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Packing and Minkowski Covering of Congruent Spherical Caps on a Sphere for $N = 2, ..., 9$

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Abstract. Let C_i ($i = 1, ..., N$) be the i-th open spherical cap of angular radius r and let M_i be its center under the condition that none of the spherical caps contains the center of another one in its interior. We consider the upper bound, r_N (not the lower bound!) of r of the case in which the whole spherical surface of a unit sphere is completely covered with N congruent open spherical caps under the condition, sequentially for $i = 2, ..., N - 1$, that M_i is set on the perimeter of C_{i-1}, and that each area of the set $(\cup_{\nu=1}^{i-1} C_\nu) \cap C_i$ becomes maximum. In this study, for $N = 2, ..., 9$, we found out that the solutions of the above covering and the solutions of Tammes problem were strictly correspondent.

1. Introduction

"How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible?" This packing problem is also called the Tammes problem and mathematically proved solutions were known for $N = 1, ..., 12$, and 24 (SCHÜTTE and VAN DER WAERDEN, 1951; DANZER, 1963; FEJES TÓTH, 1969, 1972; TESHIMA and OGAWA, 2000). On the other hand, the problem "How must the covering of a unit sphere be formed by N congruent spherical caps so that the angular radius of the spherical caps will be as small as possible?" is also important. It can be considered that this problem is dual to the problem of packing of Tammes (FEJES TÓTH, 1969). Among the problems of packing and covering on the spherical surface, the Tammes problem is the most famous. However, the systematic method of attaining these solutions has not been given.

In this paper, we would like to think of the covering in connection with the packing problem. Therefore, we consider the covering of the spherical caps such that none of them contains the center of another one in its interior. Such a set of centers is said to be a Minkowski set (FEJES TÓTH, 1999). Hereafter, we call the condition of Minkowski set of centers "Minkowski condition." In addition, the covering which satisfies the Minkowski condition is called "Minkowski covering." If angular radii of spherical caps which cover the unit sphere under the Minkowski condition are concentrically reduced to half, the resulting spherical caps do not overlap. Then, our purpose in this paper is to obtain the upper bound (not the lower bound!) of angular radius of spherical caps which cover the unit sphere under the Minkowski condition.

Suppose we have N congruent open spherical caps with angular radius r on the surface S of the unit sphere and suppose that these spherical caps cover the whole spherical surface without any gap and that the Minkowski covering is realized. Further we suppose the spherical caps are put on S sequentially in the manner which is described just below. Let C_i be the i-th open spherical cap and let M_i be its center (i = 1, ..., N). Our problem is to calculate the upper bound of r for the sequential covering, such that N congruent open spherical caps cover the whole spherical surface S under the condition that M_i is set on the perimeter of C_{i-1}, and that each area of set $(\cup_{\nu=1}^{i-1} C_\nu) \cap C_i$ become maximum in sequence for i = 2, ..., N – 1 (SUGIMOTO and TANEMURA, 2002, 2003, 2004). In this paper, we calculate the upper bound of r for N = 2, ..., 9 theoretically; the case N = 1 is self-evident. It is shown in this paper that the solutions of our problem are strictly correspondent to those of the Tammes problem for N = 2, ..., 9. Further, it should be said that our method is a systematic and a different approach to the Tammes problem from the works by SCHÜTTE and VAN DER WAERDEN (1951), etc.

In Sec. 2, to solve our problem, we consider the properties of spherical caps under the Minkowski condition. Then, we explain the procedure of our sequential covering and define the upper bouds r_N and $\bar{r}_{N-1}$. In Sec. 3 we give the solutions of our problem successively for N = 2, ..., 9. Our conclusion is summarized in Sec. 4.

2. Sequential Covering

Throughout this paper we assume that the center of the unit sphere is the origin O = (0, 0, 0). Hereafter, we represent the surface of this unit sphere by the symbol S. In the following, open spherical caps are simply written as spherical caps unless otherwise stated. We define the geodesic arc between an arbitrary pair of points T_i and T_j on S as the inferior arc of the great circle determined by T_i and T_j. Then the spherical distance between T_i and T_j is defined by the length of geodesic arc of this pair of points.

2.1. Relationship between the kissing number and the angular radius of spherical caps

First of all, we consider how many centers of congruent spherical caps can be placed on the perimeter of a spherical cap under the Minkowski condition. This problem is related to the kissing number of spherical caps. In the plane, one circle can contact simultaneously with six other congruent circles. Then the kissing number is always six in the plane. On the sphere, on the contrary, the kissing number of a circle (spherical cap) on the spherical surface changes with its angular radius. Its maximum value is five as we will see below. Here, we define a "half-cap" as the spherical cap whose angular radius is $r/2$ and which is concentric with that of the original cap. When the kissing number is k, there can k half-caps contact with the central half-cap and there is no space for another half-cap to enter. At this instance, let us increase r until the peripheries of k half-caps contact with one another. Then, if the centers of two half-caps in contact are joined by a geodesic arc, there arise k spherical equilateral triangles of side-length r and of inner angle $\sigma = 2\pi/k$ around the central

Table 1. Relationship between k and r.

k	Range of r
1	$(2\pi/3, \pi]$
2	$(\pi - \cos^{-1}(1/3), 2\pi/3]$
3	$(\pi/2, \pi - \cos^{-1}(1/3)]$
4	$(\tan^{-1}2, \pi/2]$
5	$(0, \tan^{-1}2]$

half-cap. Thus by applying the spherical cosine theorem to one of the spherical equilateral triangles, we have

$$r = \cos^{-1}\left(\frac{\cos\sigma}{1-\cos\sigma}\right). \tag{1}$$

By substituting $\sigma = 2\pi/k$ into (1), we can calculate the upper bound value of r for a given kissing number k (>1). Note that, for $k = 1$, it is clear that the upper bound value is π. For $k \geq 6$, Eq. (1) cannot have a solution. Namely, for $k = 6$, we get at once $r = \cos^{-1}1 = 0$, and for $k > 6$, we get the inequality $\cos\sigma/(1-\cos\sigma) > 1$ indicating that no real value solution exists for r. Therefore the maximum value of the kissing number of a spherical cap (circle on the spherical surface) is five. As a result, we obtain the relationship between k and r as is shown in Table 1 (SUGIMOTO and TANEMURA, 2001, 2002, 2003; SUGIMOTO, 2002). For example, from Table 1, the kissing number k is four when the angular diameter of a spherical cap is in the range $(\tan^{-1}2, \pi/2]$. Note that inclusion of the upper bound in the range of r in Table 1 is correspondent to the Minkowski condition.

Next, let us consider the packing with half-caps and the covering with corresponding spherical caps (of angular radius r) where each half-cap is concentric with correspondent spherical caps. Then it is observed that this covering satisfies the Minkowski condition; namely, this is the Minkowski covering. For the covering that the centers of spherical caps are chosen on the perimeters of other spherical caps under the Minkowski condition, we see from Table 1 that, for example, four spherical caps can be placed on the perimeter of a spherical cap when angular radius is in the range $(\tan^{-1}2, \pi/2]$. We note that the discussions of this subsection are valid for both of open and closed spherical caps.

2.2. Overlapping area of congruent spherical caps

In order to solve our problem, we consider the overlapping area of congruent spherical caps under the Minkowski condition.

Assume C_i and C_j be two congruent spherical caps, of angular radius r, which are mutually overlapping under the Minkowski condition, and let $A_{ij} = A(C_i \cap C_j)$ be the overlapping area where $A(X)$ is the area of X. Further, when T_i and T_j are the points on S, let $d_s(T_i, T_j)$ denote the spherical distance between points T_i and T_j. Especially, we denote $s_{ij} = d_s(M_i, M_j)$ as the spherical distance between centers of C_i and C_j ($r \leq s_{ij} \leq 2r$). Then let

h_{ij} be the spherical distance between cross points of perimeters of C_i and C_j. By using the spherical cosine formula about a spherical right triangle, we have

$$h_{ij} = 2\cos^{-1}\left(\frac{\cos r}{\cos\left(s_{ij}/2\right)}\right).$$

Then the overlapping area of C_i and C_j is given by

$$A_{ij} = -2\cos^{-1}\left(\frac{\cos h_{ij} - \cos^2 r}{\sin^2 r}\right)\cos r - 4\cos^{-1}\left(\frac{1-\cos h_{ij}}{\tan r \cdot \sin h_{ij}}\right) + 2\pi. \tag{2}$$

For the detailed derivation of (2), see Appendix A.2. It is obvious that A_{ij} is a continuous function of r and s_{ij} (SUGIMOTO and TANEMURA, 2001, 2002; SUGIMOTO, 2002). Then, we get the following Lemma (SUGIMOTO and TANEMURA, 2004).

Lemma. *If the range of angular radius r is $0 < r \le 2\pi/3$, then the overlapping area A_{ij} of the set $C_i \cap C_j$ is a monotone decreasing function of $s_{ij} = d_s(M_i, M_j)$ when r is fixed. Then, A_{ij} is maximum when $s_{ij} = r$.*

Proof. At first, we show that the range of r should be $0 < r \le 2\pi/3$ in order A_{ij} to be a function of s_{ij}. Under the Minkowski condition, if $2\pi/3 < r \le \pi$, the area of the set $C_i \cap C_j$ is always constant $4\pi \cos r$ and, if $r > \pi$, two spherical caps cannot be put on the spherical surface S. Therefore, the range of angular radius is limited to $0 < r \le 2\pi/3$. Let $G(a)$ and $G(b)$ be the set $C_i \cap C_j$ for $s_{ij} = a$ and b ($a < b$), respectively. We assume that C_i and C_j contact with each other first. Let e be the geodesic arc between fixed centers of these spherical caps. Next, let C_i be fixed and let us move C_j by moving M_j along e toward M_i. Then, it is obvious that $G(a) \supset G(b)$ holds during this movement. Therefore, for r fixed, A_{ij} is a monotone decreasing function of s_{ij}. Thus, A_{ij} is maximum when $s_{ij} = r$. □

On the contrary, A_{ij} is a monotone increasing function of h_{ij} when r is fixed. Further, when $s_{ij} = r$, we find that the area of set $C_i \cup C_j$ is minimum.

Let N be the number of spherical caps when the whole spherical surface is completely covered. And, let ∂C_i be the perimeter of C_i ($i = 1, ..., N$). Then we define:

$$W_i = \left\{ W_{i-1} \cup C_i \;\middle|\; \max_{M_i \in \partial C_{i-1}} A\left(W_{i-1} \cap C_i\right) \right\}, \quad i = 2, \ldots, N-1;\; W_1 = C_1. \tag{3}$$

In other words, W_i is the union of W_{i-1} and C_i satisfying the condition that the area $A(W_{i-1} \cap C_i)$ is maximum with the restriction $M_i \in \partial C_{i-1}$ (SUGIMOTO and TANEMURA, 2002, 2003). Hereafter, we call that "$\cup_{v=1}^{i} C_v$ is in an extreme state" when the set of spherical caps $C_1, ..., C_i$ possesses the property (3). We are always necessary to examine

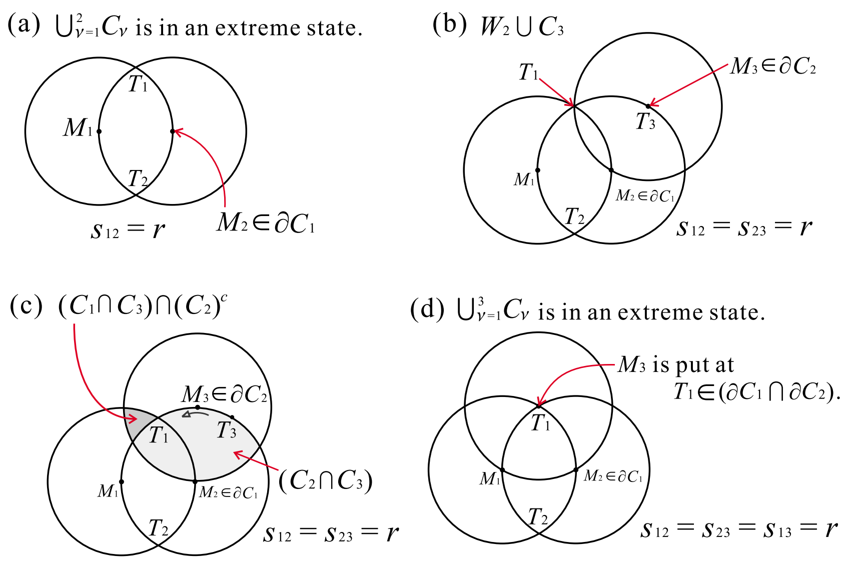

Fig. 1. The sketch of W_2 and $W_2 \cup C_3$.

whether $\bigcup_{\nu=1}^{i} C_\nu$ is in an extreme state in the sequential covering procedure of our problem mentioned in Sec. 1. For this purpose, we calculate the area $A(W_{i-1} \cap C_i)$ by using (2) and the area formula of spherical triangle. Here, we consider W_2 defined in (3). From Lemma, the area $A(C_1 \cap C_2)$ will become maximum when M_2, the center of C_2, is put on ∂C_1. Therefore, for $i = 2$, the set $W_2 = \bigcup_{\nu=1}^{2} C_\nu$ is in an extreme state when $s_{12} = r$ for $0 < r \le 2\pi/3$ as shown in Fig. 1(a).

