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Shapes of Knot Patterns

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Abstract. In the mathematical knot theory, knot can be defined as an embedding of a circle in the three dimensional Euclidean space. In the plastic sense, as 3-dimensional objects, knots represent a very interesting phenomenon. This paper focuses on shapes of knots, using computer graphics, and attempts to present variations of the form of knot patterns. At the first stage, we constructed only alternating knots. At the second stage knot patterns are formed by using combinations of similar shapes. At the third stage knot patterns are formed as combinations of different shapes. The basis for the study of these designs is the mathematical knot theory.

1. Introduction

In general, knots can be presented as a closed rope or string. They are very interesting three dimensional objects in the plastic sense. When focusing on the structure of knots, they can be divided into structures with a single component, and those consisting from several components, links, which can be imagined as several interlaced knots in space. The both types of structures, knots and links, are the object of study of the mathematical knot theory. This paper focuses on the shapes of knots, using computer graphics, and attempts to show different form variations of knot patterns which are interesting in visual and plastic sense.

2. Basic Definitions

2.1. Knots and links

A simple closed curve without self-intersections, placed in the three-dimensional Euclidean space R^3, will be called a knot and denoted by K (OCHIAI *et al*., 1996a). A link L is the collection of such interlaced closed curves. Assuming that knots K_1, ..., Kr ... are its components, the link L = K_1 ∪ ... Kr will be denoted by L.

2.2. Regular projection and regular notation

Let be given a link L in R^3 on the (x, y) plane. Its orthogonal projection P without non-regular crossings is ...

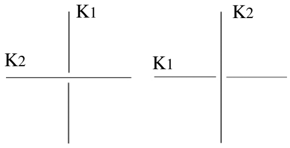


Fig. 1. Regular representation of knot.

Fig. 2. (a) Regular representation. (b) Regular projection.

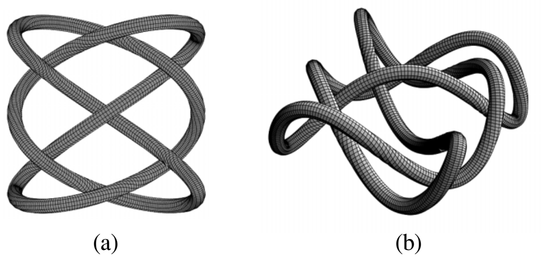


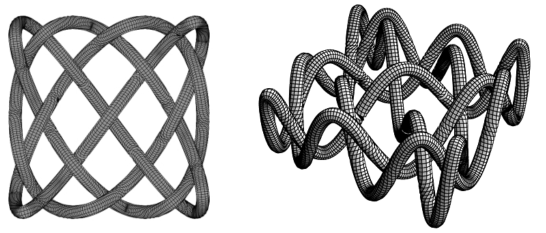
Fig. 3. Knot 1 obtained from lissajous curve (7) [1,4,3,2,7].

$$\mathrm{P}:R^3 \to \mathrm{E} \quad \mathrm{P}(x, y, z) = (x, y)$$

P will be called the regular projection of L and denoted by P(L) (OCHIAI *et al.*, 1996b).

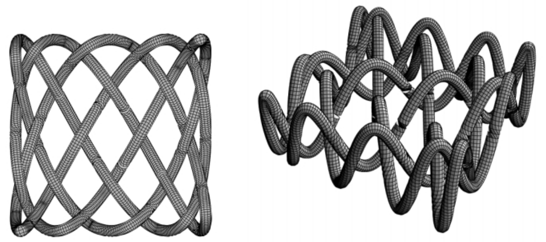
Figure 1 shows the relationship "over-under" in every crossing of P(L) (OCHIAI *et al.*, 1996b; ADAMS, 1998). To every crossing in the plane (x, y) are assigned two z-coordinates, $z_1 > z_2$, where z_1 corresponds to the over strand, and z_2 to the under strand. In a regular projection P(K) of a knot K we choose an orientation for travelling around the diagram. A knot or link is called alternating when in this travel each overcrossing is followed by undercrossing and vice versa, and non-alternating otherwise (OCHIAI *et al.*, 1996c).

Figure 1 overcrossing and undercrossing in a regular diagram of knot or link.



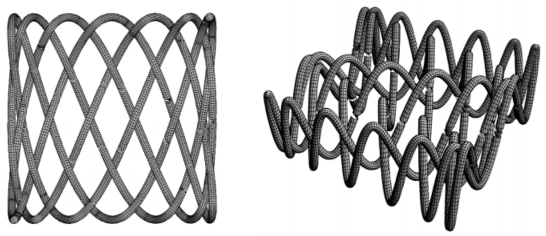
(a) (b)

Fig. 4. Knot 2 obtained from lissajous curve (17) [1,4,3,3,17].



(a) (b)

Fig. 5. Knot 3 obtained from lissajous curve (22) [1,4,3,5,22].



(a) (b)

Fig. 6. Knot 4 obtained from lissajous curve (32) [1,4,3,7,32].

3. Basic Pattern of Knots

The simplest form of a knot is an unknot a circle without crossings. The unknot can be also represented as a torus. Figures 1 and 2 show the regular projection of an unknot and its representation as a torus.

