Original Paper *Forma*, **22**, 141–155, 2007

Large Deviations, Gibbs Measures, and Topologically Conjugate Transforms in Network Dynamics

Syuji MIYAZAKI

Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
syuji@acs.i.kyoto-u.ac.jp

(Received March 6, 2007; Accepted May 9, 2007)

Keywords: Deterministic Chaos, Transition Matrix, Frobenius-Perron Operator, Large Deviation, Gibbs Measure

Abstract. A chaotic piecewise linear map whose statistical properties are identical to those of a random walk on directed graphs such as the world wide web (WWW) is constructed, and the dynamic quantity is analyzed in the framework of large deviation statistics. Gibbs measures include the weight factor appearing in the weighted average of the dynamic quantity, which can also quantitatively measure the importance of web sites. Currently used levels of importance in the commercial search engines are independent of search terms, which correspond to the stationary visiting frequency of each node obtained from a random walk on the network or equivalent chaotic dynamics. Levels of importance based on the Gibbs measure depend on each search term which is specified by the searcher. The topological conjugate transformation between one dynamical system with a Gibbs measure and another dynamical system whose standard invariant probability measure is identical to the Gibbs measure is also discussed.

1. Introduction

One of the most remarkable points about deterministic chaos is the duality consisting of irregular dynamics and fractal structure of the attractor in the phase space. Amplifying this relationship between dynamics and geometry, we will try to construct dynamics corresponding to a directed network structure such as the WWW. A directed network or graph can be represented by a transition matrix. On the other hand, temporal evolution of a chaotic piecewise-linear one-dimensional map with Markov partition can be governed by a Frobenius-Perron matrix. Both transition matrices and Frobenius-Perron matrices belong to a class of transition matrices sharing the same mathematical properties. The maximum eigenvalue is equal to unity. The corresponding eigenvector is always a real vector, and evaluates the probability density of visiting a subinterval of the map or a site of the network, which is commercially valuable information in the field of the WWW (PAGE *et al*., 1988). Relating these two matrices to each other, we are able to represent the structure of the directed network as a dynamical system. Once we relate the directed network to chaotic dynamics, several approaches to deterministic chaos can be also applied to graph theory.

In chaotic dynamical systems, local expansion rates which evaluate an orbital instability fluctuate largely in time, reflecting a complex structure in the phase space. Its average is called the Lyapunov exponent, whose positive sign is a practical criterion of chaos. There exist numerous investigations based on large deviation statistics in which one considers distributions of coarse-grained expansion rates (finite-time Lyapunov exponent) in order to extract large deviations caused by non-hyperbolicities or long correlations in the vicinity of bifurcation points (MORI and KURAMOTO, 1998).

In general, statistical structure functions consisting of weighted averages, variances, and these partition functions as well as fluctuation spectra of coarse-grained dynamic variables can be obtained by processing the time series numerically. In the case of the piecewise-linear map with Markov partition, we can obtain these structure functions analytically. This is one of the reasons why we correspond a directed network to a piecewise-linear map. We herein try to apply an approach based on large deviation statistics in the research field of chaotic dynamical systems to network analyses. We discuss mainly the Gibbs measure in the present paper. And we also discuss topological conjugate transformation between one dynamical system with a Gibbs measure and another dynamical system whose standard invariant probability measure is identical to the Gibbs measure.

2. One-Dimensional Map Corresponding to Directed Network

Let us define this adjacency matrix as A, where A_{ij} is equal to unity if the node j is linked to i. If not, A_{ij} is equal to zero. In the case of an undirected network, $A_{ij} = A_{ji} = 1$ holds if the nodes i and j are linked to each other. If not, A_{ij} is equal to zero. Transition matrix H can be derived straightforwardly from the adjacency matrix. The element H_{ij} is equal to A_{ij} divided by the number of nonzero elements of column j, which is equal to output degree $d_j = \sum_i \mathrm{A}_{ij}$ of node j, so that we have $H = AW$, where W is a diagonal matrix with $W_{ii} = 1/d_i$. The maximum eigenvalue is always equal to unity. The right eigenvector h with $Hh = h$ measures site importance in the context of the web network (PAGE *et al*., 1988). The left eigenvector v with $vH = v$ satisfies $v_1 = v_2 = \ldots$, since all the nonzero elements are equal in the same column of H. We will consider the very simple example whose transition matrix is given by

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}.$$

This example corresponds to a directed network consisting of two nodes in which node 1 is always directed to node 2, and node 2 to both with equal probability as shown in Fig. 1. In other words, this is a special coin tossing in which heads (node 1) are always followed by tails (node 2), and tails by heads or tails with equal probability.

This directed network can be represented by a map f from unit interval $I = [0,1)$ to I. We divide I into I_1 and I_2 with $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \phi$. The most simple choice of the

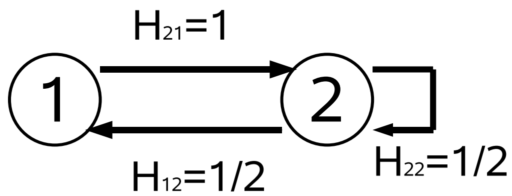

Fig. 1. Directed network consisting of two nodes in which node 1 is always directed to node 2, and node 2 to both with equal probability.