Next, we consider W_3. In order to make the situation that $\bigcup_{\nu=1}^{3} C_\nu$ is in an extreme state ($A(W_2 \cap C_3)$ is maximum with the restriction $M_3 \in \partial C_2$), from the definition of (3), we can assume $N \ge 4$. Therefore, the whole spherical surface must be covered by four or more spherical caps under the Minkowski condition. Then, we present the following theorem (SUGIMOTO and TANEMURA, 2003).

Theorem 1. *If the range of angular radius r is* $0 < r \le \pi - \cos^{-1}(1/3)$, *then* $\bigcup_{\nu=1}^{3} C_\nu$ *is in an extreme state when* $s_{12} = s_{13} = s_{23} = r$.

Proof. We first examine the range of r. From the consideration of kissing number in the foregoing Subsection, four or more spherical caps cannot be placed on S under the Minkowski condition when $r > \pi - \cos^{-1}(1/3)$. Therefore, the range of r should be $0 < r \le \pi - \cos^{-1}(1/3)$.
From Lemma, $\bigcup_{\nu=1}^{2} C_\nu$ is in an extreme state when $s_{12} = r$ (see Fig. 1(a)). Namely, at this time, $\bigcup_{\nu=1}^{2} C_\nu$ is identical to W_2. Then, we define T_1 and T_2 as the two cross points of perimeters ∂C_1 and ∂C_2. Now, from (3), the center M_3 is set on the perimeter of C_2 outside

of C_1 and we need to consider the area $A(W_2 \cap C_3)$. At this time, $s_{23} = r$. Hence, from Lemma, the area $A_{23} = A(C_2 \cap C_3)$ is always fixed and maximum for any $M_3 \in \partial C_2$. Here, as shown in Fig. 1(b), let T_3 be the fixed point on ∂C_2 such that $d_s(T_1, T_3) = r$ and that it is outside C_1. First, we put M_3 at T_3. Next, let us move M_3 along ∂C_2 toward T_1. Then, as shown in Fig. 1(c), we see the relation $A(W_2 \cap C_3) = A((C_1 \cap C_3) \cap (C_2)^c) + A_{23}$ holds. Therefore, under the condition $M_3 \in \partial C_2$, $A(W_2 \cap C_3)$ is maximum when $A((C_1 \cap C_3) \cap (C_2)^c)$ attains its maximum. Then, we seek for the position of M_3 when $A((C_1 \cap C_3) \cap (C_2)^c)$ is maximum. Let $G(a)$ and $G(b)$ be the set $(C_1 \cap C_3) \cap (C_2)^c$ for $s_{13} = a$ and b $(a < b)$, respectively. Then, it is obvious that $G(a) \supset G(b)$ holds. Therefore, for r fixed, $A((C_1 \cap C_3) \cap (C_2)^c)$ is a monotone decreasing function of s_{13}. Hence, $A((C_1 \cap C_3) \cap (C_2)^c)$ is maximum when M_3 is put at T_1. Therefore, $A(W_2 \cap C_3)$ attains its maximum when M_3 is selected on T_1. Then, $s_{13} = r$ holds. Thus, as shown in Fig. 1(d), $\cup_{\nu=1}^{3} C_\nu$ is in an extreme state when $s_{12} = s_{13} = s_{23} = r$ for $0 < r \le \pi - \cos^{-1}(1/3)$. □

2.3. Procedure of sequential covering

As mentioned in Sec. 1, our problem is to calculate the upper bound of r for the sequential covering, such that N congruent open spherical caps cover the whole spherical surface under the condition that M_i is set on the perimeter ∂C_{i-1}, and that each area $A(W_{i-1} \cap C_i)$ becomes maximum in sequence for $i = 2, ..., N - 1$. Note that, although N spherical caps are needed in our problem, $N - 1$ spherical caps are used in the sequential covering since we want to make the situation that $\cup_{\nu=1}^{N-1} C_\nu$ is in an extreme state. First, before beginning covering, the angular radius r of spherical caps is chosen sufficiently small so that $\cup_{\nu=1}^{N-1} C_\nu$ cannot cover the whole spherical surface in the Minkowski covering. Note that, in the result which will be obtained in the procedure below, the set W_{N-1} does not cover the whole spherical surface. Algorithmically, the procedure of sequential covering is the followings (SUGIMOTO and TANEMURA, 2003):

STEP 1: The center M_1 of the first spherical cap C_1 is put at $(x, y, z) = (0, 0, -1)$. Then, from (3), $W_1 = C_1$ holds.

STEP 2: The center M_2 of C_2 is put at a certain point on the perimeter ∂C_1. As a result, $\cup_{\nu=1}^{2} C_\nu$ is in an extreme state ($A(W_1 \cap C_2)$ is maximum with the restriction $M_2 \in \partial C_1$) since $s_{12} = r$. If $(N - 1) \ge 3$, go to the next step; otherwise the procedure ends.

STEP 3: The center M_3 of C_3 is put on one of the cross points of ∂C_1 and ∂C_2. Then $\cup_{\nu=1}^{3} C_\nu$ is in an extreme state since $s_{12} = s_{13} = s_{23} = r$. If $(N - 1) \ge 4$, put $i = 4$ and go to the next step; otherwise the procedure ends.

STEP 4: First, the center M_i of C_i is placed at a certain point on the acceptable part of ∂C_{i-1} which is outside the other spherical caps. Next, move M_i among the acceptable points on ∂C_{i-1} and compute $A(W_{i-1} \cap C_i)$ for respective points of M_i. Then, M_i is fixed at the position where $A(W_{i-1} \cap C_i)$ attains its maximum (i.e. the set W_i is formed on S). Go to STEP 5.

STEP 5: If $i = N - 1$, the procedure ends; otherwise put $i \leftarrow i + 1$ and go to STEP 4.

Therefore, our sequential covering satisfies the condition that, sequentially for $i = 2, ..., N - 1$, each $A(W_{i-1} \cap C_i)$ is maximum with the restriction $M_i \in \partial C_{i-1}$ ($\cup_{\nu=1}^{i} C_\nu$ is in an extreme state).

2.4. Upper bounds r_N and $\bar{r}_{N-1}$

We define r_N as the upper bound of r which is mentioned at the top of Subsec. 2.3. Next, we define another upper bound of r, $\bar{r}_{N-1}$, such that the set $\cup_{\nu=1}^{N-1} C_\nu$ which contains W_{N-2} cannot cover S under the Minkowski condition. Then $\bar{r}_{N-1}$ should be equal to the spherical distance of the largest interval in the uncovered region $(W_{N-2})^c$ of S. It is because, when the angular radius r is equal to $\bar{r}_{N-1}$, the set $\cup_{\nu=1}^{N-1} C_\nu$ which contains W_{N-2} can cover S except for a finite number of points or a line segment under our sequential covering. Therefore, $\cup_{\nu=1}^{N-1} C_\nu$ is in an extreme state if and only if at least one endpoint of the interval, which has the above mentioned spherical distance $\bar{r}_{N-1}$, comes on the perimeter ∂C_{N-2}. Further, when there are two or more uncovered points, the spherical distance of any pair of these uncovered points is less or equal to $\bar{r}_{N-1}$ since the largest interval is assumed to be $\bar{r}_{N-1}$. Then, we can put the center M_N of C_N at one of the uncoverd points. At this moment, we see that $\cup_{\nu=1}^{N} C_\nu$ which contains W_{N-1} covers S without any gap. Then, we notice the fact that r_N is equal to $\bar{r}_{N-1}$.

Therefore, for our problem, it is necessary to know the spherical distance of the largest interval in the uncovered region $(W_{N-2})^c$. In our cases in Sec. 3, it becomes important to consider triangles or quadrangles as the shape of $(W_{N-2})^c$ in the final steps of sequential covering. For this purpose, we investigate the farthest pair of points in spherical triangle and quadrangle so that it is useful for later considerations.

First, we consider the spherical triangle. Let T_1, T_2, and T_3 be the points on S and let the spherical triangle $T_1T_2T_3$ be such that the side T_1T_2 of the triangle $T_1T_2T_3$ is an inferior arc of the great circle determined by T_1 and T_2: namely the side T_1T_2 is the geodesic arc and satisfies $0 < d_s(T_1, T_2) \le \pi$. Next, we define the point H as the middle point of the geodesic arc T_1T_2. Then, we get the position of H as follows:

$$\mathbf{H} = \frac{\mathbf{T}_1 + \mathbf{T}_2}{2\cos\left(\cos^{-1}\left((\mathbf{T}_1, \mathbf{T}_2)\right)/2\right)}$$

where a bold symbols $\mathbf{H}$, $\mathbf{T}_1$, and $\mathbf{T}_2$ are unit vectors from the origin O to the points H, T_1, and T_2 on the unit sphere, respectively, and where $(\mathbf{T}_1, \mathbf{T}_2)$ is the inner product of vectors $\mathbf{T}_1$ and $\mathbf{T}_2$. Then, we get the following theorem.

Theorem 2. *If the spherical triangle $T_1T_2T_3$ satisfies the condition $\pi/2 \ge d_s(T_1, H) = d_s(H, T_2) = d_s(T_1, T_2)/2 \ge d_s(H, T_3) > 0$, then the farthest pair of points inside the spherical triangle $T_1T_2T_3$ is the pair of T_1 and T_2.*

Proof. Let C' be the closed spherical cap with its center at H and with its angular radius as $d_s(T_1, H)$. From the condition that $d_s(T_1, H) \ge d_s(H, T_3)$, T_3 is inside C'. In order to prove Theorem 2, we need to consider three cases: (I) $d_s(T_1, H) = d_s(H, T_3) = \pi/2$ and $T_3 \in \partial C'$; (II) $d_s(T_1, H) = d_s(H, T_3) < \pi/2$ and $T_3 \in \partial C'$; (III) $d_s(T_1, H) \le \pi/2$ and $T_3 \notin \partial C'$. First, for the case (I), let T_1, T_2, and H be (0, 1, 0), (0, –1, 0) and (0, 0, 1), respectively. Namely, the geodesic arc T_1T_2 is the half of the great circle. Therefore, the perimeter of C' is the equator of unit sphere. Then, all the sides of spherical triangle $T_1T_2T_3$ are the geodesic arcs.

Furthermore, the spherical triangle $T_1T_2T_3$ is formed with two sectors T_1HT_3 and T_2T_3H, and is contained in C'. Therefore, it is clear that the spherical distance $d_s(T_1, T_2) = \pi$ is the largest distance. Next, for the case (II), $d_s(T_1, T_3)$ is shorter than the length of arc T_1T_3 of perimeter $\partial C'$ and the geodesic arc T_1T_3 is inside of C'. Therefore, the spherical triangle T_1HT_3 is contained in the sector T_1HT_3. Hence, it is entirely contained in C'. Similarly, we find that the spherical triangle T_2T_3H is contained in C'. Thus, the closed spherical cap C' entirely contains the spherical triangle $T_1T_2T_3$. The proof for the case (III) is the same procedure as in the case (II). Therefore, in the cases (II) and (III), the spherical distance $d_s(T_1, T_2)$ is the largest distance. Thus, the farthest pair of points in spherical triangle $T_1T_2T_3$ is the pair of T_1 and T_2. □

Note that if the condition of Theorem 2 is not satisfied, the claim of Theorem 2 does not hold. For example, when the spherical triangle is an equilateral triangle of side lengths between $\pi/2$ and π, its height is larger than the sides: namely the farthest pair of points in spherical equilateral triangle is not the pairs of spherical triangular vertices.