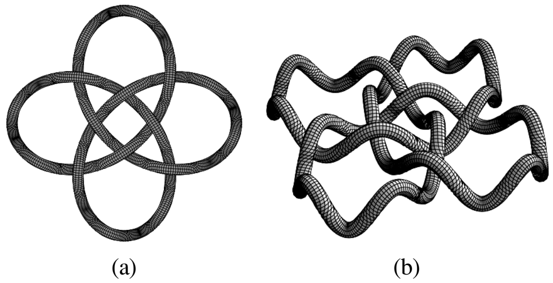


Fig. 7. Knot 1 obtained from hypotorochoid curve (8) [16,4,18,4,16].

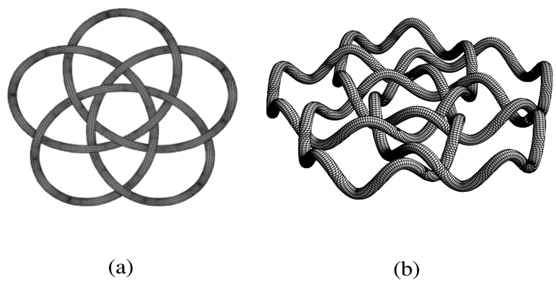


Fig. 8. Knot 2 obtained from hypotorochoid curve (15) [15,3,20,4,15].

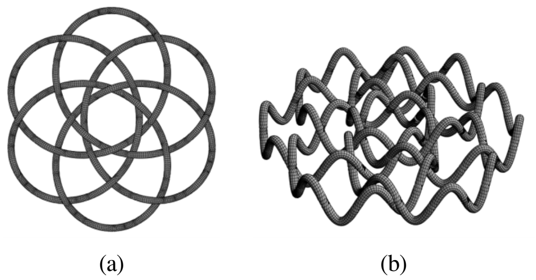


Fig. 9. Knot 3 obtained from hypotorochoid curve (24) [12,2,16.9,4,36].

4. Forms of Alternating Knots

4.1. Conditions for alternating knots

In this section we will consider alternating knots and links satisfying the following conditions:

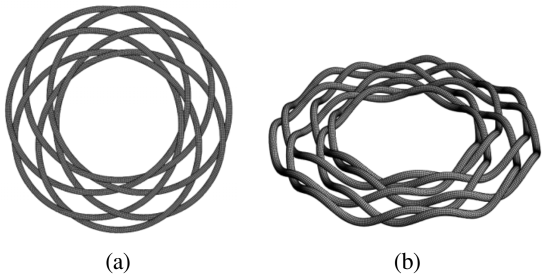


Fig. 10. Knot 4 obtained from hypotorochoid curve (21) [14,2,42,3,42].

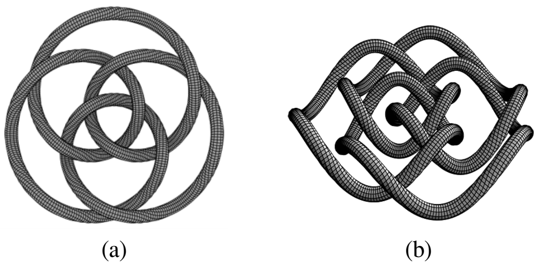


Fig. 11. Knot 1 obtained from epitrochoid curve (9) [1.5,.5,3,1,12].

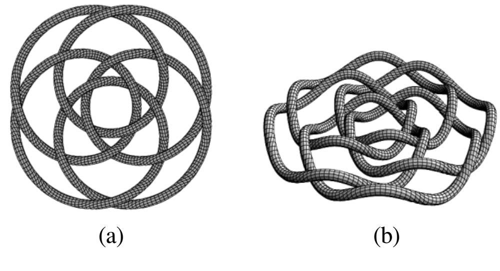
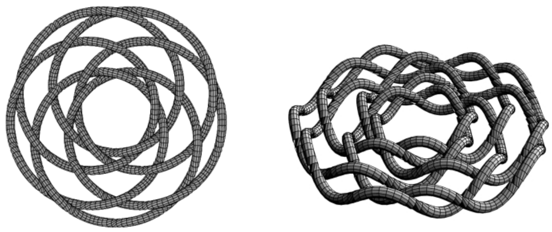


Fig. 12. Knot 2 obtained from epitrochoid curve (16) [12,.3,25,4,20].

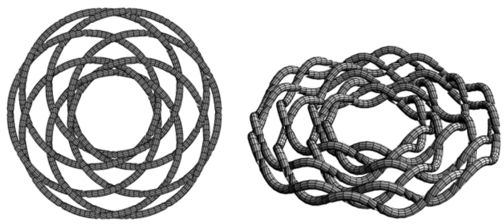
(1) They are derived from plane closed curves, as curved surfaces corresponding to them. Lissajous, trochoid, and toroidal plane curves are used as the basis for the construction of the corresponding knots or links;

(2) The crossing number of a knot or link is denoted by k. Every knot diagram can be reduced by using reidemeister moves to a diagram with a minimum number of crossings, so the crossing number k is the basic invariant for every classification of knots or links. For example, Fig. 3 shows ...