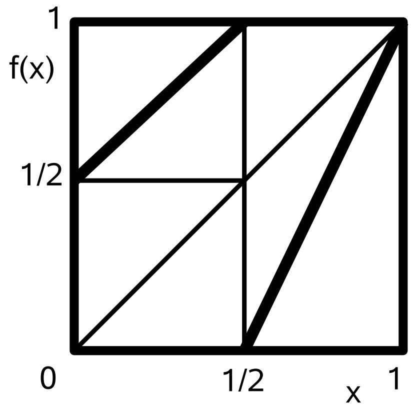

Fig. 2. Piecewise-linear map corresponding to the directed network consisting of two nodes in which node 1 is always directed to node 2, and node 2 to both with equal probability.

map satisfying $f(I_1) = I_2$, $f(I_2) = I_1 \cup I_2$ is piecewise linear, and is given by $f(x) = x + 1/2$ $(0 \le x < 1/2)$, $2x - 1$ $(1/2 \le x \le 1)$ with $I_1 = [0,1/2)$ and $I_2 = [1/2,1)$ as shown in Fig. 2. This is a simple example of Markov partition. The slope $f'(I_1) = 1$ is equal to the output degree of node 1, $f'(I_2) = 2$ to that of node 2. Note that the slope of the map is equal to the output degree of the graph in this way and that the element of H is equal to the inverse of a slope or of an output degree. This map is a special case of Kalman's map, whose static and dynamical properties reproduce those of a discrete-time discrete-state Markov process characterized by the aforementioned transition matrix H (KALMAN, 1956; KOHDA and FUJISAKI, 1999).

In the case of chaotic dynamics caused by a one-dimensional map f, the trajectory is given by iteration. Its distribution at time n, $\rho_n(x)$, is given by the average of this delta function $\langle \delta(x_n - x) \rangle$, where $\langle \cdots \rangle$ denotes the average with respect to initial points x_0. The temporal evolution of ρ is given by the following relation

$$\rho_{n+1}(x) = \int_0^1 \delta\big(f(y) - x\big)\rho_n(y)dy \equiv \mathcal{H}\rho_n(x).$$

This operator $\mathcal{H}$, called the Frobenius-Perron operator, is explicitly given as

$$\mathcal{H}G(x) = \sum_j \frac{G(y_j)}{\left|f'(y_j)\right|},$$

where the sum is taken over all solutions $y_j(x)$ satisfying $f(y_j) = x$.

In the case of a piecewise-linear map with Markov partition, invariant density is constant for each interval. Taking these 2 functions as a basis, we can represent the Frobenius-Perron operator as this 2 by 2 matrix H. This is nothing but the transition matrix of the directed graph consisting of 2 nodes mentioned before. For an arbitrary transition matrix, all the elements are summed up to unity along the same column. For a Frobenius-Perron matrix of a piecewise linear Markov map corresponding to a unweighted network, this sum rule is also satisfied, and furthermore, all the nonzero elements are equal to the inverse of output degree in the same column. In this sense, our Frobenius-Perron matrix belongs to a subset of the whole transition matrix (stochastic matrix). For a weighted network, where the transition probabilities between the nodes distribute, Kalman's map (KALMAN, 1956; KOHDA and FUJISAKI, 1999) must be considered. In this case, the sum rule along the column is not satisfied, as in the case of general Frobenius-Perron matrices.

Right and left eigenvectors corresponding to eigenvalue 1 of the aforementioned Frobenius-Perron matrix H are determined. The left eigenvector v is given by $v = (1/2, 1/2)$, where it is so normalized that the sum of all elements is equal to unity. Note that the element is equal to the width of the subintervals I_1 and I_2 of the Markov partition. The right eigenvector h gives the probability density to visit each subinterval, and is equal to $h = (2/3, 4/3)$, where it is so normalized that the inner product of the right and the left eigenvectors is equal to unity.

The Lyapunov exponent of the one-dimensional map is an average of the logarithm of the slope of the map with respect to its invariant density. Comparing the directed network, the map f, the matrix H, we find that the output degree of node k is equal to the slope of the interval I_k. Thus the Lyapunov exponent of the network is found to be an average of the logarithm of the output degree of each node. This exponent quantifies the complexity of link relations. Degree distribution, a function describing the total number of vertices in a graph with a given degree (number of connections to other vertices), is often used to characterize link relations. The network Lyapunov exponent and its fluctuation are also supposed to be useful in the context of the network. In this way, we can relate the network structure itself to a chaotic dynamical system, and we try to characterize the network based on an approach to deterministic chaos, namely large deviation statistics, in other words, thermodynamical formalism. Large deviation statistics of Lyapunov exponents of chaotic dynamical systems are intensively discussed in the literature (MORI and KURAMOTO, 1998).