Next, we consider the case of quadrangle. Let us given the spherical triangle $T_1T_2T_3$ which satisfies the condition of Theorem 2. Then, we add the point T_4 on S to the opposite side of T_3 against the great circle which passes T_1 and T_2. By joining T_4 with T_1 and T_2 by the geodesic arcs, respectively, we make the spherical quadrangle $T_1T_4T_2T_3$ that is formed with two spherical triangles $T_1T_2T_3$ and $T_1T_4T_2$. Then, we have the following Corollary from Theorem 2.

Corollary. *If the spherical quadrangle $T_1T_4T_2T_3$ satisfies the following conditions*

$$\begin{cases} d_s(T_1, T_2) \ge d_s(T_3, T_4), \\ \dfrac{\pi}{2} \ge d_s(T_1, H) = d_s(H, T_2) = \dfrac{d_s(T_1, T_2)}{2} \ge d_s(H, T_3) > 0, \\ d_s(T_1, H) \ge d_s(H, T_4) > 0, \end{cases}$$

then the farthest pair of points in the spherical quadrangle $T_1T_4T_2T_3$ is the pair of T_1 and T_2.

Proof. Let C' be the closed spherical cap whose center is H and whose angular radius is $d_s(T_1, H)$. From Theorem 2, the spherical triangle $T_1T_2T_3$ is entirely contained in C' and its farthest pair is the pair of T_1 and T_2. Similarly, the spherical triangle $T_1T_4T_2$ is contained in C' and its farthest pair is the pair of T_1 and T_2. Therefore, the spherical quadrangle $T_1T_4T_2T_3$ is entirely contained in C' whose angular diameter is determined by $d_s(T_1, T_2)$. Thus, the pair of T_1 and T_2 is the farthest pair of points in spherical quadrangle $T_1T_4T_2T_3$. □

Now, we consider the meanings of the upper bounds r_N and $\bar{r}_{N-1}$ in an illustrative example. Figure 2 shows the distinction between two upper bounds r_N and $\bar{r}_{N-1}$. In Fig.

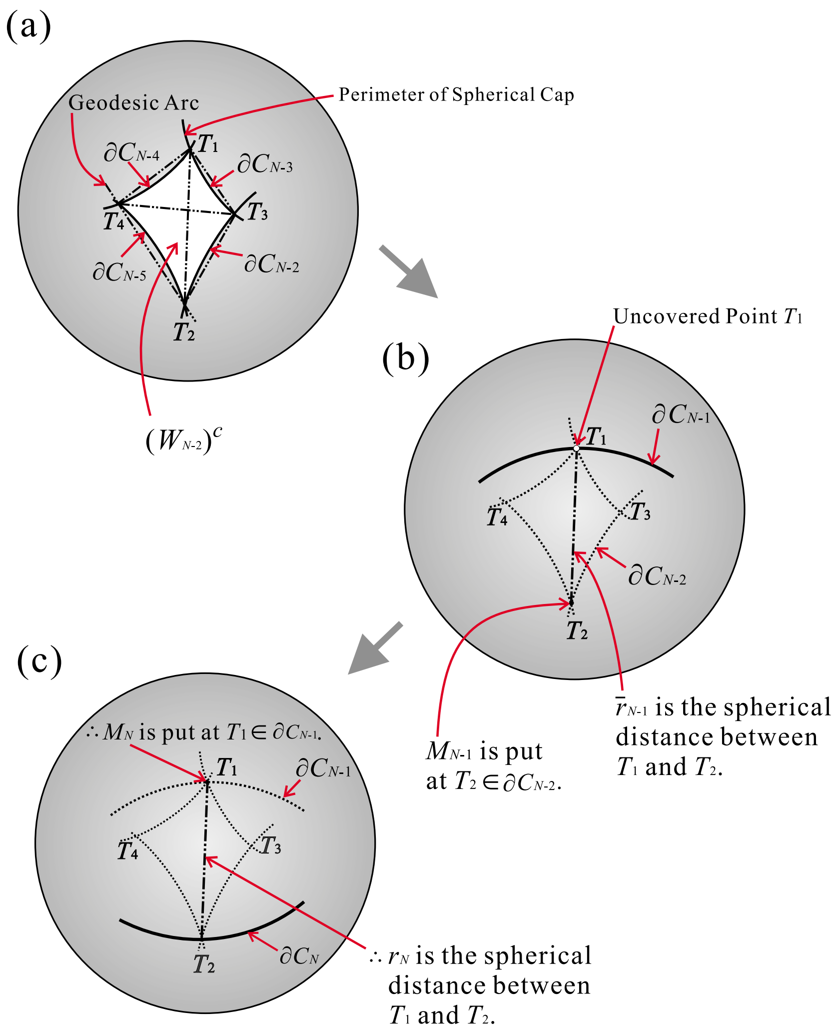

Fig. 2. Upper bounds r_N and $\bar{r}_{N-1}$.

2(a), the shaded region is the covered region W_{N-2} and the white region is the uncovered region $(W_{N-2})^c$. At this time, the shape of $(W_{N-2})^c$ is the quadrangle $T_1T_4T_2T_3$ on S. We note that the sides of uncovered region $(W_{N-2})^c$ are not geodesic arcs but are perimeters of spherical caps. Here, we suppose that the spherical quadrilateral $T_1T_4T_2T_3$ (the quadrilateral enclosed by dotted-and-dashed segments in Fig. 2(a)) satisfies the condition of Corollary of Theorem 2. Then, it is obvious that the quadrangle $T_1T_4T_2T_3$ (the quadrangle enclosed by solid segments in Fig. 2(a)) on S is inside the spherical quadrilateral $T_1T_4T_2T_3$. Therefore, from Corollary of Theorem 2, we find that the pair of T_1 and T_2 is the farthest pair of points in the spherical quadrilateral $T_1T_4T_2T_3$ and in the quadrangle $T_1T_4T_2T_3$ on S.

Namely, the spherical distance of the largest interval in $(W_{N-2})^c$ is $d_s(T_1, T_2)$. Then, when the radii r of $N-1$ spherical caps are all equal to $d_s(T_1, T_2)$ and the center M_{N-1} is put on T_1 or T_2, the set $\cup_{\nu=1}^{N-1} C_\nu$ which contains W_{N-2} can cover S except for a point (T_2 or T_1). Therefore, the upper bound, $\bar{r}_{N-1}$, of r such that the set $\cup_{\nu=1}^{N-1} C_\nu$ which contains W_{N-2} cannot cover S under the Minkowski condition is $d_s(T_1, T_2)$. At this time, as shown in Fig. 2(b), if $T_2 \in \partial C_{N-2}$ is selected as M_{N-1}, the set $\cup_{\nu=1}^{N-1} C_\nu$ is in an extreme state where T_1 is the unique uncovered point. Therefore, when M_N is put at $T_1 \in \partial C_{N-1}$, the set $\cup_{\nu=1}^{N} C_\nu$ which contains W_{N-1} covers S without any gap. Thus, as shown in Fig. 2(c), our upper bound r_N is equal to $d_s(T_1, T_2)$. Namely, in Fig. 2, we see the fact that $r_N = \bar{r}_{N-1} = d_s(T_1, T_2)$.

Here, we note the advantage of using the upper bound $\bar{r}_{N-1}$. The value of $\bar{r}_{N-1}$ is easier to calculate than r_N, since it is better to examine the extreme situation where the spherical surface S cannot be covered by $N-1$ spherical caps than the situation where S is covered by N caps. Then, at the last stage of the process of covering, we only need to observe the situation where a few uncovered regions remain since our covering is sequential. Moreover, the covering of our problem is finished in fact when $N-1$ spherical caps cover S except for a finite number of points or a line segment since open spherical caps are considered. In such a case, as shown in the cases below, the position of the center of the N-th spherical cap is almost unique. Although $\bar{r}_{N-1}$ and r_N are not necessarily coincident, the value of $\bar{r}_{N-1}$ will give a strong candidate for r_N.

3. Results

3.1. N = 2, 3, 4, 5 and 6

For cases of $N = 2, \ldots, 6$, we find that r_N is equal to the upper bound of the range of r for k in Table 1.

For $N = 2$, r_2 (i.e. r_N for $N = 2$) is equal to π. It is because that the radius of the first spherical cap C_1 should be equal to $\bar{r}_1 = \pi$ in order C_1 to cover the whole S except for a point. From the STEP 1 as described in Subsec. 2.3, the center M_1 of C_1 is put at the south pole $(0, 0, -1)$. In this case, the north pole $(0, 0, 1)$ is open for $\bar{r}_1 = \pi$. Then it is obvious that we can put M_2, the center of the second spherical cap C_2, at the north pole. At this time, S is covered without any gap by C_1 and C_2 which satisfy the Minkowski condition. Thus we see $r_2 = \bar{r}_1 = \pi$. We note that this value π is the upper bound of the range of r for $k = 1$ as shown in Table 1.

For $N = 3$, r_3 should be equal to the upper bound of the range of r for $k = 2$. It is because that, when $r = 2\pi/3$, the set $\cup_{\nu=1}^{2} C_\nu$ under the condition $M_2 \in \partial C_1$ can cover S except for a point. First, let M_1 be the south pole as described above. Then, if M_2 is put at the point $(\sin(2\pi/3), 0, -\cos(2\pi/3)) = (\sqrt{3}/2, 0, 1/2)$ on ∂C_1, the unique uncovered point P of S will be $(-\sqrt{3}/2, 0, 1/2)$. Then, $\cup_{\nu=1}^{2} C_\nu$ is in an extreme state. Therefore, we get $\bar{r}_2 = 2\pi/3$, and we can put the center M_3 of C_3 at P. At this time, $\cup_{\nu=1}^{3} C_\nu$ which contains W_2 covers the whole of S under the Minkowski condition. It is obvious that the position of centers is the trisection point of a great circle. Thus the correspondent angular radius r_3 is equal to $2\pi/3$.

For $N = 4$, r_4 should be equal to the upper bound of the range of r for $k = 3$. The reason is the following. First, we can assume that M_3 is put on one of the cross points of perimeters ∂C_1 and ∂C_2 under the condition $M_2 \in \partial C_1$. It is because that, from Theorem 1 in Subsec.

2.2, $\cup_{v=1}^{3} C_v$ is in an extreme state when $s_{12} = s_{13} = s_{23} = r$. If r is equal to the spherical distance between the cross points of ∂C_1 and ∂C_2, the spherical surface S except for a point is covered by the set W_3. As before, let M_1 be the south pole and let M_2 be at $(\sin r, 0, -\cos r)$. At this time, let us assume that the angular radius r is equal to the upper bound of the range r for $k = 3$ in Table 1. Thus, when $r = \pi - \cos^{-1}(1/3)$, M_3 is selected as one of the trisection point of ∂C_1 and let it be $((-1/2)\sin r, (-\sqrt{3}/2)\sin r, -\cos r) = (-\sqrt{2}/3, -\sqrt{2}/\sqrt{3}, 1/3)$. Then, it is easy to see that the point $P = ((-1/2)\sin r, (\sqrt{3}/2)\sin r, -\cos r) = (-\sqrt{2}/3, \sqrt{2}/\sqrt{3}, 1/3)$ is the unique uncovered point on S. We note, at the same time, that the coordinates of M_2 turns out to be $(2\sqrt{2}/3, 0, 1/3)$. Therefore, we get $\bar{r}_3 = \pi - \cos^{-1}(1/3)$, and we can put M_4 on that point P. As a result, the set $\cup_{v=1}^{4} C_v$ which contains W_3 can cover the whole of S when $r = \bar{r}_3$ and we get finally $r_4 = \pi - \cos^{-1}(1/3)$. Then, we find that the position of centers of these four spherical caps is in accord with the vertices of regular tetrahedron.

