\mathbf{y}'_j = (1, j/2m, 1)$ , where  $0 \leq i, j \leq 2m$ .

We are to show that cube  $K$  is completely filled up by Delone polytopes spanned by the following sets of vertices.

I. The first group of Delone tiles consists of  $4m^2$  simplices spanned by the following 4-sets of vertices:

$$\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{y}_j, \mathbf{y}_{j+1}, \text{ where } 0 \leq i \leq m-1, 0 \leq j \leq m-1; \quad (13)$$

$$\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{y}'_j, \mathbf{y}'_{j+1}, \text{ where } m \leq i \leq 2m-1, 0 \leq j \leq m-1; \quad (14)$$

$$\mathbf{x}'_i, \mathbf{x}'_{i+1}, \mathbf{y}_j, \mathbf{y}_{j+1}, \text{ where } 0 \leq i \leq m-1, m \leq j \leq 2m-1; \quad (15)$$

$$\mathbf{x}'_i, \mathbf{x}'_{i+1}, \mathbf{y}'_j, \mathbf{y}'_{j+1}, \text{ where } m \leq i \leq 2m-1, m \leq j \leq 2m-1. \quad (16)$$

II. The second group of Delone tiles consists of  $4m$  4-gonal pyramids with rectangular bases and apexes as follows:

$$\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}'_i, \mathbf{x}'_{i+1}, \text{ and the apex } \mathbf{y}_m, 0 \leq i \leq m-1; \quad (17)$$

$$\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}'_i, \mathbf{x}'_{i+1}, \text{ and the apex } \mathbf{y}'_m, m \leq i \leq 2m-1; \quad (18)$$

$$\mathbf{y}_j, \mathbf{y}_{j+1}, \mathbf{y}'_j, \mathbf{y}'_{j+1}, \text{ and the apex } \mathbf{x}_m, 0 \leq j \leq m-1; \quad (19)$$

$$\mathbf{y}_j, \mathbf{y}_{j+1}, \mathbf{y}'_j, \mathbf{y}'_{j+1}, \text{ and the apex } \mathbf{x}'_m, m \leq j \leq 2m-1. \quad (20)$$

III. The third group of Delone tiles consists of a single simplex spanned by the 4 vertices:

$$\mathbf{x}_m = (1/2, 0, 0), \mathbf{x}'_m = (1/2, 1, 0), \mathbf{y}_m = (0, 1/2, 1), \mathbf{y}'_m = (1, 1/2, 1). \quad (21)$$

1. For the first group of  $4m^2$  polytopes it suffices to prove that the simplices presented just in (13) are Delone simplices, i.e. when  $0 \leq i, j \leq m-1$ . Then all the rest simplices from the first group (presented in (14), (15), and (16)) will be Delone simplices by symmetry arguments. Given  $i$  and  $j$ , the center  $O_{ij}$  of the circumscribed sphere for a simplex in (13) lies on the intersection line  $l$  of two planes: one of the planes is the bisector perpendicular to segment  $[i, i+1]$  on the axis  $Ox$ , another one is the bisector perpendicular to segment  $[j, j+1]$  on the axis  $Oy$ . Therefore the circumcenter  $O_{ij}$  for this simplex has coordinates

$$\left( \frac{i + \frac{1}{2}}{2m}, \frac{j + \frac{1}{2}}{2m}, z_{ij} \right), \quad 0 \leq j, j \leq m-1.$$

Now we show that  $0 < z_{ij} < 1$ . Indeed, a line  $l$  given by equations

$$x = \frac{i + \frac{1}{2}}{2m}, \quad y = \frac{j + \frac{1}{2}}{2m}$$

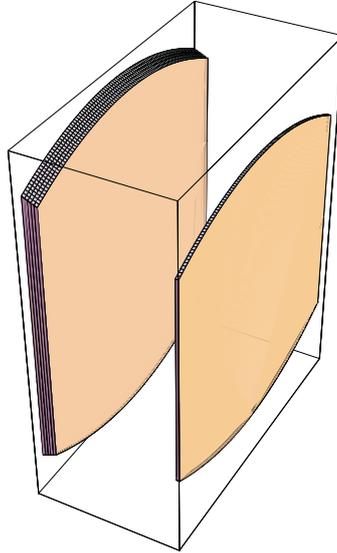


Fig. 13. Samples of Voronoi cells for the case  $m = 50$ . Five consecutive cells of the points  $((m + i)/2m, 0, 0)$ ,  $i = 3, 4, 5, 6, 7$  and one cell of the point  $((2m - 1)/2m, 0, 0)$  are shown.

crosses the bottom and top faces of the chosen cube  $K$  in points  $\mathbf{t}_0 = ((i+1/2)/2m, (j+1/2)/2m, 0)$  and  $\mathbf{t}_1 = ((i+1/2)/2m, (j+1/2)/2m, 1)$ , respectively. Consider function  $f(z) := |\mathbf{t}, \mathbf{x}_i|^2 - |\mathbf{t}, \mathbf{y}_j|^2$ , where  $\mathbf{t} = ((i+1/2)/2m, (j+1/2)/2m, z) \in l$ . It is equal to

$$f(z) = 2z - 1 + \frac{j^2 + 2j - i^2 - 2i}{4m^2}.$$

It is easy to check that  $f(0) < 0$  and  $f(1) > 0$ . Therefore,  $f(z_{ij}) = 0$  in some intermediate point  $z_{ij}$ :  $0 < z_{ij} < 1$ . This point corresponds to the circumcenter  $O_{ij}$  of a simplex in (13). Thus the circumcenter has to be located inside segment  $[0, 1]$  of line  $l$ .