3. Large Deviation Statistics

Let us briefly describe large deviation statistics following the series of studies by

Fujisaka and his coworkers (FUJISAKA and INOUE, 1987, 1989; FUJISAKA and SHIBATA, 1991). Consider a stationary time series of a dynamic variable $\tilde{u}\{t\}$ at time t. The average over time interval T is given by this formula,

$$\bar{u}_T(t) = \frac{1}{T}\int_t^{t+T} \tilde{u}\{s\}ds,$$

which distributes when T is finite. When T is much larger than the correlation time of $\tilde{u}$, the distribution $P_T(u)$ of coarse-grained $u = \bar{u}_T$ is assumed to be an exponential form $P_T(u) \propto \mathrm{e}^{-S(u)T}$. Here we can introduce fluctuation $S(u)$ as

$$S(u) = -\lim_{T\to\infty}\frac{1}{T}\log P_T(u).$$

When T is comparable to the correlation time, correlation cannot be ignored, so non-exponential or non-extensive statistics will be a problem, but here we do not discuss this point any further. Let q be a real parameter. We introduce the generating function M_q of T by this definition:

$$M_q(T) \equiv \left\langle \mathrm{e}^{qT\bar{u}_T} \right\rangle = \int_{-\infty}^{\infty} P_T(u)\mathrm{e}^{qTu}du.$$

We can also here assume the exponential distribution and introduce a characteristic function $\phi(q)$ as

$$\phi(q) = \lim_{T\to\infty}\frac{1}{T}\log M_q(T).$$

The Legendre transform holds between fluctuation spectrum $S(u)$ and characteristic function $\phi(q)$, which is obtained from saddle-point calculations. $dS(u)/du = q$, $\phi(q) = -S(u(q)) + qu(q)$. In this transform a derivative $d\phi/dq$ appears, and is a weighted average of $\bar{u}_T$,

$$u(q) = \frac{d\phi(q)}{dq} = \lim_{T\to\infty}\frac{\left\langle \bar{u}_T\mathrm{e}^{qT\bar{u}_T} \right\rangle}{M_q(T)},$$

so we find that q is a kind of weight index. We can also introduce susceptibility $\chi(q) = du(q)/dq$ as a weighted variance. These statistical structure functions $S(u)$, $\phi(q)$, $u(q)$, $\chi(q)$ constitute the framework of statistical thermodynamics of temporal fluctuation, which characterize the static properties of chaotic dynamics. In order to consider dynamic properties, we can introduce this generalized spectrum density as a weighted average of conventional spectrum density as

$$I_q(\omega) = \lim_{T\to\infty} \left\langle \left| \int_0^T \left[\tilde{u}\{t+s\} - u(q)\right] \mathrm{e}^{-i\omega s} ds \right|^2 \mathrm{e}^{qT\overline{u}_T} \right\rangle \Big/ \left(T\, M_q(T)\right).$$

In the same way, the generalized double time correlation function is

$$C_q(T) = \lim_{T\to\infty} \lim_{\tau\to\infty} \left\langle \left(\tilde{u}\{t+\tau\} - u(q)\right)\left(\tilde{u}\{\tau\} - u(q)\right) \mathrm{e}^{qT\overline{u}_T} \right\rangle \Big/ M_q(T).$$

The relation between the two is given by the Wiener-Khintchine theorem:

$$C_q(T) = \int_{-\infty}^{\infty} I_q(\omega) \mathrm{e}^{-i\omega t} d\omega \big/ (2\pi) \text{ and } I_q(\omega) = \sum_{t=-\infty}^{\infty} C_q(t) \mathrm{e}^{-i\omega t}.$$

In the case of chaotic dynamics, q of thermodynamical formalism is merely a weight index to process time series. This index q can be used to control traffic in the context of the web network.

Let us consider the case of a one-dimensional map. Let $\tilde{u}[x_n]$ be a unique function of x, which is governed by the map $x_{n+1} = f(x_n)$. The question is how to obtain statistical structure functions and generalized spectral densities of $\tilde{u}$. The answer is to solve the eigenvalue problems of a generalized Frobenius-Perron operator. As we mentioned before, the characteristic function $\phi(q)$ is given by the asymptotic form of the generating function $M_q(n)$ in the limit of $n \to \infty$ corresponding to the temporal coarse-grained quantity

$$\overline{u}_n = \frac{1}{n} \sum_{j=0}^{n-1} \tilde{u}\left[x_{j+m}\right],$$

where we assume an exponential fast decay of time correlations of u. A generating function can be expressed in terms of invariant density,

$$M_q(n) = \int \rho_\infty(x) \exp\left[q \sum_{j=0}^{n-1} \tilde{u}\left[f^j(x)\right] \right] dx = \int \mathcal{H}_q^n \rho_\infty(x) dx,$$

where the generalized Frobenius-Perron operator $\mathcal{H}_q$ is defined and related to the original one as

$$\mathcal{H}_q G(x) = \mathcal{H}\left[\mathrm{e}^{q\tilde{u}[x]} G(x)\right] = \sum_k \frac{\mathrm{e}^{q\tilde{u}[y_k]} G(y_k)}{\left|f'(y_k)\right|}$$

for an arbitrary function $G(x)$ ($\mathcal{H}_0 = \mathcal{H}$). To obtain the above equation, the following relation is repeatedly used:

$$\mathcal{H}\left\{G(x)\exp\left[q\sum_{j=0}^{m}\tilde{u}\left[f^{j}(x)\right]\right]\right\} = \left(\mathcal{H}_{q}G(x)\right)\exp\left[q\sum_{j=0}^{m-1}\tilde{u}\left[f^{j}(x)\right]\right].$$