For $N = 5$, before deriving r_5, $\bar{r}_4 = \pi/2$ is shown first. When $r = \pi/2$, from Theorem 1, the centers M_1, M_2, and M_3 are put, for example, at $(0, 0, -1)$, $(1, 0, 0)$, and $(0, -1, 0)$, respectively, in order for $\cup_{v=1}^{3} C_v$ to be in an extreme state. Then, the uncovered region $(W_3)^c$ is the spherical equilateral triangle of side-length $\pi/2$ and vertices $(0, 0, 1)$, $(-1, 0, 0)$, and $(0, 1, 0)$. Therefore, from the discussion in Subsec. 2.4, we find that the largest spherical distance in $(W_3)^c$ is equal to $\pi/2$; namely $\bar{r}_4 = \pi/2$. Next, when M_4 is put on the point $(0, 0, 1) \in \partial C_3$ or $(-1, 0, 0) \in \partial C_3$ (in this paper, we choose $(-1, 0, 0)$), the set $\cup_{v=1}^{4} C_v$ is in an extreme state. As a result, the set W_4 can cover S except for a line segment which is an inferior great circle connecting $(0, 1, 0)$ and $(0, 0, 1)$ and whose length is $\pi/2$. Therefore, when M_5 is put at any point on this line segment except for points $(0, 1, 0)$ and $(0, 0, 1)$, we find that $\cup_{v=1}^{5} C_v$ covers S without gap under the Minkowski condition. Thus, $r_5 = \bar{r}_4 = \pi/2$. Namely, r_5 is equal to the upper bound of the range of r for $k = 4$.

For $N = 6$, we can assume that r_6 is equal to the result of the case $N = 5$. First, similar to the case $N = 5$, we put the centers of C_1, C_2, C_3, and C_4 of radius $r = \pi/2$ at $(0, 0, -1)$, $(1, 0, 0)$, $(0, -1, 0)$, and $(-1, 0, 0)$, respectively. If we put M_5 on $(0, 0, 1)$ or $(0, 1, 0)$ (in the following, we assume $(0, 1, 0)$ is chosen as M_5), the set $\cup_{v=1}^{5} C_v$ is in an extreme state and the spherical surface S except for a point, namely the point $(0, 0, 1)$, can be covered by the set W_5. As a result, the set $\cup_{v=1}^{6} C_v$ containing W_5 covers the whole of S when we put M_6 on $(0, 0\ 1)$. Thus, our assumption $r_5 = r_6 = \pi/2$ is confirmed. Then, we find that the position of centers of these spherical caps for $N = 6$ is in accord with the vertices of regular octahedron. Therefore, if all spherical caps of our covering for $N = 6$ are replaced by half-caps (the spherical cap whose angular radius is $r/2$ and which is concentric with that of the original cap), all of those half-caps contact other four half-caps and there is no space for those half-caps to move.

3.2. $N = 7$

From the considerations for the cases $N \le 6$ and Subsec. 2.1, we can assume that, for $N = 7$, the angular radius r should satisfy the inequalities $\tan^{-1}2 < r < \pi/2$. The reason is the following. First, the center M_1 of the first spherical cap C_1 is set at the south pole $(0, 0, -1)$ as before. Now, we remind the cases of $N \le 6$ in Subsec. 3.1, in order to investigate the position of centers of spherical caps for $N = 7$. For $N = 2$, M_2 is placed on the perimeter ∂C_1. For $N = 3$, M_2 and M_3 are both on ∂C_1. For $N = 4$, the centers M_2, M_3, and M_4, are placed on ∂C_1. For $N = 5$ and 6, there are four centers of other spherical caps on ∂C_1. Then, from these facts and from Subsec. 2.1, we can consider two cases for $N = 7$ as follows: four centers M_2, M_3, M_4, and M_5 are on ∂C_1; and five centers M_2, M_3, M_4, M_5, and M_6 are on ∂C_1. For $N = 7$, we find that the second case is excluded because of the following reason. When five centers M_2, M_3, M_4, M_5, and M_6 are placed on ∂C_1 according to our sequential covering, the range of angular radius r must be $0 < r \le \tan^{-1}2$ from the consideration in Subsec. 2.1. Here, we consider the case that r is equal to $\tan^{-1}2 \approx 1.10715$. Then, it is obvious that the spherical distance between M_1: (0, 0, –1) and any point of the set which is covered by our six spherical caps is smaller than $2\tan^{-1}2$, while the length of longitude line joining south and north poles is larger than $2\tan^{-1}2$. Thus, the union of our six spherical caps would leave a big open area (whose size is at least comparable to the area of spherical cap of angular radius $\pi - 2\tan^{-1}2$) near the north pole of the unit sphere. Furthermore, the open area near the north pole would become still bigger for $0 < r < \tan^{-1}2$. Hence, the set $\cup_{\nu=1}^{6} C_\nu$ cannot cover S at all. Thus, we should exclude the second case. Therefore, we have to consider the first case that four centers M_2, M_3, M_4, and M_5 of spherical caps are placed on ∂C_1. Then, below, we investigate the range of r under this condition. From the consideration in Subsec. 2.1, when $r \le \pi/2$, it is possible to put four spherical caps on the perimeter of a spherical cap. Especially, when the range of r is $(\tan^{-1}2, \pi/2]$, four spherical caps can be placed on the perimeter of a spherical cap and, at the same time, five spherical caps cannot be placed on the perimeter of a spherical cap. Further, from the setup of our problem, we want to make r the biggest possible. Therefore, we can assume that r is larger than $\tan^{-1}2$. On the other hand, if $r \ge \pi/2$, we cannot cover S by seven spherical caps without breaking the Minkowski condition due to the results of cases $N \le 6$. Thus, r should be in the range $\tan^{-1}2 < r < \pi/2$. We note that the equality sign does not enter in these inequalities.

Let (x, y, z) be the coordinates of cross points where the perimeters of C_a (the coordinates of the center: (a_1, a_2, a_3)) and C_b (the coordinates of the center: (b_1, b_2, b_3)) intersect. By solving the simultaneous equations

$$\begin{cases} a_1x + a_2y + a_3z = \cos r, \\ b_1x + b_2y + b_3z = \cos r, \\ x^2 + y^2 + z^2 = 1, \end{cases} \tag{4}$$

we will have the coordinates of the cross points. Thus, in the case where the centers of two spherical caps are put respectively at (0, 0, –1) and $(\sin r, 0, -\cos r)$, we get

$$\begin{cases} -z = \cos r, \\ \sin r \cdot x - \cos r \cdot z = \cos r, \\ x^2 + y^2 + z^2 = 1. \end{cases} \tag{5}$$

Let the points $K_1 = (x_1, y_1, z_1)$ and $K_2 = (x_2, y_2, z_2)$ be the solutions of simultaneous equation (5). Solving (5), we obtain

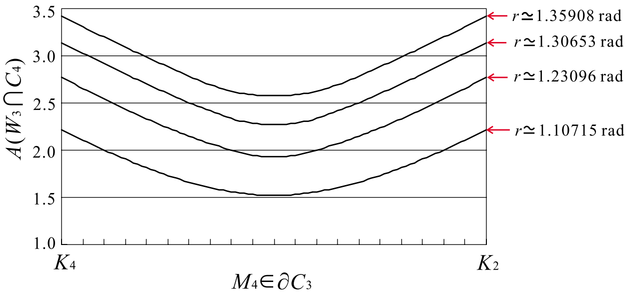

Fig. 3. The curve of $A(W_3 \cap C_4)$ when M_4 is moved on the arc K_4K_2 of C_3. Here, the arc K_4K_2 is divided into 100 equal intervals and the area $A(W_3 \cap C_4)$ is calculated on 101 end points of the intervals. Note that the curve of $r \simeq 1.10715$ corresponds to the case of $r = \tan^{-1}2$. The values of r of other curves are described in the text.

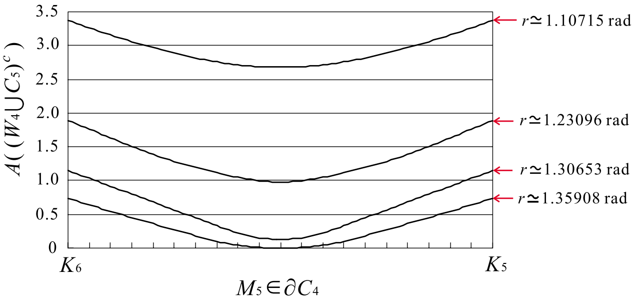

Fig. 4. The curve of $A((W_4 \cup C_5)^c)$ when M_5 is moved on the arc K_6K_5 of C_4. The similar computation method as in Fig. 3 is taken. See the legend in Fig. 3.

$$(x_1, y_1, z_1) = \left(-\frac{\cos r(\cos r - 1)}{\sin r}, \frac{(\cos r - 1)\sqrt{2\cos r + 1}}{\sin r}, -\cos r\right), \tag{6}$$

$$(x_2, y_2, z_2) = \left(-\frac{\cos r(\cos r - 1)}{\sin r}, -\frac{(\cos r - 1)\sqrt{2\cos r + 1}}{\sin r}, -\cos r\right). \tag{7}$$

Here, the center M_1 of C_1 is set at the south pole (0, 0, –1) as already mentioned. We put the centers M_2 and M_3 of C_2 and C_3 each, at K_1 and ($\sin r$, 0, $-\cos r$), respectively. Note that K_1 and K_2 are the points where the perimeters of C_3 and C_1 intersect. We also note here that the locations of centers of the second and the third spherical caps defined as above are different from the cases $N = 4, 5, 6$ because of the convenience of our computation. We will use this convention for all cases of $N \geq 7$ hereafter. Then, taking into account Theorem 1

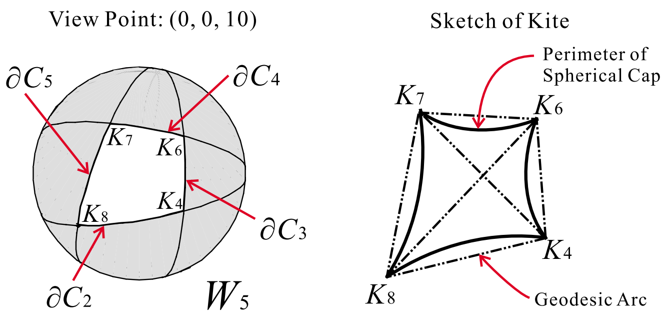

Fig. 5. The kite $K_8K_4K_6K_7$ on unit sphere.

in Subsec. 2.2, $\cup_{v=1}^{3} C_v$ is in an extreme state. At this time, similarly, we calculate the points where the perimeters of C_2 and C_1 intersect. Among these cross points, let $K_3 = (x_3, y_3, z_3)$ be the point which is outside C_3. Let $K_4 = (x_4, y_4, z_4)$ be one of the cross points between ∂C_2 and ∂C_3 and let it not be the south pole (0, 0, –1). The explicit expressions of coordinates of cross points K_3 and K_4 are shown in Appendix A.1.

Let us place the center M_4 at K_4 and move it to K_2 along the arc K_4K_2 of C_3. We note that, during this movement, the distance s_{43} between M_4 and M_3 is equal to the angular radius r and the overlapping area A_{43} of C_4 and C_3 is invariant while the area $A(W_3 \cap C_4)$ is variable. Then, we want to know the position of M_4 where the area $A(W_3 \cap C_4)$ is maximum. Therefore, we calculate the area $A(W_3 \cap C_4)$ against the moving point M_4 numerically for several fixed values of r among $\tan^{-1}2 \le r < \pi/2$. In order to use the result later, the computation for $r = \tan^{-1}2$ is also performed. The results are shown in Fig. 3. In this figure, the horizontal axis is the position of M_4 on the arc K_4K_2 of C_3 and the vertical axis is the area $A(W_3 \cap C_4)$. In this computation, the arc K_4K_2 is divided into 100 equal intervals and the area $A(W_3 \cap C_4)$ is calculated on 101 end points of the intervals. Hereafter, the similar computations are performed for determination of centers of spherical caps (see Figs. 4, 7, and 10). As a result, for the four values of r in Fig. 3, we find that this curve of $A(W_3 \cap C_4)$ is symmetrical at the center of the arc K_4K_2 (it is evident from the spherical symmetry) and $A(W_3 \cap C_4)$ is maximum at both ends. The same fact as above would hold for every values of r in the range $\tan^{-1}2 \le r < \pi/2$. Therefore, we expect that $\cup_{v=1}^{4} C_v$ is in an extreme state if and only if M_4 is put at K_2 or K_4 for $\tan^{-1}2 \le r < \pi/2$. To make sure, we shall check that these points K_2 and K_4 satisfy the condition that $\cup_{v=1}^{4} C_v$ is in an extreme state after obtaining the exact values of the angular radius r at the last paragraph in this subsection. At the moment, we choose M_4 on the point K_2. Then, let $K_5 = (x_5, y_5, z_5)$ be one of the cross points of perimeters ∂C_4 and ∂C_1, and let it be outside C_3. Further, let $K_6 = (x_6, y_6, z_6)$ be one of the cross points of ∂C_4 and ∂C_3, and let it not be the south pole (0, 0, –1). The explicit expressions of cross points K_5 and K_6 are given in Appendix A.1.