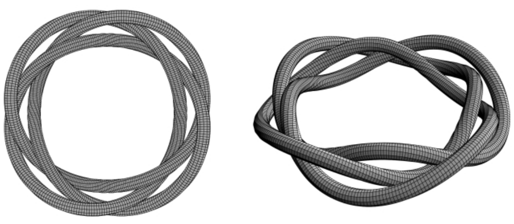
(a) (b)

Fig. 13. Knot 3 obtained from epitrochoid curve (25) [10,2,25,4,30].



(a) (b)

Fig. 14. Knot 4 obtained from epitrochoid curve (36) [3,.5,8,1,42].



(a) (b)

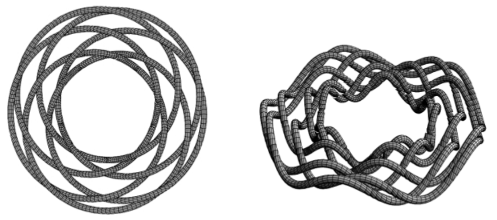
Fig. 15. Knot 1 obtained from torus (8) [6,1,8,6,16].

4.2. Forms of alternating knots by lissajous curves

Lissajous curves are defined by the following parametric representation (KOBAYASHI, 1997):

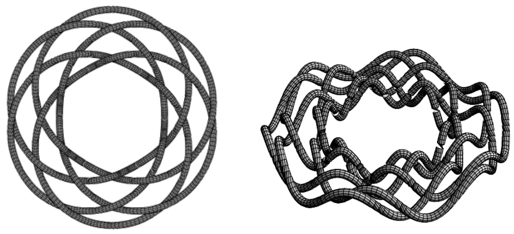
By introducing the third coordinate z added to (1), we obtain the following expressions:

$$x = a \sin lt$$
$$y = a \sin mt \quad (1)$$



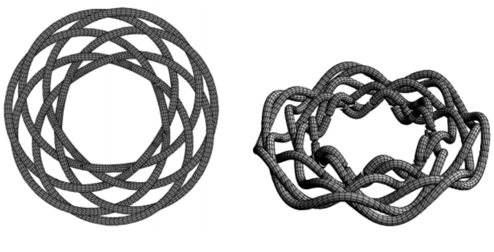
(a) (b)

Fig. 16. Knot 2 obtained from torus (25) [6,2,5,6,35].



(a) (b)

Fig. 17. Knot 3 obtained from torus (24) [6,2,6,5,36].



(a) (b)

Fig. 18. Knot 4 obtained from torus (25) [10,2,25,4,30].

or,

$$x = a \cos lt$$
$$y = a \cos mt. \tag{2}$$

By introducing the third coordinate z added to (1), we obtain the following expressions:

Alternating knots are obtained from (3) by varying parameters (MORITA, 2005).

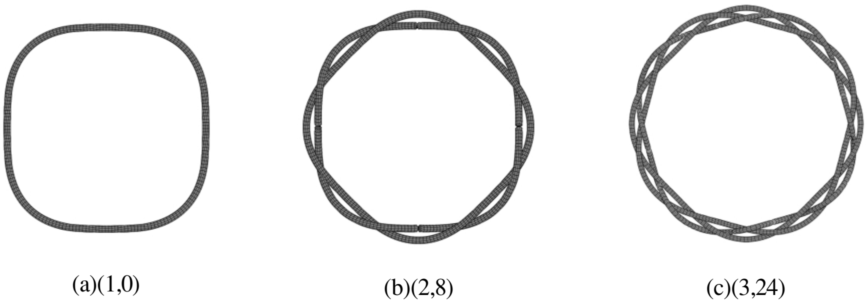


Fig. 19. Figures 19(b) and (c) are formed with units of Fig. 19(a).

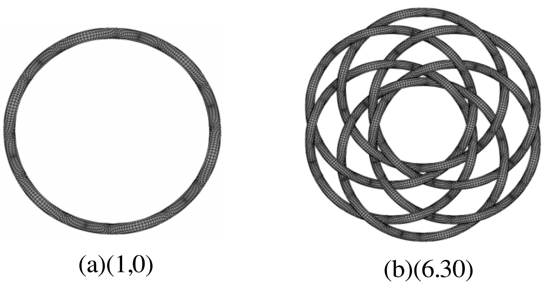


Fig. 20. Figure 20(b) was formed with units of Fig. 20(a).

$$x = a \sin lt$$
$$y = a \sin mt$$
$$z = b \sin nt. \quad (3)$$

Interpretation of each of the figures is the following: (a) is the knot diagram, i.e., the plane drawing of a knot, and (b) is its perspective drawing. For each knot is given the list of its corresponding parameters.

The result of forming alternating knots from lissajous curves is the following: alternating knots can be obtained from lissajous curves if $k = n$.