The normal Frobenius-Perron operator $\mathcal{H}$ depends on the map f only. The generalized one $\mathcal{H}_q$ depends also on a dynamic variable u and determines statistical structure functions and generalized spectral densities of u. For example, in the case of local expansion rates $\tilde{u}[x] = \log|f'(x)|$ whose average is the Lyapunov exponent, the generalized operator is explicitly given by

$$\mathcal{H}_{q}G(x) = \sum_{k}\frac{G(y_k)}{\left|f'(y_k)\right|^{1-q}}.$$

In the case of the simple network mentioned earlier, two subintervals constitute the Markov partition, such that $\mathcal{H}_q$ can be represented by a 2×2 matrix as

$$H_q = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} e^{q\tilde{u}[I_1]} & 0 \\ 0 & e^{q\tilde{u}[I_2]} \end{pmatrix}$$

in the same way as $\mathcal{H}$, where $\tilde{u}[I_k] = \tilde{u}[x]$ with $x \in I_k$. Let $\nu_q^{(+)}$ be the maximum eigenvalue of H_q. The statistical structure functions and the generalized spectral density are given by

$$\phi(q) = \log\nu_q^{(+)}, \quad u(q) = \frac{d\phi(q)}{dq}, \quad \chi(q) = \frac{du(q)}{dq},$$

$$I_q(\omega) = \int v^{(+)}(x)\left[\tilde{u}[x]-u(q)\right]\left[J_q(\omega)+J_q(-\omega)-1\right]\left[\tilde{u}[x]-u(q)\right]h^{(+)}(x)dx,$$

where $J_q(\omega) = 1/[1 - (e^{i\omega}/\nu_q^{(+)})H_q]$, $v^{(+)}(x)$ and $h^{(+)}(x)$ are respectively the left and right eigenfunctions corresponding to the maximum eigenvalue $\nu_q^{(+)}$ of H_q. The generalized double time correlation function is given by

$$C_q(t) = \int v^{(+)}(x)\left[\tilde{u}[x]-u(q)\right]\left[H_q/\nu_q^{(+)}\right]^{t}\left[\tilde{u}[x]-u(q)\right]h^{(+)}(x)dx.$$

4. Gibbs Measures

Let us analyze the directed network based on the large deviation statistics. We consider an arbitrary dynamic variable and set $\tilde{u}(I_1) = u_1$ and $\tilde{u}(I_2) = u_2$. The generalized Frobenius-Perron matrix H_q can be represented as

$$H_q = \begin{pmatrix} 0 & e^{qu_2}/2 \\ e^{qu_1} & e^{qu_2}/2 \end{pmatrix}.$$

The largest eigenvalue $\nu_q^{(+)}$ and the second one $\nu_q^{(-)}$ are given by

$$\nu_q^{(\pm)} = \frac{e^{qu_2}}{4} \pm \frac{e^{qu_2}}{4}\sqrt{1+8e^{q(u_1-u_2)}},$$

whose right and left eigenvectors

$$\begin{pmatrix} h_1^{(\pm)} & (q) \\ h_2^{(\pm)} & (q) \end{pmatrix}, \quad \left(v_1^{(\pm)}(q) v_2^{(\pm)}(q)\right)$$

are also given by

$$v_1^{(\pm)}(q) = \frac{e^{qu_1}}{e^{qu_1} + \nu_q^{(\pm)}},$$

$$v_2^{(\pm)}(q) = \frac{\nu_q^{(+)}}{e^{qu_1} + \nu_q^{(\pm)}},$$

$$h_1^{(\pm)}(q) = \frac{e^{qu_2}\left(e^{qu_1} + \nu_q^{(\pm)}\right)}{e^{q(u_1+u_2)} + 2\left[\nu_q^{(\pm)}\right]^2},$$

$$h_2^{(\pm)}(q) = \frac{2\nu_q^{(\pm)}\left(e^{qu_1} + \nu_q^{(\pm)}\right)}{e^{q(u_1+u_2)} + 2\left[\nu_q^{(\pm)}\right]^2}$$

satisfying $v_1^{(\pm)}(q) + v_2^{(\pm)}(q) = 1$ and $h_1^{(\pm)}(q)v_1^{(\pm)}(q) + h_2^{(\pm)}(q)v_2^{(\pm)}(q) = 1$.

The characteristic function is given by $\phi(q) = \log\nu_q^{(+)}$. The weighted average $u(q) = d\phi(q)/dq$ is explicitly given by

$$u(q) = u_2 + \frac{u_1 - u_2}{2}\left(1 - \frac{1}{2\sqrt{1+8e^{q(u_1-u_2)}}}\right).$$