Next, the center M_5 is put at a certain point on the arc K_6K_5 of C_4. Then, we need to calculate the area $A(W_4 \cap C_5)$ when M_5 is moved on the arc K_6K_5 of C_4. However, in order to simplify calculation, we pay attention to the area $A((W_4 \cup C_5)^c)$. It is because, for $i \ge 2$, we find the relation that the area $A(W_{i-1} \cap C_i)$ is maximum with the restriction

$M_i \in \partial C_{i-1}$ ($\cup_{\nu=1}^{i} C_\nu$ is in an extreme state) is the same as that the area $A((W_{i-1} \cup C_i)^c)$ is maximum with the restriction $M_i \in \partial C_{i-1}$. Therefore, for $i = 5$, we calculate $A((W_4 \cup C_5)^c)$ against the moving point M_5 numerically for several fixed values of r among $\tan^{-1}2 \le r < \pi/2$. Here, the computation is performed as in the determination of M_4. Figure 4 shows the results. In this figure, the horizontal axis is the position of M_5 on the arc K_6K_5 of C_4 and the vertical axis is the area $A((W_4 \cup C_5)^c)$. As a result, for the four values of r in Fig. 4, the curve of $A((W_4 \cup C_5)^c)$ is symmetrical at the center of the arc K_6K_5 (it is evident from the spherical symmetry) and $A((W_4 \cup C_5)^c)$ is maximum when M_5 is placed on K_6 or K_5. The same fact as above would hold for every values of r in the range $\tan^{-1}2 \le r < \pi/2$. Therefore, we expect that $\cup_{\nu=1}^{5} C_\nu$ is in an extreme state if and only if M_5 is put at K_5 or K_6 for $\tan^{-1}2 \le r < \pi/2$. To make sure, we shall check whether the points K_5 and K_6 are such points after obtaining the exact values of the angular radius r at the last paragraph in this subsection like the case of M_4. Here, we choose M_5 on the point K_5. Then, let $K_7 = (x_7, y_7, z_7)$ be one of the cross points of the perimeters ∂C_5 and ∂C_4, and let it be outside of C_1. Similarly, let $K_8 = (x_8, y_8, z_8)$ be one of the cross points of ∂C_5 and ∂C_2, and let it be outside of C_1. The exact coordinates of K_7 and K_8 are also given in Appendix A.1.

At the time when C_5 is put on the sphere, the shape of the uncovered region $(W_5)^c$ becomes a kite on the unit sphere (see Fig. 5). We note that the sides of the kite $K_8K_4K_6K_7$ are not geodesic arcs but are perimeters of spherical caps. Then, we see that there are four centers M_2, M_3, M_4, and M_5 on the perimeter ∂C_1.

From the configuration of the vertices K_8, K_4, K_6 and K_7 of the kite $K_8K_4K_6K_7$, for the range $\tan^{-1}2 \le r < \pi/2$, we find that there hold always the following relations of the spherical distance between each vertices.

$$\begin{cases} d_s(K_8, K_4) = d_s(K_8, K_7), \\ d_s(K_6, K_4) = d_s(K_6, K_7), \\ d_s(K_6, K_4) < d_s(K_8, K_4) < d_s(K_6, K_8), \\ d_s(K_6, K_4) < d_s(K_4, K_7) < d_s(K_6, K_8)\,. \end{cases} \tag{8}$$

Note that, as defined in Subsec. 2.2, $d_s(K_i, K_j)$ is the spherical distance between points $K_i = (x_i, y_i, z_i)$ and $K_j = (x_j, y_j, z_j)$; namely

$$d_s(K_i, K_j) = \cos^{-1}\left(x_i \cdot x_j + y_i \cdot y_j + z_i \cdot z_j\right). \tag{9}$$

Refer to Appendix A.1 for the explicit coordinates of K_4, K_6, K_7, and K_8. We produced the relations (8) by using mathematical software. Especially four inequalities in (8) are obtained numerically.

So far, we have arranged five spherical caps. Next, let us consider the sixth and seventh spherical caps. As mentioned in Subsec. 2.4, if we take the angular radius of spherical caps to be equal to the largest spherical distance in the kite $K_8K_4K_6K_7$, the sixth spherical cap C_6

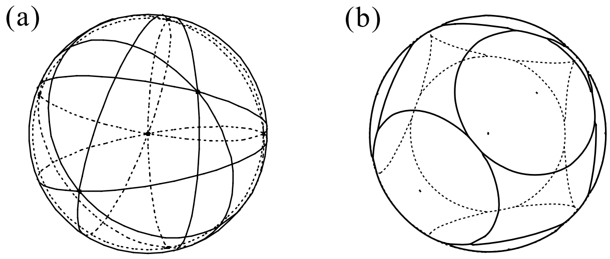

Fig. 6. (a) Our sequential covering for $N = 7$. (b) Our solution of Tammes problem for $N = 7$. Both viewpoints are (0, 0, 10). In this example, the coordinates of the centers are respectively (0, 0, –1), (0.16977, –0.96282, –0.21014), (0.97767, 0, –0.21014), (0.16977, 0.96282, –0.21014), (–0.91871, 0.33438, –0.21014), (–0.55588, –0.46644, 0.68806), and (0.39850, 0.33438, 0.85404).

can cover the region except for a point in the kite under the Minkowski condition. Therefore, we find that $\bar{r}_6$ is the largest spherical distance in the kite. It is obvious that the kite $K_8K_4K_6K_7$ on the sphere is inside the spherical quadrilateral $K_8K_4K_6K_7$ for fixed vertices K_8, K_4, K_6, and K_7 (see Fig. 5). From (8) and the coordinates of K_8, K_4, K_6, and K_7 in Appendix A.1, we find numerically that four vertices of the spherical quadrilateral $K_8K_4K_6K_7$ satisfies the relations $d_s(K_6, K_8) > d_s(K_4, K_7)$, and $\pi/2 \geq d_s(K_6, H) = d_s(H, K_8) = d_s(K_6, K_8)/2 > d_s(H, K_4) = d_s(H, K_7) > 0$ for $\tan^{-1}2 \leq r < \pi/2$. Note that H is the middle point of the geodesic arc K_6K_8. Therefore, from Corollary of Theorem 2, the farthest pair of points in the spherical quadrilateral $K_8K_4K_6K_7$ is the pair of K_6 and K_8; namely d_s (K_6, K_8) is the largest spherical distance in the kite $K_8K_4K_6K_7$ and is equal to $\bar{r}_6$. Then, from (9) and the coordinates of K_6 and K_8 in Appendix A.1, we have

$$r = d_s(K_6, K_8) = \cos^{-1}\left(\frac{41\cos^4 r - 8\cos^3 r - 18\cos^2 r + 1}{9\cos^4 r + 8\cos^3 r - 2\cos^2 r + 1}\right). \tag{10}$$

In addition, we note $K_8 \in \partial C_5$. Thus, $\cup_{v=1}^{6} C_v$ is in an extreme state (the set $\cup_{v=1}^{6} C_v$ covers S except for the point K_6) if and only if M_6 is put on the point K_8. At this time, K_6 is the unique uncovered point. Equation (10) is solved against r by using mathematical software. As a result, the value of the upper bound for $N = 7$ is obtained

$$r_7 = \bar{r}_6 = \cos^{-1}\left(1 - \frac{4}{\sqrt{3}}\cos\left(\frac{7\pi}{18}\right)\right) \approx 1.35908 \quad \text{rad.} \tag{11}$$

Therefore, when M_6 and M_7 are put at K_8 and K_6, respectively, then $\cup_{v=1}^{7} C_v$ which contains W_6 covers the whole of S (see Fig. 6(a)). Namely, our sequential covering for $N = 7$ is completed.

Here, we check whether the position of points K_2 and K_5 for M_4 and M_5 satisfy the condition that $\cup_{v=1}^{4} C_v$ and $\cup_{v=1}^{5} C_v$ are in an extreme state, respectively. For that purpose,

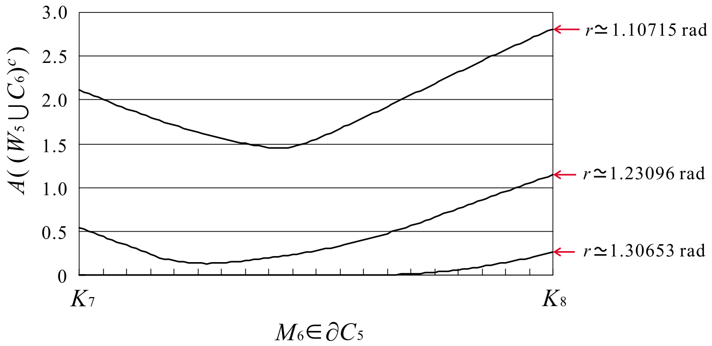

Fig. 7. The change of $A((W_5 \cup C_6)^c)$ when M_6 is moved on the arc K_7K_8 of C_5. The similar computation method as in Fig. 3 is taken. See the legend in Fig. 3.

when r is equal to the value of (11), we examine the position where the area $A(W_{i-1} \cap C_i)$ is maximum with the restriction $M_i \in \partial C_{i-1}$ (i = 4 and 5). First, about M_4, we calculate numerically the area $A(W_3 \cap C_4)$ against the moving point M_4 for $r = \cos^{-1}(1 - (4/\sqrt{3})\cos(7\pi/18))$. As mentioned above, we checked numerically that $A(W_3 \cap C_4)$ is maximum with the restriction $M_4 \in \partial C_3$ ($\cup_{\nu=1}^{4} C_\nu$ is in an extreme) if and only if M_4 is put at K_2 or K_4 for $r = \cos^{-1}(1 - (4/\sqrt{3})\cos(7\pi/18))$. The fact is indicated by the curve of $r \simeq 1.10715$ corresponding to $r = \cos^{-1}(1 - (4/\sqrt{3})\cos(7\pi/18))$ in Fig. 3. Next, about M_5, we use the fact that $A(W_4 \cap C_5)$ is maximum with the restriction $M_5 \in \partial C_4$ is the same as that $A((W_4 \cup C_5)^c)$ is maximum with the restriction $M_5 \in \partial C_4$. As mentioned above, we checked numerically that $A((W_4 \cup C_5)^c)$ is maximum with the restriction $M_5 \in \partial C_4$ ($\cup_{\nu=1}^{5} C_\nu$ is in an extreme) if and only if M_5 is put at K_5 or K_6 for $r = \cos^{-1}(1 - (4/\sqrt{3})\cos(7\pi/18))$. The result is graphically presented by the curve of $r \simeq 1.10715$ corresponding to $r = \cos^{-1}(1 - (4/\sqrt{3})\cos(7\pi/18))$ in Fig. 4. Thus, our choice for M_4 and M_5 is justified.

3.3. $N = 8$

It is expected that the solution r_8 for $N = 8$ should not be larger than r_7. Then, we assume $r \le r_7$. Further, at the moment, we assume $\tan^{-1}2 \le r$. If our answer for $N = 8$ is not obtained by this assumption $\tan^{-1}2 \le r$, we will consider the range of $r < \tan^{-1}2$ next. First, as in the case of $N = 7$, the centers M_1, M_2, and M_3 are placed at $(0, 0, -1)$, K_1, and $(\sin r, 0, -\cos r)$, respectively. Since, from Theorem 1 in Subsec. 2.2, the set $\cup_{\nu=1}^{3} C_\nu$ is in an extreme state when the centers M_1, M_2, and M_3 satisfy the relations $s_{12} = s_{13} = s_{23} = r$. Next, from the considerations for determinations of M_4 and M_5 in Subsec. 3.2, we can assume that the allocation of points K_2 and K_5 for M_4 and M_5 respectively satisfy the condition that $\cup_{\nu=1}^{4} C_\nu$ and C_ν each are in an extreme state. To make sure, we shall check that K_2 and K_5 are such points after obtaining the exact values of r_8. Therefore, we use the same positions of the first five spherical caps for the case $N = 7$. When the fifth spherical cap C_5 is put on the sphere, in the same way as the foregoing Subsection, a quadrilateral kite $K_8K_4K_6K_7$ on the sphere might be formed as the uncovered region. Hence, the relations (8) hold.