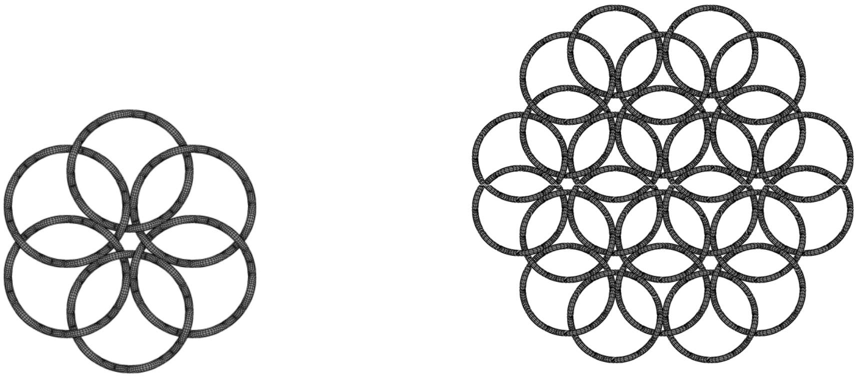
4.3. Form of alternating knots created from trochoid curves

4.3.1 Forms of alternating knots obtained from hypotrochoid curves

Hypotrochoid curves are defined by the following parametric representation:

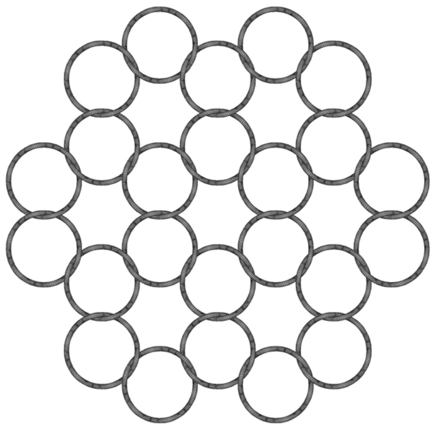
$$x = (a - b) \cos t + c \cos(a - b)t/b$$
$$y = (a - b) \sin t - c \sin(a - b)t/b. \quad (4)$$

Adding the coordinate z to (4), it becomes:



(a)(6,12) (b)(24,60)

Fig. 21. Figure 21(b) was formed based on Fig. 21(a).



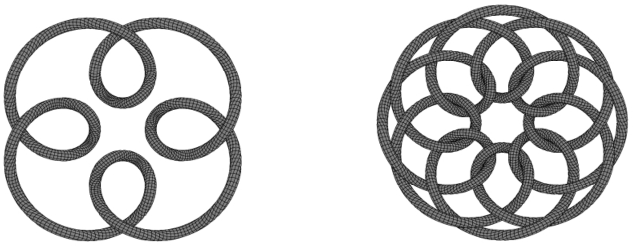
(a)(6,12) (b)(24,60)

Fig. 22. Figure 22(b) was formed based on Fig. 22(a).

$$x = (a - b)\cos t + c \cos(a - b)t/b$$
$$y = (a - b)\sin t - c \sin(a - b)t/b$$
$$z = d \sin et. \qquad (5)$$

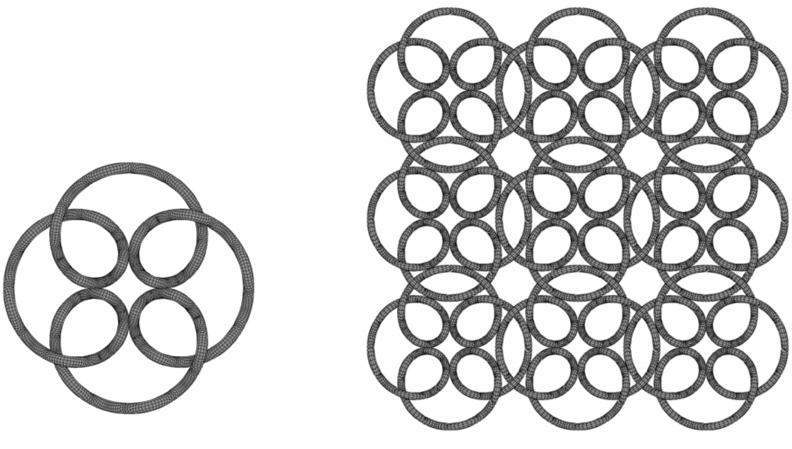
Alternating knots are obtained from (5) by varying parameters.

Examples of alternating knots obtained from hypotrochoid curves are shown in Figs. 7–10.



(a)(1,4) (b)(2,24)

Fig. 23. Figure 23(b) was formed with units of Fig. 23(a).



(a)(1,4) (b)(9,48)

Fig. 24. Figure 24(b) was formed with units of Fig. 24(a).