Its asymptotic behaviors depend on the sign of $u_2 - u_1$. For $u_1 < u_2$ ($u_1 > u_2$), we have $u(\infty) = u_2$ ($u(\infty) = (u_1 + u_2)/2$) and $u(-\infty) = (u_1 + u_2)/2$ ($u(-\infty) = u_2$). Let us define the scaled variable $U(q) \equiv (u(q) - u(-\infty))/(u(\infty) - u(-\infty))$ and the scaled weighted variance $\sigma(q) \equiv \chi(q)/[(u(\infty) - u(-\infty))^2]$ with $0 < U < 1$. The scaled weighted average and the scaled weighted variance are respectively given by

$$U(q) = \frac{1}{\sqrt{1 + 8e^{-Q}}}, \qquad (u_1 < u_2),$$

$$U(q) = 1 - \frac{1}{\sqrt{1 + 8e^{Q}}}, \qquad (u_1 > u_2),$$

$$\sigma(q) = 8e^{-Q}\left(1 + 8e^{-Q}\right)^{-3/2}, \quad (u_1 < u_2),$$

$$\sigma(q) = 8e^{Q}\left(1 + 8e^{Q}\right)^{-3/2}, \qquad (u_1 > u_2)$$

with $Q \equiv 2q(u(\infty) - u(-\infty))$. The fluctuation spectrum is also explicitly given by

$$S(U) = \frac{U}{2}\log\frac{8U^2}{1 - U^2} + \frac{1}{2}\log\frac{2(1 - U)}{1 + U}, \qquad (u_1 < u_2),$$

$$S(U) = \frac{U}{2}\log\frac{1 - (1 - U)^2}{8\left(1 - U^2\right)} + \log\frac{4(1 - U)}{2 - U}, \qquad (u_1 > u_2).$$

Expanding the spectrum around the average up to the quadratic term, we have a parabola indicating the central limit theorem. The scaled average $U_0 \equiv ((u(0) - u(-\infty))/(u(\infty) - u(-\infty))$ is given by $U_0 = 1/3$ (2/3) for $u_1 < u_2$ ($u_1 > u_2$). The parabola is given by $S_{CLT}(U) = (27/16)(U - U_0)^2$ for both cases. Large deviation statistics obviously do not coincide with the central limit theorem.

The generalized correlation function $C_q(t)$ is given by

$$C_q(t) = K_q \exp\left[-\left(\gamma_q + i\pi\right)t\right],$$

with

$$\gamma_q = \log\frac{\sqrt{1 + 8e^{q(u_1 - u_2)}} + 1}{\sqrt{1 + 8e^{q(u_1 - u_2)}} - 1},$$

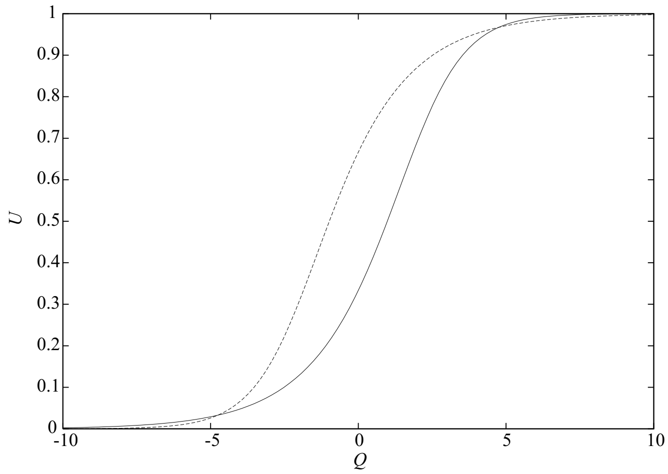

Fig. 3. Scaled weighted average U plotted against the scaled weight index Q for $u_1 < u_2$ (solid line) and $u_1 > u_2$ (dashed line).

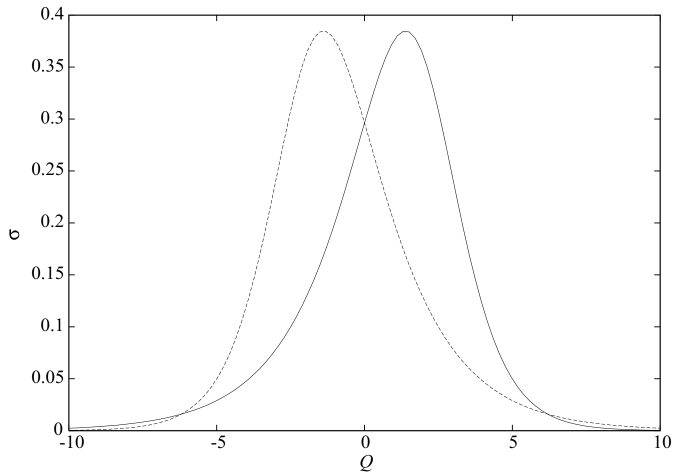

Fig. 4. Scaled weighted variance σ plotted against the scaled weight index Q for $u_1 < u_2$ (solid line) and $u_1 > u_2$ (dashed line).

$$K_q = \left[v_1^{(+)}\left(u_1 - u(q)\right)h_1^{(-)} + v_2^{(+)}\left(u_2 - u(q)\right)h_2^{(-)}\right] \times \left[v_1^{(-)}\left(u_1 - u(q)\right)h_1^{(+)} + v_2^{(-)}\left(u_2 - u(q)\right)h_2^{(+)}\right].$$

The oscillatory factor $\exp[-i\pi t] = (-1)^t$ corresponds to the unstable periodic orbit with period 2. These statistical structure functions are obtained for other simple networks