Next, we search the position of M_6 which satisfies the condition that the area $A(W_5 \cap C_6)$ is maximum with the restriction $M_6 \in \partial C_5$ ($\cup_{v=1}^{6} C_v$ is in an extreme state). From the restriction $M_6 \in \partial C_5$, M_6 is put at a certain point on the arc K_7K_8 of C_5 and, during M_6 is moved on the arc K_7K_8 of C_5, the area $A(W_5 \cap C_6)$ is calculated. At this time, in order to simplify calculation, we use the relation that the area $A(W_5 \cap C_6)$ is maximum with the restriction $M_6 \in \partial C_5$ is the same as that the area $A((W_5 \cup C_6)^c)$ is maximum with the restriction $M_6 \in \partial C_5$. Hence, we calculate the area $A((W_5 \cup C_6)^c)$ against the moving point M_6 numerically for several fixed values of r among $\tan^{-1}2 \le r < r_7$. As a result, for the four values of r in Fig. 7, we find that $A((W_5 \cup C_6)^c)$ is maximum when M_6 is put on K_8. Figure 7 shows the graph of computational results. In this figure, the horizontal axis is the position of M_6 on the arc K_7K_8 of C_5 and the vertical axis is the area $A((W_5 \cup C_6)^c)$. Then, we expect that $\cup_{v=1}^{6} C_v$ is in an extreme state if and only if M_6 is put at K_8 in the range $\tan^{-1}2 \le r < r_7$. We shall check that K_8 is such a point after obtaining the exact values of the angular radius r as in the cases of M_4 and M_5. We choose here M_6 on the point K_8.

Then, when M_6 is put at K_8, we notice that three cases are possible to be considered for the relation between r and $d_s(K_8, K_4)$: $r = d_s(K_8, K_4)$; $r < d_s(K_8, K_4)$; and $r > d_s(K_8, K_4)$. First, we consider the case $r = d_s(K_8, K_4)$. From (9) and the coordinates of K_4 and K_8 in Appendix A.1, we have

$$r = d_s(K_8, K_4) = \cos^{-1}\left(\frac{16\cos^4 r + \cos^3 r - 9\cos^2 r - \cos r + 1}{9\cos^3 r - \cos^2 r - \cos r + 1}\right). \tag{12}$$

Equation (12) is the quartic equation of $\cos r$ and can be solved by using the algebraic formula of Ferrari. As a result, we get the following value of the angular radius r.

$$r = \cos^{-1}\left(-\frac{1}{7} + \frac{2\sqrt{2}}{7}\right) \approx 1.30653 \quad \text{rad.} \tag{13}$$

Then, we find the following relation among $d_s(K_8, K_4)$, $d_s(K_4, K_7)$, and r by using mathematical software.

$$d_s(K_4, K_7) = d_s(K_8, K_4) = r. \tag{14}$$

Therefore, from (8) and (14), we see that the spherical triangle $K_8K_4K_7$ is equilateral and the uncovered region $(W_6)^c$ is the concave isosceles triangle $K_4K_6K_7$ on S inside the spherical isosceles triangle $K_4K_6K_7$. From the value of (13) and the coordinates of K_4, K_6, and K_7 in Appendix A.1, we find that the spherical isosceles triangle $K_4K_6K_7$ satisfies the relations $\pi/2 \ge d_s(K_4, H) = d_s(H, K_7) = d_s(K_4, K_7)/2 \ge d_s(H, K_6) > 0$. Note that H is the middle point of the geodesic arc K_4K_7. Therefore, from Theorem 2 in Subsec. 2.4, the largest spherical distance in $(W_6)^c$ is $d_s(K_4, K_7)$. Hence, $\cup_{v=1}^{7} C_v$ is in an extreme state (the set $\cup_{v=1}^{7} C_v$ covers S except for a point) if and only if $r = d_s(K_4, K_7)$ and M_7 is chosen on one of the points K_7 or K_4. Here we put M_7 on K_7; namely the spherical surface S is covered by

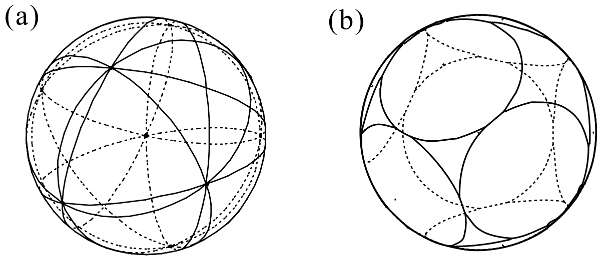

Fig. 8. (a) Our sequential covering for $N = 8$. (b) Our solution of Tammes problem for $N = 8$. Both viewpoints are (0, 0, 10). In this example, the coordinates of the centers are respectively (0, 0, –1), (0.19992, –0.94435, –0.26120), (0.96528, 0, –0.26120), (0.19992, 0.94435, –0.26120), (–0.88248, 0.39116, –0.26120), (–0.68256, –0.55319, 0.47759), (–0.28273, 0.55319, 0.78361), and (0.48264, –0.39116, 0.78361).

the set $\cup_{\nu=1}^{7} C_\nu$ except for the point K_4. Then, from the relation (14), we find $\bar{r}_7 = \cos^{-1}(-1/7 + 2\sqrt{2}/7)$. Thus, from the fact $K_7 \in \partial C_6$, it is obvious that $\bar{r}_7$ is equal to the upper bound r_8 for $N = 8$. Finally, M_8 is uniquely determined to be the uncovered point K_4, and then the whole of S is covered by $\cup_{\nu=1}^{8} C_\nu$ which contains W_7 (see Fig. 8(a)).

Next, we consider the other two cases $r < d_s(K_8, K_4)$ and $r > d_s(K_8, K_4)$ by way of precaution. Here, we examine the relations among $d_s(K_8, K_4)$, $d_s(K_4, K_7)$, and r, and find numerically that the following relations hold by using mathematical software.

$$\text{For } \tan^{-1} 2 \le r < \cos^{-1}\left(-\frac{1}{7} + \frac{2\sqrt{2}}{7}\right),\ d_s(K_6, K_4) \le r < d_s(K_4, K_7) < d_s(K_8, K_4). \tag{15}$$

$$\text{For } \cos^{-1}\left(-\frac{1}{7} + \frac{2\sqrt{2}}{7}\right) < r < \frac{\pi}{2},\ d_s(K_8, K_4) < d_s(K_4, K_7) < r. \tag{16}$$

In the case (15), since the angular radius r is smaller than $d_s(K_8, K_4)$, the uncovered region $(W_6)^c$ is reduced to the pentagon which is bounded by perimeters of spherical caps and $d_s(K_4, K_7)$ inside this pentagon is larger than r due to (8) and (15). As was described in Subsec. 2.4, we expect the situation that the set W_7 covers S except for finite points. Therefore, in the uncovered pentagon $(W_6)^c$, we take $\bar{r}_7$ to be the spherical distance of the largest interval, such that at least one endpoint of the interval comes on the perimeter ∂C_6. However, from the relation (15), we cannot take r to be the largest spherical distance in $(W_6)^c$. Further, we find that an uncovered region is left on S when M_7 is put on the uncovered pentagon $(W_6)^c$ according to our sequential covering. Refer to the considerations of determination for M_7 and the fact that the uncovered region is left when W_7 is formed on S for $\tan^{-1}2 \le r < \cos^{-1}(-1/7 + 2\sqrt{2}/7)$ in the following Subsections. In the case (16), on the other hand, since the angular radius r is larger than $d_s(K_8, K_4)$, the uncovered region $(W_6)^c$ is reduced to the triangle which is bounded by perimeters of spherical caps and the spherical distance of the largest interval inside this triangle is smaller than r due to (8) and

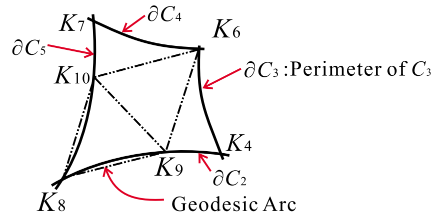

Fig. 9. The sketch of the kite $K_8K_4K_6K_7$ and the spherical rhombus $K_8K_9K_6K_{10}$.

(16). Then, the seventh spherical cap C_7 covers the uncovered triangle $(W_6)^c$ completely when the center M_7 is put in this triangle. Namely, the center of C_8 cannot be placed on S under the Minkowski condition. Therefore, in two cases above, the range of r is not suitable for our upper bound.

Now, when r is equal to the value of (13), we check whether the positions of points K_2, K_5, and K_8 for M_4, M_5, and M_6 satisfy the condition that $\cup_{\nu=1}^4 C_\nu$, $\cup_{\nu=1}^5 C_\nu$, and $\cup_{\nu=1}^6 C_\nu$ are in an extreme state, respectively. For i = 4, 5, and 6, as mentioned above, we checked numerically that the points K_2, K_5, and K_8 are the positions where the area $A(W_{i-1} \cap C_i)$ (or $A((W_{i-1} \cup C_i)^c)$) are maximum with the restriction $M_i \in \partial C_{i-1}$ ($\cup_{\nu=1}^i C_\nu$ is in an extreme state), respectively, when r is equal to $\cos^{-1}(-1/7 + 2\sqrt{2}/7)$. Then, the facts are indicated by the curve of $r \simeq 1.30653$ corresponding to $r = \cos^{-1}(-1/7 + 2\sqrt{2}/7)$ in Figs. 3, 4, and 7. Thus, for the case $N = 8$, our choice of M_4, M_5, and M_6 is justified.

At the beginning of this subsection, we initially assumed that r should be in the range $(\tan^{-1}2, r_7]$. In fact, after the investigation, our upper bound r_8 (the value of (13)) has fallen within the range $(\tan^{-1}2, r_7]$. However, one might suspect that the fact is due to the assumption. So, if r is in the range $(0, \tan^{-1}2]$, we examine whether W_7 is able to cover S except for finite points. From the results of $r \simeq 1.10715$ in Figs. 3, 4, and 7, we find that the set W_7 must leave an uncovered region on S when r is equal to $\tan^{-1}2 \approx 1.10715$. Therefore, from the setup of our problem, $r = \tan^{-1}2$ cannot be r_8. Furthermore, for $0 < r < \tan^{-1}2$, the uncovered region would become still bigger. Thus, our assumption that r_8 is in the range $(\tan^{-1}2, r_7]$ is confirmed ($r = \tan^{-1}2$ is just excluded from the above consideration) and $\cos^{-1}(-1/7 + 2\sqrt{2}/7)$ is certainly a solution for $N = 8$.

3.4. $N = 9$

According to the considerations for $N = 7$ and 8, the angular radius for $N = 9$ should be smaller than r_8. First, at the moment, we expect that the inequalities $\tan^{-1}2 \le r < r_8$ hold like the case of $N = 8$. Then, the same positional relation of the first five spherical caps for the case of $N = 7$ is used again. From Theorem 1 in Subsec. 2.2 and from the considerations for determinations of M_4 and M_5 in Subsec. 3.2, we can consider that the set $\cup_{\nu=1}^5 C_\nu$ is in an extreme state when the centers M_1, M_2, M_3, M_4, and M_5 are placed at the points $(0, 0, -1)$, K_1, $(\sin r, 0, -\cos r)$, K_2 and K_5, respectively. To make sure, after obtaining the exact value of r_9, we shall check that the allocations of K_2 and K_5 to M_4 and M_5, respectively, satisfy the condition that $\cup_{\nu=1}^4 C_\nu$ and $\cup_{\nu=1}^5 C_\nu$ are in an extreme state. In the same way as in Subsec. 3.2, the shape of the set $(W_5)^c$ is again the kite $K_8K_4K_6K_7$ on the sphere. Then, the relations (8) and (15) hold. We should cover this kite $K_8K_4K_6K_7$ except for a point by the set $C_6 \cup C_7 \cup C_8$ under the conditions $M_6 \in \partial C_5$, $M_7 \in \partial C_6$, and $M_8 \in \partial C_7$.