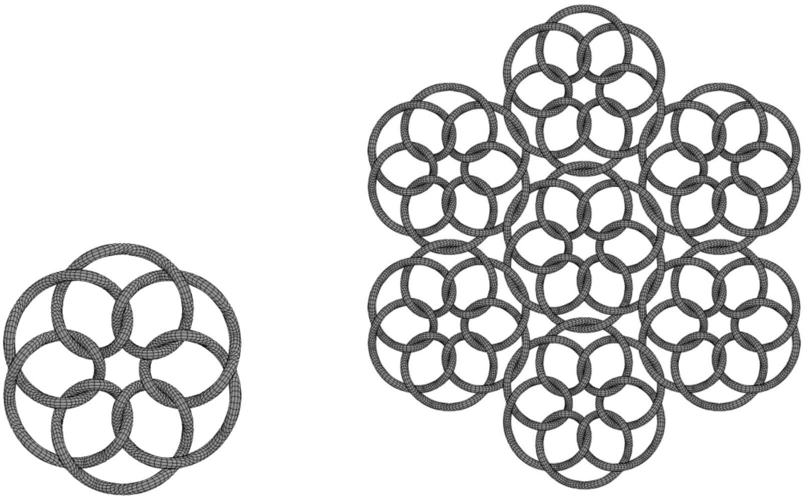
4.3.2 Forms of altermating knots obtained from epipotrochoid curves

Epitrochoid curves can be defined by the following parametric representation

$$x = (a + b)\cos t - c\cos(a + b)t/b$$
$$y = (a + b)\sin t - c\sin(a + b)t/b. \quad (6)$$

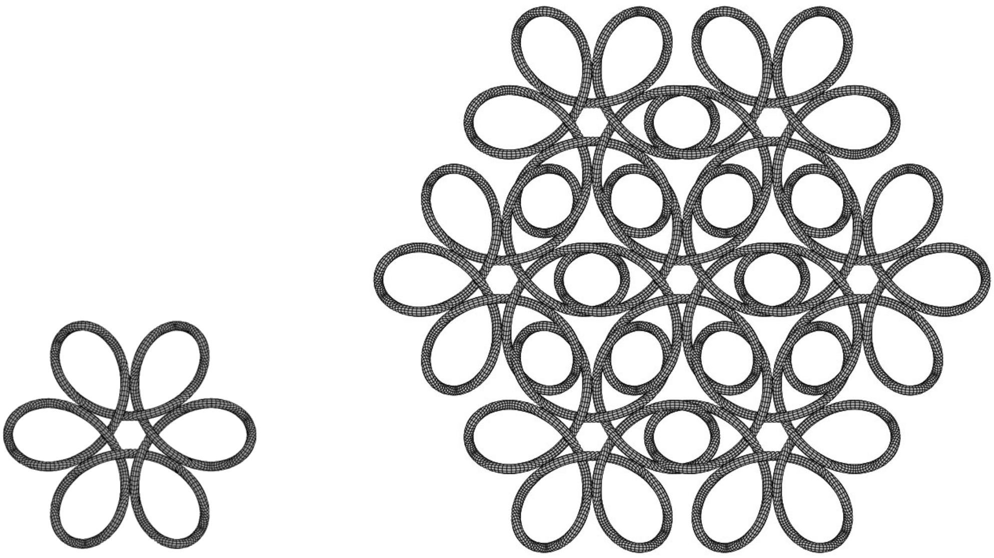
Adding the coordinate z to (6), it becomes:

$$x = (a + b)\cos t - c\cos(a + b)t/b$$
$$y = (a + b)\sin t - c\sin(a + b)t/b$$
$$z = d\sin et. \quad (7)$$



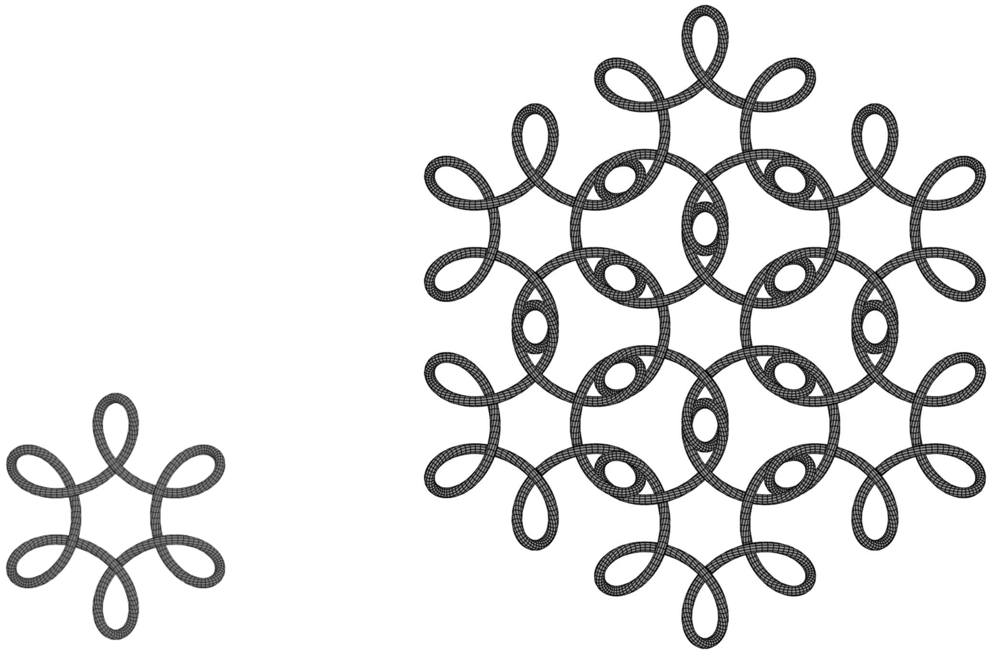
(a)(1,18) (b)(7,132)

Fig. 25. Figure 25(b) was formed with units of Fig. 25(a).