Thus, the circumball has no points of  $X$  except for the four points mentioned in (13). In fact, in an open layer  $0 < z < 1$  there are no points of  $X$  at all because the  $z$ -coordinate of any point of  $X$  is integer. Now plane  $z = 1$  crosses the circumball along a circle centered at point  $((i+1/2)/2m, (j+1/2)/2m, 1)$  and coming through two points from  $X$ :  $(0, j/2m, 1)$  and  $(0, (j+1)/2m, 1)$ . Since  $0 \leq i \leq m-1$  this circle contains no other points of  $X$  in plane  $z = 1$ . The analogous fact is true for plane  $z = 0$ . Points of  $X$  which are placed on planes  $z = k$ , where integer  $k \neq 0, 1$ , do not enter the circumball at all, because the circumcenter  $O_{ij}$  lies between planes  $z = 0$  and  $z = 1$ . This geometrical fact implies that any point  $\mathbf{x} = (l/2m, n, k) \in \Lambda_0$  ( $l, n$  are integer and  $k$  is even) if  $k \neq 0$  is situated further away  $O_{ij}$  than point  $\mathbf{x}_i$ . Respectively, a point from  $\Lambda_1$ , if it does not lie in plane  $z = 1$ , is further away  $O_{ij}$  than  $\mathbf{y}_j$ . Inspect this fact for point  $\mathbf{x} \in \Lambda_0$ . Since  $k \neq 0, 1$  we have

$$|\mathbf{x}, O_{ij}|^2 = \left( \frac{l-i-\frac{1}{2}}{2m} \right)^2 + \left( n - \frac{j+\frac{1}{2}}{2m} \right)^2 + (k-z_{ij})^2 > \left( \frac{1}{4m} \right)^2 + \left( \frac{1}{4m} \right)^2 + z_{ij}^2 = |\mathbf{x}, O_{ij}|^2,$$

where  $|\mathbf{x}, O_{ij}|$  is the circumradius of the simplex spanned by vertices  $\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{y}_j, \mathbf{y}_{j+1}$ . Therefore points of  $\Lambda_0$  except for  $\mathbf{x}_i$  and  $\mathbf{x}_{i+1}$  do not enter the circumball. Consequently, simplex spanned by  $\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{y}_j, \mathbf{y}_{j+1}$  is really a Delone simplex.

2. By means of similar arguments one can establish why 4-gonal pyramids from the second group of polytopes are Delone tiles too. Indeed, as before, for all  $0 \leq i \leq m-1$ , the center  $O_i$  of the circumsphere for a pyramid in (17) lies on the intersection line of two planes: of bisector perpendicular to segment  $[i, i+1]$  on the axis  $Ox$  and of plane  $y = 1/2$ . So the circumcenter has coordinates  $((i+1/2)/2m, 1/2, z_i), 0 \leq i \leq m-1$ . And again, as before, it is easy to show that  $0 < z_i < 1$ . Therefore, all circle sections of the circumsphere by planes  $z = k, k \neq 0, 1$  are free of points of  $X$ . The corresponding circles in planes  $z = 0$  and  $z = 1$  are easily seen to come through the vertices of the pyramid and contain no other points of  $X$ . Proving that the circumball around the simplex spanned by vertices from (21) is pretty simple exercise too.

Thus, for an arbitrarily large integer  $m$  there is a Delone set  $X$  (even periodic one) such that *in any vertex of the Delone tiling for point set  $X$  at least  $4m + 6$  edges meet.*

This is equivalent to: *any cell of the Voronoi tiling has at least  $4m+6$  faces* (see Fig. 13).

## 6. Concluding Remarks

In the conclusion we again emphasize that the problem of the upper bound for the number of faces in an Euclidean spacefiller remains open. So far this question is solved for stereotopes (Sec. 2). However, even in this particular case the problem turns out to be very sensitive to conditions. As mentioned in our paper, the answer immediately alternates (no upper bound) if spaces are of either positive or negative curvature, or if in Euclidean case we cancel the strict convexity (Sec. 3). As shown in Sec. 4, no upper bound exists also if we keep the strict convexity condition but replace the congruence condition by a weaker condition: by combinatorial equivalence (Sec. 4). Moreover, no upper bound exists even in much more particular class of tilings, namely in a class of Voronoi tilings (Sec. 5).

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