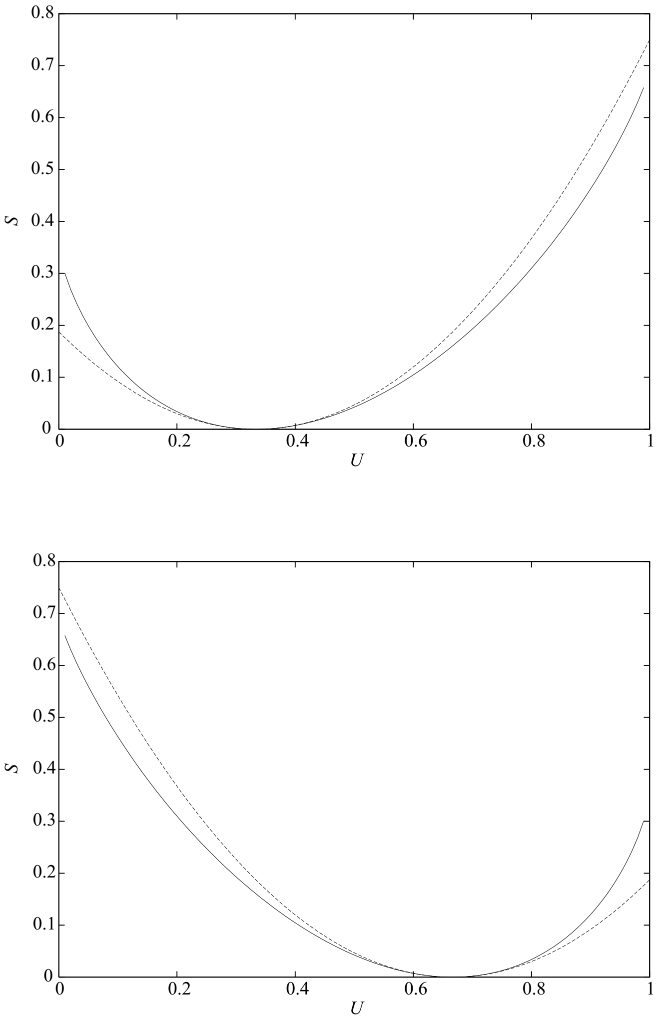

Fig. 5. Fluctuation spectrum S (solid line) and parabola indicating the central limit theorem (dashed line) plotted against U for $u_1 < u_2$ (*u*pper panel) and $u_1 > u_2$ (lower panel).

(MIYAZAKI, 2006a, b, c).

In thermodynamic formalism in the context of the mathematical formulation of equilibrium statistical mechanics (SINAI, 1972; BOWEN, 1975; RUELLE, 1976, 1978) and the ergodic theory of dynamical systems (ECKMANN and PROCACCIA, 1986; PARADIN and VULPIANI, 1987; GRASSBERGER *et al.*, 1988; BESSIS *et al.*, 1988; MORI *et al.*, 1989), the variational principle (WALTERS, 1982)

$$P_f(\varphi) = \sup\left(h_\mu(f) + \int \varphi d\mu\right)$$

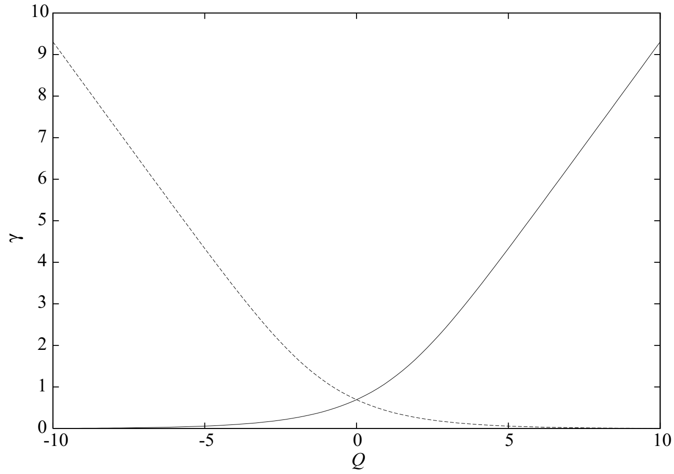

Fig. 6. Damping factor γ plotted against the scaled weight index Q for $u_1 < u_2$ (solid line) and $u_1 > u_2$ (dashed line).

is often used, where φ denotes a piecewise continuous function, $h_\mu(f)$ the Kolmogorov-Sinai entropy and the supremum is attained at a measure μ_φ called Gibbs measure, which depends on chosen function φ. Hereafter we confine ourselves to hyperbolic one-dimensional dynamical systems. The topological pressure $P_f(\varphi)$ may admit different analytical branches corresponding to different local structures of the invariant set reflected by different Gibbs measure (MORI *et al.*, 1989; FEIGENBAUM *et al.*, 1989; JUST and FUJISAKA, 1993). The topological pressure $P_f(\varphi)$ and the characteristic function $\phi(q)$ are identical if one identifies $\varphi = qu - \log|f'|$. The Gibbs measure corresponds to $h_1^{(+)}(q)v_1^{(+)}(q)$ and $h_2^{(+)}(q)v_2^{(+)}(q)$.