(a)(1,6) (b)(7,66)

Fig. 26. Figure 26(b) was formed with units of Fig. 26(a).



(a)(1,6) (b)(7,64)

Fig. 27. Figure 27(b) was formed with units of Fig. 27(a).

Alternating knots are formed from (7) by changing parameters. Examples of alternating knots obtained from epitrochoid curves are shown in Figs. 11–14.

The results of the formation of alternating knots from hypotrochoid and epitrochoid curves are the following: if we denote by s the crosing number of a central figure in the knot, alternating knots can be formed from trochoid curves if $k = sx$ and $e = sy$.

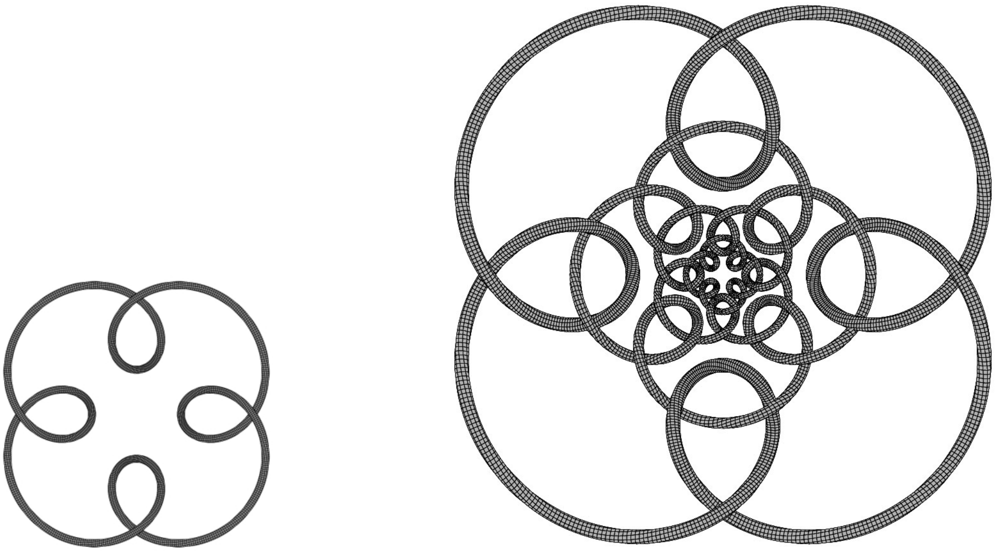
4.4. Forms of alternating knots obtained from toruses

A toroidal curve can be defined by the following parametric representation:

$$x = (a + b \cos ct)\cos ct$$
$$y = (a + b \cos ct)\sin dt$$
$$z = d \sin et. \tag{8}$$

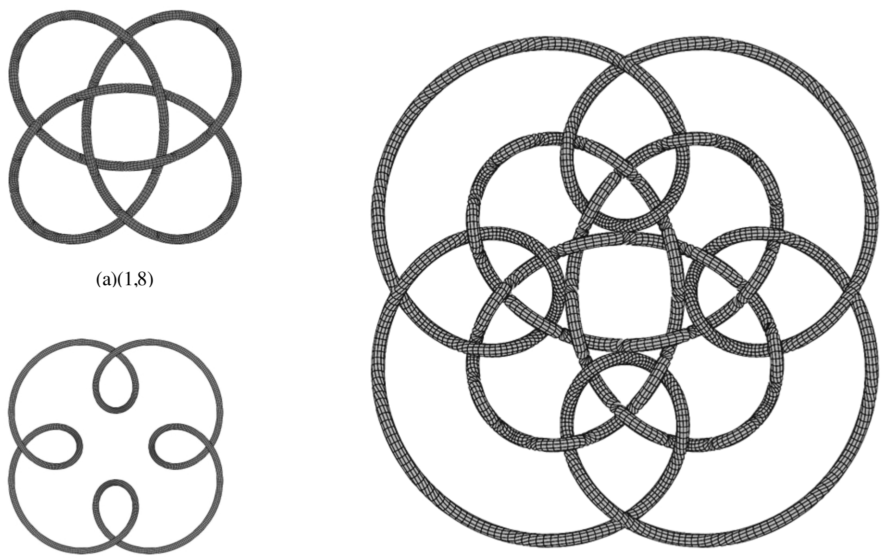
Alternating torus knots are obtained from (8) by varying the parameters. Examples of alternating torus knots are shown in Figs. 15–18.

The results of the formation of alternating knots from toroidal curves are the following: alternating knots can be formed from toroidal curves if $k = sx$ and $e = sy$.



(a)(1,4) (b) (4,40)

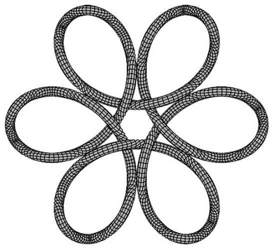
Fig. 28. Figure 28(b) shows examples of knot patterns of similar figures having epitrochoid covers.