From the consideration for determination of M_6 in Subsec. 3.3, and from the condition that $\cup_{\nu=1}^{6} C_\nu$ is in an extreme state, we can assume that the center M_6 is placed on the point K_8 as in the case of $N = 8$. At this time, the uncovered region $(W_6)^c$ is reduced to the pentagon which is bounded by perimeters of spherical caps (see Fig. 9). Here, let K_9 be one of the cross points of the perimeters ∂C_6 and ∂C_2, and let it be outside C_1. Further, let K_{10} be one of the cross points of ∂C_6 and ∂C_5, and let it be outside C_1. Obviously the relation $d_s(K_8, K_9) = d_s(K_8, K_{10})$ holds. Now, we want to place three centers of C_7, C_8, and C_9 in the uncovered pentagon $K_9K_4K_6K_7K_{10}$ under the Minkowski condition. Then, we consider an spherical equilateral triangle of side-length $r = d_s(K_8, K_9)$ in the pentagon $K_9K_4K_6K_7K_{10}$ since three centers M_7, M_8, and M_9 could be arranged on the vertices of the spherical equilateral triangle under the Minkowski condition. Therefore, we can assume that K_9 and K_{10} satisfy

$$d_s(K_8, K_9)\left(= d_s(K_8, K_{10})\right) = d_s(K_9, K_{10}) = d_s(K_6, K_9) = d_s(K_{10}, K_6). \tag{17}$$

As shown in Fig. 9, we see the spherical rhombus $K_8K_9K_6K_{10}$ which is formed with two spherical equilateral triangles $K_8K_9K_{10}$ and $K_9K_6K_{10}$. If the spherical rhombus $K_8K_9K_6K_{10}$ which satisfies (17) is possible to be formed, we can consider that the angular radius r_9 ($\bar{r}_8$) is equal to a side-length (e.g. the spherical distance between the points K_8 and K_9) of the spherical rhombus $K_8K_9K_6K_{10}$ if and only if M_7 is placed on the point K_9 or K_{10}. We shall show that this allocation of M_7 is actually possible, after obtaining the solution r_9. Then, due to $d_s(K_9, K_2) = 2r$ (the proof is given in Appendix A.3), C_7 is in contact with C_4 at the point K_6 and the shape of the uncovered region should be a triangle $K_{10}K_6K_7$ when M_7 is placed on the point K_9. At this time, we will see later that the side-length of spherical rhombus $K_8K_9K_6K_{10}$ is the largest spherical distance in the uncovered triangular region. Thus, if M_8 is chosen either on the point K_6 or K_{10}, the triangle is covered by C_8 except for a point in the triangle. Note that we supposed that the points K_8 and K_9 are the positions such that $\cup_{\nu=1}^{6} C_\nu$ and $\cup_{\nu=1}^{7} C_\nu$ are in an extreme state, respectively. However, it is not checked yet that K_8 and K_9 are such positions in this step. We shall check these points after obtaining the exact value of r_9 and the coordinates of K_9.

Now we examine whether the point K_9 or K_{10} which satisfies (17) is possible to exist. To begin with, we calculate the coordinates of the point K_9. Since K_9 is one of the cross points of the perimeters ∂C_6 and ∂C_2, the coordinates of K_9 can be obtained by using the simultaneous equation (4). Refer to the coordinates of $K_8 = M_6$ in Appendix A.1 and Eq. (6) (the coordinates of $K_1 = M_2$). The exact coordinates of $K_9 = (x_9, y_9, z_9)$ are given in Appendix A.1. From our consideration that r_9 ($\bar{r}_8$) is equal to a side-length of spherical rhombus $K_8K_9K_6K_{10}$ which satisfies (17), we will solve the equation

$$r = d_s(K_6, K_9) = \cos^{-1}\left(\frac{\cos r\left(81\cos^4 r - 28\cos^3 r - 46\cos^2 r - 4\cos r + 5\right)}{(\cos r + 1)\left(9\cos^3 r - \cos^2 r - \cos r + 1\right)}\right) \tag{18}$$

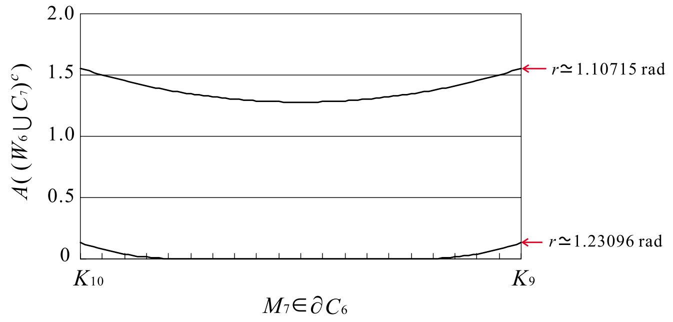

Fig. 10. The change of $A((W_6 \cup C_7)^c)$ when M_7 is moved on the arc $K_{10}K_9$ of C_6. The similar computation method as in Fig. 3 is taken. See the legend in Fig. 3.

against r. By developing Eq. (18), we obtain the quartic equation of $\cos r$. Therefore, (18) can be solved by using the algebraic formula of Ferrari as in the case of $N = 8$. As a result, the value of the angular radius r is obtained as

$$r_9 = \bar{r}_8 = \cos^{-1}\left(\frac{1}{3}\right) \approx 1.23096 \quad \text{rad.} \tag{19}$$

From (19) and the coordinates of K_6, K_8, K_9, and K_{10}, we checked that the relation (17) strictly holds; namely $d_s(K_8, K_9) = d_s(K_8, K_{10}) = d_s(K_9, K_{10}) = d_s(K_6, K_9) = d_s(K_{10}, K_6) = \cos^{-1}(1/3)$. Then, K_{10} is obtained by using the fact that it is one of the cross points of the perimeters ∂C_6 and ∂C_5. Note that the coordinates of the point K_{10} are omitted due to their long expressions. As we have noted, we check here whether the positions of the point K_2, K_5, K_8, and K_9 for M_4, M_5, M_6, and M_7 satisfy the condition that $\cup_{\nu=1}^{4} C_\nu$, $\cup_{\nu=1}^{5} C_\nu$, $\cup_{\nu=1}^{6} C_\nu$, and $\cup_{\nu=1}^{7} C_\nu$ are in an extreme state, respectively, by using the value of (19). First, we checked numerically that $A(W_{i-1} \cap C_i)$ (or $A((W_{i-1} \cup C_i)^c)$) are maximum when M_i ($i = 4$, 5, and 6) are put at K_2, K_5, and K_8, respectively. These facts are indicated by the curve of $r \simeq 1.23096$ corresponding to $r = \cos^{-1}(1/3)$ in Figs. 3, 4, and 7. Therefore, we find numerically that the positions of K_2, K_5, and K_8 satisfy the condition that $\cup_{\nu=1}^{4} C_\nu$, $\cup_{\nu=1}^{5} C_\nu$, and $\cup_{\nu=1}^{6} C_\nu$ are in an extreme state, respectively. Next, we examine the position of M_7 where the area $A(W_6 \cap C_7)$ is maximum. Then, in order to simplify calculation, we use the relation that the area $A(W_6 \cap C_7)$ is maximum with the restriction $M_7 \in \partial C_6$ is the same as that the area $A((W_6 \cup C_7)^c)$ is maximum with the restriction $M_7 \in \partial C_6$. Then, we calculate the area $A((W_6 \cup C_7)^c)$ against the moving point M_7 on the arc $K_{10}K_9$ of C_6 numerically for several fixed values of r among $\tan^{-1}2 \le r < r_8$. As a result, the curve of $A((W_6 \cup C_7)^c)$ is symmetrical at the center of the arc $K_{10}K_9$ (it is evident from the shape of uncovered region $(W_6)^c$) and $A((W_6 \cup C_7)^c)$ is maximum at both end for the two values of r in Fig. 10. The same fact as above would hold for every values of r in the range $\tan^{-1}2 \le r < r_8$. Figure 10 illustrates the result of computation for M_7. In this figure, the horizontal axis is the position of M_7 on the arc $K_{10}K_9$ of C_6 and the vertical axis is the area $A((W_6 \cup C_7)^c)$. Then, the result

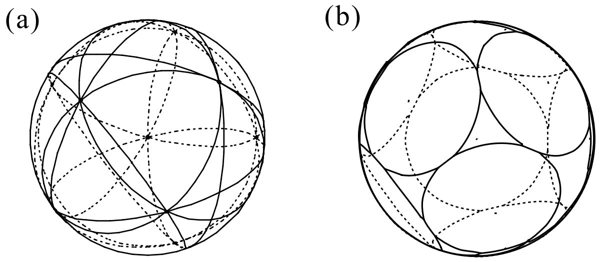

Fig. 11. (a) Our sequential covering for $N = 9$. (b) Our solution of Tammes problem for $N = 9$. Both viewpoints are (0, 0, 10). In this example, the coordinates of the centers are respectively (0, 0, –1), (0.23570, –0.91287, –0.33333), (0.94281, 0, –0.33333), (0.23570, 0.91287, –0.33333), (–0.82496, 0.45644, –0.33333), (–0.78567, –0.60858, 0.11111), (0.15713, –0.60858, 0.77778), (0.58926, 0.45644, 0.66667), and (–0.54997, 0.30429, 0.77778).

is graphically presented by the curve of $r \simeq 1.23096$ corresponding to $r = \cos^{-1}(1/3)$ in Fig. 10. Therefore, for $r = \cos^{-1}(1/3)$, we check numerically that $\cup_{\nu=1}^{7} C_\nu$ are in an extreme state if and only if M_7 is put at K_9 or K_{10}.

As mentioned above, in this paper, we put M_7 at K_9. Then, the uncovered region $(W_7)^c$ is the triangle $K_{10}K_6K_7$ on S that satisfies the relations $\pi/2 \geq d_s(K_{10}, H) = d_s(H, K_6) = d_s(K_{10}, K_6)/2 \geq d_s(H, K_7) > 0$. We note that H is the middle point of the geodesic arc $K_{10}K_6$. From Theorem 2 in Subsec. 2.4, the largest spherical distance in $(W_7)^c$ is $d_s(K_{10}, K_6)$. Namely, we find $\bar{r}_8 = \cos^{-1}(1/3)$. Hence, if M_8 is put at K_{10} or K_6, the set $\cup_{\nu=1}^{8} C_\nu$ which contains W_7 covers the spherical surface S except for a point. Then, due to the facts $K_6 \in \partial C_7$ and $K_{10} \in \partial C_7$, we find that $\cup_{\nu=1}^{8} C_\nu$ is in an extreme state. In this paper, we choose M_8 on K_6.

Then, $\cos^{-1}(1/3)$ satisfies the initial assumption $\tan^{-1}2 \leq r < r_8$. However, one can suspect this result is owing to the initial assumption. When r is in the range $(0, \tan^{-1}2]$, we check whether W_8 is able to cover S except for finite points. From the results of $r \simeq 1.10715$ in Figs. 3, 4, 7, and 10, we find the fact that our first to eighth spherical caps must leave an uncovered region on S when r is equal to $\tan^{-1}2 \approx 1.10715$. Hence, for $0 < r < \tan^{-1}2$, the uncovered region would become still bigger. Therefore, our upper bound r_9 for $N = 9$ does not exist in the range $(0, \tan^{-1}2]$ like the case of $N = 8$. Thus, we note that $\tan^{-1}2 < r < r_8$ is confirmed ($r = \tan^{-1}2$ is just excluded from the above consideration).

Finally, M_9 is put on the unique uncovered point K_{10}, and then $\cup_{\nu=1}^{9} C_\nu$ which contains W_8 covers the whole of S (see Fig. 11(a)). Thus, our consideration that the angular radius r_9 ($\bar{r}_8$) is equal to a side-length of spherical rhombus $K_8K_9K_6K_{10}$ which satisfies (17) is confirmed and $\cos^{-1}(1/3)$ is certainly a solution for $N = 9$.

4. Conclusion

In Sec. 3, we calculated the upper bound of r for $N = 2, ..., 9$, such that the set $\cup_{\nu=1}^{N} C_\nu$ which contains W_{N-1} covers the whole spherical surface S (see Table 2).

Table 2. Upper bound r_N of our problem.

Number of spherical caps N	Upper bound of angular radius of our problem r_N [rad]
2	$\pi \approx 3.14159$
3	$2\pi/3 \approx 2.09440$
4	$\pi - \cos^{-1}(1/3) \approx 1.91063$
5	$\pi/2 \approx 1.57080$
6	$\pi/2 \approx 1.57080$
7	$\cos^{-1}(1 - (4/\sqrt{3})\cos(7\pi/18)) \approx 1.35908$
8	$\cos^{-1}(-1/7 + 2\sqrt{2}/7) \approx 1.30653$
9	$\cos^{-1}(1/3) \approx 1.23096$

It was described, in Subsec. 2.1, that the covering with spherical caps of angular radius r is correspondent with the packing with half-caps (see Subsec. 2.1 for its definition) according to our method that the centers of spherical caps are chosen on the perimeters of other spherical caps under the Minkowski condition. Let us suppose the centers of N half-caps are placed on the positions of the centers of spherical caps C_i ($i = 1, ..., N$) which are considered in Sec. 3. At this time, we get the packing with N congruent half-caps (see Figs. 6(b), 8(b), and 11(b)). Then, for $N = 2, ..., 9$, we find that the upper bound of angular radius of our problem with N congruent spherical caps and the value of angular diameter of Tammes problem with N congruent spherical caps are equivalent. In addition, we find that the location of centers of our problem is correspondent with that of the Tammes problem for $N = 2, ..., 9$, respectively (SCHÜTTE and VAN DER WAERDEN, 1951; DANZER, 1963; FEJES TÓTH, 1972).