Note that $H_q/v_q^{(+)}$ has unit eigenvalue and corresponds to a conventional Frobenius-Perron matrix of the induced map $\tilde{f}$

$$\tilde{f}(x;q,u) = \begin{cases} \dfrac{1-a}{a}x + a, & \left(x \in \tilde{I}_1 = [0,a]\right) \\ \dfrac{1}{1-a}(x-1)+1, & \left(x \in \tilde{I}_2 = [a,1]\right) \end{cases}$$

with

$$a = \frac{e^{qu_1}}{e^{qu_1} + v_q^{(+)}}.$$

Note that the same conditions $[0,1) = \tilde{I}_1 \cup \tilde{I}_2$, $\tilde{I}_1 \cap \tilde{I}_2 = \phi$, $\tilde{f}(\tilde{I}_1) = \tilde{I}_2$ and $\tilde{f}(\tilde{I}_2) = \tilde{I}_1 \cup \tilde{I}_2$ on the Markov partition as the original dynamics f satisfy. The Frobenius-Perron matrix H can be given by a product of the adjacency matrix A and the diagonal matrix W

$$H = AW = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

where the diagonal elements W_{ii} are equal to $1/|f'(I_i)|$. The generalized Frobenius-Perron matrix H_q divided by its largest eigenvalue $\nu_q^{(+)}$ can also be given by a product of the adjacency matrix A and the other diagonal matrix W_q

$$\begin{aligned} \tilde{H}_q &= H_q / \nu_q^{(+)} = AW_q \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{qu_1} / \nu_q^{(+)} & 0 \\ 0 & e^{qu_2} \Big/ \left(2\nu_q^{(+)}\right) \end{pmatrix}, \end{aligned}$$

where the matrix $\tilde{H}_q$ has an unit eigenvalue as the largest one, and the diagonal elements $(W_q)_{ii}$ are equal to $1/|\tilde{f}'(I_i)|$. The two matrices H and $\tilde{H}_q$ share the same adjacency matrix A, the logarithm of the largest eigenvalue of which gives topological entropy, so that the transformation between the maps f and $\tilde{f}$ is topologically conjugate.

Let us consider the concrete dynamic variable $u_1 = 0$ and $u_2 = 1$ as an example of the case of $u_1 < u_2$. The stationary probability density to visit node 1 (heads) and node 2 (tails) are respectively equal to $h_1^{(+)}(0) = 2/3$ and $h_2^{(+)}(0) = 4/3$. The stationary visiting frequencies are $h_1^{(+)}(0)v_1^{(+)}(0) = 1/3$ and $h_2^{(+)}(0)v_2^{(+)}(0) = 2/3$, which is used in the context of the WWW. For $q \to +\infty$ $(-\infty)$, the weighted average $u(q) = v_1^{(+)}(q)u_1h_1^{(+)}(q) + v_2^{(+)}(q)u_2h_2^{(+)}(q)$ is equal to the largest (smallest) possible local average value, i.e., $u(+\infty) = 1$ and $u(-\infty) = 1/2$. The former and the latter correspond respectively to the unstable fixed point of f yielding the time series of $\tilde{u}$: 111 ... with average 1 (only tails) and the unstable periodic point 010101 ... with average 1/2 (heads and tails alternate). In the limit $q \to +\infty$, $\tilde{I}_2$ approaches [0,1) and the width of $\tilde{I}_1$ shrinks, and $\tilde{f}(x;q,u) \to x$ for $x \in \tilde{I}_2$ which implies that all points in [0,1) are marginal fixed points in this limit. The second iteration $\tilde{f} \circ \tilde{f}$ is given by

$$\begin{aligned} \tilde{f} \circ \tilde{f}(x;q,u) = \frac{x}{a}, & \qquad \left(x \in \tilde{I}_1 = [0,a)\right), \\ \frac{x-1}{a} + \frac{1-a+a^2}{a}, & \qquad \left(x \in \tilde{I}_{21} = \left[a, 1-(1-a)^2\right)\right), \\ \frac{x-1}{(1-a)^2} + 1, & \qquad \left(x \in \tilde{I}_{22} = \left[1-(1-a)^2, 1\right)\right). \end{aligned}$$

In the limit $q \to -\infty$, $\tilde{I}_1$ approaches [0,1) and the widths of $\tilde{I}_{21}$ and $\tilde{I}_{22}$ shrink, and $\tilde{f}(x;q,u) \to x$ for $x \in \tilde{I}_1$ which implies that all points in [0,1) are marginal periodic points with period 2 in this limit.

5. Concluding Remarks

As explained in the preceding sections, the differing local structures of the invariant set are reflected by different Gibbs measures. For this dynamical system f, there exist different Gibbs measures. For fixed Gibbs measures $(v_1^{(+)}(q)h_1^{(+)}(q), v_2^{(+)}(q)h_2^{(+)}(q))$, one can construct a induced dynamical system $\tilde{f}$ whose invariant measure (SRB measure) $(\tilde{v}_1^{(+)}(0)\tilde{h}_1^{(+)}(0), \tilde{v}_2^{(+)}(0)\tilde{h}_2^{(+)}(0))$ is identical to the above-chosen Gibbs measure $(v_1^{(+)}(q)h_1^{(+)}(q), v_2^{(+)}(q)h_2^{(+)}(q))$.