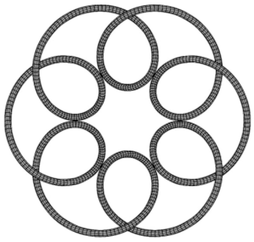
(a)(1,8)

(b)(1,4) (c) (4,24)

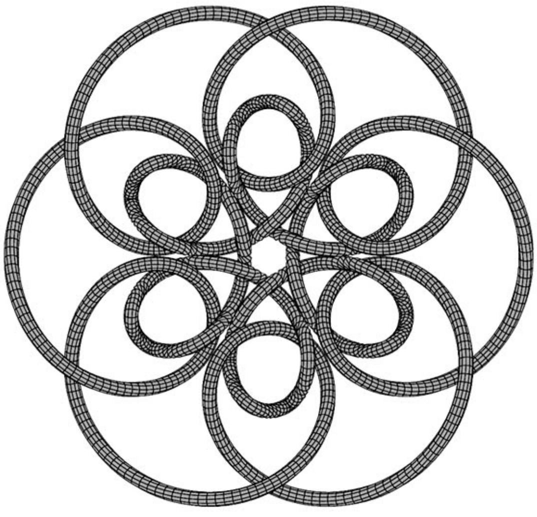
Fig. 29. Figure 29(c) was formed with Figs. 29(a) and (b).



(a)(1,6)



(b)(1,6)



(c)(2.28)

Fig. 30. Figure 30(c) was formed with Figs. 30(a) and (b).

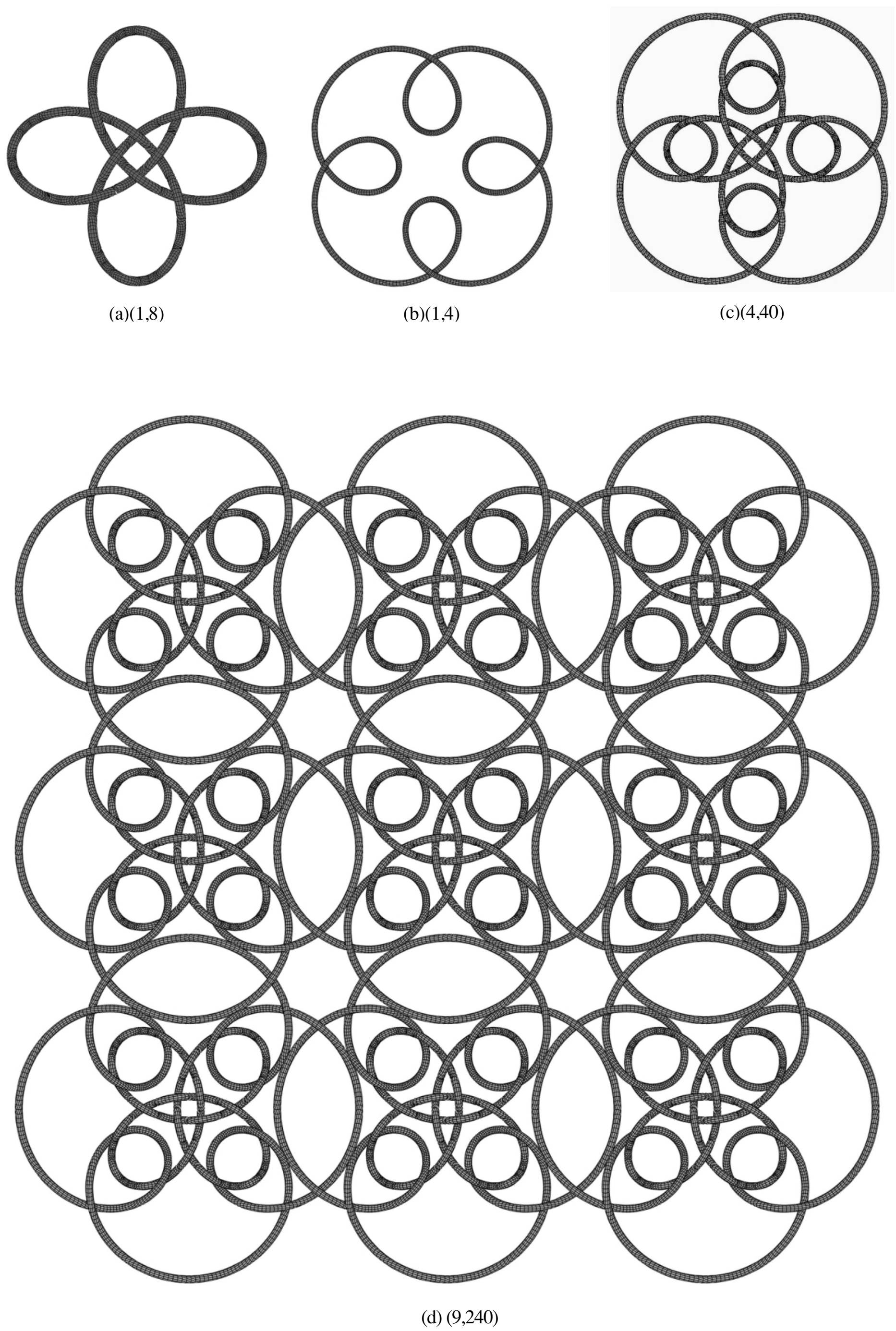
5. Forms of Knot Patterns Based on Combinations

5.1. Conditions for the formation of knots

The conditions to the form of the knots are as follows: (1) They are restricted to cases where links have more than two components. (2) In the first stage knot patterns are formed based on combinations of similar shapes. (3) In the second stage knot patterns are formed based on combinations of different shapes. (4) This paper will first intoroduce a number of components, and then a crossing number for the classification of knots. For example, Fig. 19(b) (2,8).