Accordingly, we find the fact that the results of our problem are coincident with those of Tammes problem about $N = 2, ..., 9$ at least (SUGIMOTO and TANEMURA, 2002, 2003, 2004). Further, SCHÜTTE and VAN DER WAERDEN (1951), and DANZER (1963) have solved the Tammes problem for $N = 7$, 8, and 9, through the consideration on irreducible graphs obtained by connecting those points, among N points, whose spherical distance is exactly the minimal distance. Then, after establishing the theorem which states that such irreducible graphs can only have triangles and quadrangles, Schütte, van der Waerden, and Danzer proved and obtained the minimal distance r for respective values of $N = 7$, 8, and 9. Further, they need the independent considerations for $N = 7$, 8, and 9, respectively (SCHÜTTE and VAN DER WAERDEN, 1951; DANZER, 1963; FEJES TÓTH, 1972). In contrast to this, we presented in this paper a systematic method which is different from the approach by SchÜtte, van der Waerden, and Danzer. Namely, as shown in Subsec. 2.4 and Sec. 3, our method is able to obtain a solution for N by using the results for the case $N - 1$ or $N - 2$ successively. In addition, in this study, we have considered the packing problem from the standpoint of sequential covering. The advantages of our approach are that we only need to observe uncovered region in the process of packing and that this uncovered region decreases step by step as the packing proceeds. At least, in the cases of $N \leq 9$, the solutions of Tammes problem can be found by our method. However, we may say that our method has not necessarily given a mathematical rigorous proof about our result.

In this paper, we used the mathematical software, Maple*, which is capable of manipulating complicate algebraic expressions exactly and is also useful for numerical computations. Consequently, we were able to calculate strict coordinates and solutions.

Here, we remark on the efficiency of covering. Now, let us define the efficiency of covering on spherical surface by (area of S)/($N \times$ (area of a cap)) = $2/(N \times (1 - \cos r))$. Then, from the value of r_N and from the positions of C_i ($i = 1, ..., N$), it appears that our solutions of $N = 2, ...,9$ give the worst efficient covering of S with N congruent spherical caps under the Minkowski condition respectively. At least, from Lemma and Theorem in Subsec. 2.2, it is obvious that our solutions r_2 and r_3 for $N = 2$ and 3, respectively, give the worst efficient covering under the Minkowski condition. Namely, our results $r_2 = \pi$ and $r_3 = 2\pi/3$ respectively give the efficiency of covering 1/2 and 4/9 (SUGIMOTO and TANEMURA, 2004). However, its proof for other values of N is still open and it is taken as a future subject.

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Appendix A.1: The Coordinates of the Point K_i (i = 3, 4, 5, 6, 7,8 and 9)

In the following, r ($\tan^{-1}2 \le r \le \pi/2$) is the angular radius.

$K_3 = (x_3, y_3, z_3)$:

$$\left(\frac{\sin r(\cos^2 r - 2\cos r - 1)}{(\cos r + 1)^2}, -\frac{2\cos r \sin r\sqrt{2\cos r + 1}}{(\cos r + 1)^2}, -\cos r \right).$$

$K_4 = (x_4, y_4, z_4)$:

$$\left(\frac{2\cos r \sin r(2\cos r + 1)}{(\mathrm{cosr} + 1)^2}, -\frac{2\cos r \sin r\sqrt{2\cos r + 1}}{(\cos r + 1)^2}, -\frac{4\cos^2 r - \cos r - 1}{\cos r + 1} \right).$$

$K_5 = (x_5, y_5, z_5)$:

$$\left(\frac{\sin r(\cos^2 r - 2\cos r - 1)}{(\cos r + 1)^2}, \frac{2\cos r \sin r\sqrt{2\cos r + 1}}{(\cos r + 1)^2}, -\cos r \right).$$

*Maple is a trademark of Waterloo Maple Inc., Canada.

$K_6 = (x_6, y_6, z_6)$:

$$\left(\frac{2\cos r \sin r(2\cos r+1)}{(\cos r+1)^2}, \frac{2\cos r \sin r\sqrt{2\cos r+1}}{(\cos r+1)^2}, -\frac{4\cos^2 r-\cos r-1}{\cos r+1}\right).$$

$K_7 = (x_7, y_7, z_7)$:

$$\left(\frac{2\cos r \sin r(2\cos r+1)(\cos r-1)}{(\cos r+1)^3}, \frac{2\cos r \sin r(3\cos r+1)\sqrt{2\cos r+1}}{(\cos r+1)^3}, -\frac{4\cos^2 r-\cos r-1}{\cos r+1}\right).$$

$K_8 = (x_8, y_8, z_8)$:

$$\left(\frac{2\cos r \sin r(\cos r-1)(2\cos r+1)}{9\cos^3 r-\cos^2 r-\cos r+1}, \frac{2\cos r \sin r(\cos r-1)\sqrt{2\cos r+1}}{9\cos^3 r-\cos^2 r-\cos r+1}, -\frac{4\cos^4 r-\cos^3 r+5\cos^2 r+\cos r-1}{9\cos^3 r-\cos^2 r-\cos r+1}\right).$$

$K_9 = (x_9, y_9, z_9)$:

$$\left(\frac{\sin r\left(\cos^3 r-5\cos^2 r-\cos r+1\right)}{9\cos^3 r-\cos^2 r-\cos r+1}, -\frac{4\sin r\cos^2 r\sqrt{2\cos r+1}}{9\cos^3 r-\cos^2 r-\cos r+1}, -\frac{\cos r\left(\cos^3 r+11\cos^2 r-\cos r-3\right)}{9\cos^3 r-\cos^2 r-\cos r+1}\right).$$

Appendix A.2: The Overlapping Area of Two Congruent Spherical Caps

We give the overlapping area A_{ij} of two congruent spherical caps C_i and C_j of angular radius r, where their centers are M_i and M_j, respectively (see Fig. A1). We denote by s_{ij} the spherical distance between M_i and M_j and we assume $r \le s_{ij} \le 2r$. Then, let us assume G be the middle point of the geodesic arc M_iM_j. Further, we define T_1 and T_2 as the two cross points of perimeters ∂C_i and ∂C_j. Thus, the spherical distance h_{ij} between T_1 and T_2 is expressed by using the spherical cosine formula about a spherical right triangle M_iGT_1 as follows:

$$h_{ij} = 2\cos^{-1}\left(\frac{\cos r}{\cos\left(s_{ij}/2\right)}\right).$$

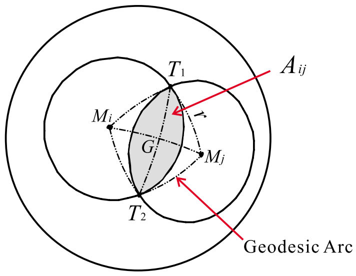

Fig. A1. Overlapping area A_{ij}.

If the points M_i, T_1 and T_2 are mutually connected by geodesic arcs, there arises a spherical isosceles triangle $M_iT_1T_2$ on the unit sphere. Then, let λ and μ be the interior angles at vertices M_i and T_1 (T_2) of this triangle, respectively. By the spherical cosine theorem, we have

$$\lambda = \cos^{-1}\left(\frac{\cos h_{ij} - \cos^2 r}{\sin^2 r}\right), \quad \mu = \cos^{-1}\left(\frac{1 - \cos h_{ij}}{\tan r \cdot \sin h_{ij}}\right).$$

Then the area A_1 of the spherical isosceles triangle $M_iT_1T_2$ turns out

$$A_1 = \lambda + 2\mu - \pi = \cos^{-1}\left(\frac{\cos h_{ij} - \cos^2 r}{\sin^2 r}\right) + 2\cos^{-1}\left(\frac{1 - \cos h_{ij}}{\tan r \cdot \sin h_{ij}}\right) - \pi .$$

On the other hand, the area of a spherical cap of the angular radius r is $|C| = 2\pi(1 - \cos r)$. Therefore the area A_2 of a sector with angle λ of the spherical cap is equal to $|C| \cdot \lambda/2\pi$, namely

$$A_2 = \cos^{-1}\left(\frac{\cos h_{ij} - \cos^2 r}{\sin^2 r}\right) \cdot (1 - \cos r).$$

Thus the overlapping area A_{ij} of C_i and C_j is

$$A_{ij} = 2(A_2 - A_1) = -2\cos^{-1}\left(\frac{\cos h_{ij} - \cos^2 r}{\sin^2 r}\right) \cdot \cos r - 4\cos^{-1}\left(\frac{1 - \cos h_{ij}}{\tan r \cdot \sin h_{ij}}\right) + 2\pi .$$

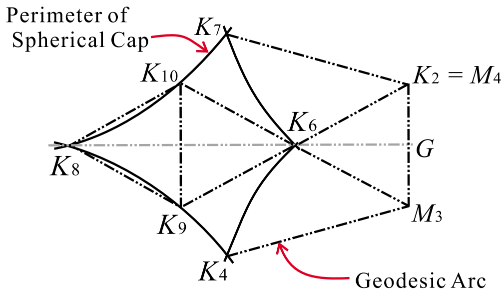

Fig. A2. The sketch of the kite $K_8K_4K_6K_7$ and three spherical equilateral triangles $K_8K_9K_{10}$, $K_9K_6K_{10}$, and $K_6K_3K_2$ in the pentagon $K_8K_4K_3K_2K_7$ on S.

Appendix A.3: The Proof of $d_s(K_9, K_2) = 2r$ for $N = 9$

For $N = 9$, the first five centers M_1, M_2, M_3, M_4, and M_5 are placed at the points $(0, 0, -1)$, K_1, $(\sin r, 0, -\cos r)$, K_2 and K_5, respectively, in order to satisfy the condition that the set $\cup_{v=1}^{5} C_v$ is in an extreme state. Then, the shape of the set $(W_5)^c$ is the kite $K_8K_4K_6K_7$ on the sphere (see Fig. A2). We note that the sides of the kite $K_8K_4K_6K_7$ are not geodesic arcs but are perimeters of spherical caps. Next, we assume that $K_9 \in \partial C_2$ and $K_{10} \in \partial C_5$ satisfy the relations (17). In order to prove the relation $d_s(K_9, K_2) = 2r$, we first note that K_4, K_6, $K_2 \in \partial C_3$ and K_7, K_6, $M_3 \in \partial C_4$. Then, obviously the relations $r = d_s(K_7, K_2) = d_s(K_2, K_6) = d_s(K_2, M_3) = d_s(K_6, M_3) = d_s(M_3, K_4)$ hold. As shown in Fig. A2, we see the pentagon $K_8K_4M_3K_2K_7$ on S contains the kite $K_8K_4K_6K_7$ and three spherical equilateral triangles $K_8K_9K_{10}$, $K_9K_6K_{10}$, and $K_6M_3K_2$. Note that our pentagon $K_8K_4M_3K_2K_7$ is not a spherical pentagon since the sides K_8K_4 and K_8K_7 are perimeters of spherical caps. Here, let G be the middle point of the geodesic arc M_3K_2. Next, let q denote the great circle determined by K_8 and G. Then, from the relations (8), (17), and $r = d_s(K_7, K_2) = d_s(K_2, K_6) = d_s(K_2, M_3) = d_s(K_6, M_3) = d_s(M_3, K_4)$, the pentagon $K_8K_4M_3K_2K_7$ and three spherical equilateral triangles $K_8K_9K_{10}$, $K_9K_6K_{10}$, and $K_6M_3K_2$ are symmetrical by reflection with respect to q. Accordingly, the great circle q is the mirror arc reflecting K_9 to K_{10}, K_4 to K_7, and M_3 to K_2, respectively. In addition, we see that K_6 is on q. Therefore, K_9, K_6, and K_2 are on one great circle. Thus, $d_s(K_9, K_2) = d_s(K_9, K_6) + d_s(K_6, K_2) = r + r = 2r$ holds.

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