In the context of WWW, one may choose a dynamic variable $\tilde{u}$ as $\tilde{u}(I_i) = 1$ if node (web site) i contains a specific keyword; if not $\tilde{u}(I_i) = 0$. Gibbs measures $h_i^{(+)}(q)v_i^{(+)}(q)$ for large positive q reflect only the web sites including the specific keyword and these closely related sites. For large negative q, Gibbs measures reflects only the sites *not* including the keyword and these closely related sites. Only the keyword independent measure $h_i^{(+)}(0)v_i^{(+)}(0)$ is widely used to estimate the importance in the web network. We believe that Gibbs measures $h_i^{(+)}(q)v_i^{(+)}(q)$ are very promising as a measure of site importance depending on search terms or as a filtering technique for harmful contents.

In the present paper, we confined ourselves to hyperbolic systems. Let us consider the quadratic map $x_{n+1} = g(x_n) = 2 - x_n^2$ as the simplest nonhyperbolic system. The statistical structure functions of the local expansion rate $\tilde{u}(x) = \log|g'(x)|$ can be obtained analytically (MORI and KURAMOTO, 1998). The weighted average $u(q)$ is given by a monotone increasing step function. The values of the steps are $-\infty$, log2, and log4, which correspond to the nonhyperbolic phase caused by $g'(0) = 0$, the hyperbolic phase characterized by the long time average called Lyapunov exponent in this case, and the nonhyperbolic crisis phase caused by the collision ($-2 = g \circ g(0)$) of the extremum ($g'(0) = 0$) with the unstable fixed point ($g'(-2) = 4$). It is known that the tent map $y_{n+1} = \tilde{g}(y_n) = 1 - 2|y_n|$ is topologically conjugate to the quadratic map. The instant values as well as the long time average of the local expansion rate of the tent map are equal to log2, which coincides with the weighted average of the hyperbolic phase of the quadratic map. We think that there exists the aforementioned topological conjugate transformation from a hyperbolic phase of a nonhyperbolic dynamical system f to another dynamical system whose standard invariant measure corresponds to the Gibbs measure of f with a certain dynamical quantity $\tilde{u}$.

This study was partially supported by the 21st Century COE Program "Center of Excellence for Research and Education on Complex Functional Mechanical Systems" at Kyoto University.

REFERENCES

BESSIS, D., PARADIN, G., TURCHETTI, G. and VALENTI, S. (1988) *J. Stat. Phys.*, **51**, 109.

BOWEN, R. (1975) *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Mathematics*, Vol. **470**, Springer, Berlin.

ECKMANN, J. P. and PROCACCIA, I. (1986) *Phys. Rev. A*, **34**, 659.

FEIGENBAUM, M. J., PROCACCIA, I. and TEL, T. (1989) *Phys. Rev. A*, **39**, 5359.
GRASSBERGER, P., BADII, R. and POLITI, A. (1988) *J. Stat. Phys.*, **51**, 135.
FUJISAKA, H. and INOUE, M. (1987) *Prog. Theor. Phys.*, **77**, 1334.
FUJISAKA, H. and INOUE, M. (1989) *Phys. Rev. A*, **39**, 1376.
FUJISAKA, H. and SHIBATA, H. (1991) *Prog. Theor. Phys.*, **85**, 187.
JUST, W. and FUJISAKA, H. (1993) *Physica D*, **64**, 98.
KALMAN, R. E. (1956) *Proc. Symp. Nonlinear Circuit Analysis VI*, 273.
KOHDA, T. and FUJISAKI, H. (1999) *IEICE Trans. Fundamentals*, **E82-A**, 1747.
MIYAZAKI, S. (2006a) *IPSJ Journal*, **47**, 795 (in Japanese).
MIYAZAKI, S. (2006b) Springer LNAI series 4012, *New Frontiers in Artificial Intelligence: Proceeding of the 19th Annual Conferences of the Japanese Society for Artificial Intelligence*, pp. 261–270.
MIYAZAKI, S. (2006c) *Prog. Theor. Phys. Suppl.*, No. 162, 147.
MORI, H. and KURAMOTO, Y. (1998) *Dissipative Structures and Chaos*, Springer, Berlin.
MORI, H., HATA, H., HORITA, T. and KOBAYASHI, T. (1989) *Prog. Theor. Phys. Suppl.*, No. 99, 1.
PAGE, L., BRIN, S., MOTWANI, R., WINOGRAD, T. (1988) *The PageRank Citation Ranking: Bringing Order to the Web* (http://www-db.stanford.edu/~backrub/pageranksub.ps).
PARADIN, G. and VULPIANI, A. (1987) *Phys. Rep.*, **156**, 147.
RUELLE, D. (1976) *Am. J. Math.*, **98**, 619.
RUELLE, D. (1978) *Thermodynamical Formalism, Encyclopedia of Mathematics and Its Applications*, Vol. 5, Addison-Wesley, Reading.
SINAI, Ya. G. (1972) *Usp. Math. Nauk.*, **27**, 21.
WALTERS, P. (1982) *An Introduction to Ergodic Theory*, Springer, Berlin.