5.2. Knot patterns based on combinations of similar shapes

Examples of knot patterns consisting of units are shown in Figs. 19–27. The knots in Figs. 19(b) and (c) are formed with units of Fig. 19(a). The two examples of knot patterns are with a fixed center for the units. Examples of knot patterns consisting of rings are shown in Figs. 20(b), 21(a), and 22(a), and Fig. 21(b) are based on Fig. 22(a); Fig. 22(b) is based on Fig. 22(a). Examples of knot patterns consisting of epitrochoid curves are shown in Figs. 23(b), 24(b), and 25(b). Figure 23(b) was formed with units of Fig. 23(a), combining two of these. Figure 24(b) consists of knots of Fig. 24(a); Fig. 25(b) of that of Fig. 25(a). Examples of knot patterns consisting of hypotrochoid curves are shown in Figs. 26(b) and 27(b). Figure 26(b) consists of knots of Fig. 26(a); Fig. 27(b) of that of Fig. 27(a). Figure 28(b) shows examples of knot patterns of similar figures having epitrochoid curves.



(a)(1,8) (b)(1,4) (c)(4,40)

(d) (9,240)

Fig. 31. Figure 31(c) was formed with Figs. 31(a) and (b). Figure 31(d) was formed with units of Fig. 31(c).

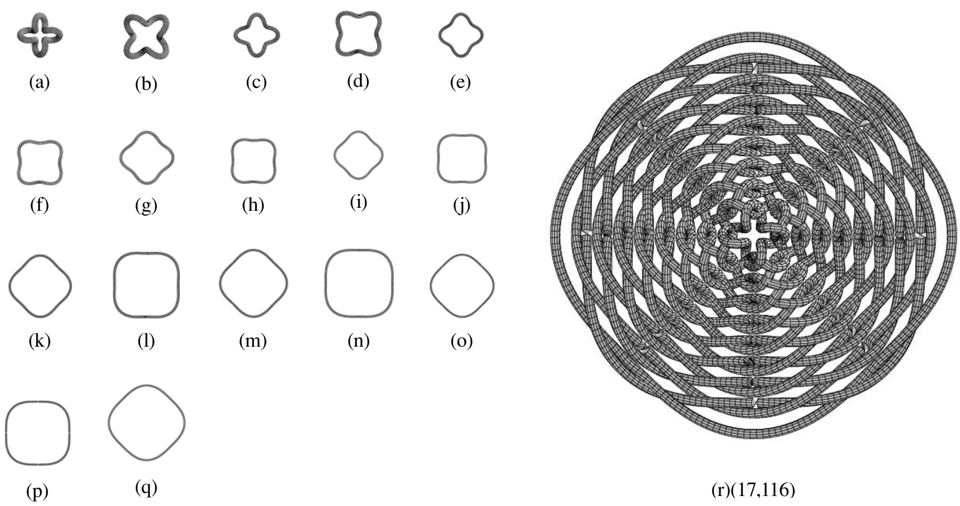


Fig. 32. Figure 32(r) is based on the combination of Figs. 32(a) to (q).

5.3. Knot patterns based on combinations of different shapes

Knot patterns can be obtained by using the combinations of epitrochoid and hypotrochoid curves. The curves in Fig. 29(c) are derived from Figs. 29(a) and (b). Figure 30(c) was formed with Figs. 30(a) and (b). Figure 31(c) was formed with Figs. 31(a) and (b), Fig. 31(d) was formed with units of Fig. 31(c). Figure 32(r) is based on the combination of Figs. 32(a) to (q). Figure 32(r) consists from torus knots.

6. Conclusions

This paper focuses on the shape of knots, using computer graphics, and represents an attempt to show different variations of the form of knots. An the first stage, knot patterns are restricted to alternating knots. An the second stage knot patterns are formed by using the combinations of similar shapes, and an the third stage knot patterns are constructed by using the combinations of different shapes.

The conclusions are the following:

1) in the case $k = n$ alternating knots can be obtained from Lissajous curves;

2) in the case $k = sx$ and $e = sy$, alternating knots can be obtained from trochoid curves and toroidal curves;

3) in knot patterns based on combinations of basic units, they appear interesting shapes with the overlapping of the units;

4) in knot patterns based on combinations of similar figures, they occur various shapes depending on basic units and their combinations;

5) in knot patterns based on combinations of different shapes it is possible to obtain new knot patterns by increasing the number of basic types of shapes that are combined.

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