Original Paper ——— *Forma*, **22**, 157–175, 2007

Packing and Minkowski Covering of Congruent Spherical Caps on a Sphere, II: Cases of $N = 10$, 11, and 12

Teruhisa SUGIMOTO[1]* and Masaharu TANEMURA[1,2]

[1]*The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan*
[2]*Department of Statistical Science, The Graduate University for Advanced Studies, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan*
**E-mail address: sugimoto@ism.ac.jp*

(Received August 6, 2007; Accepted October 3, 2007)

Keyword: Spherical Cap, Sphere, Packing, Covering, Tammes Problem

Abstract. Let C_i ($i = 1, ..., N$) be the i-th open spherical cap of angular radius r and let M_i be its center under the condition that none of the spherical caps contains the center of another one in its interior. We consider the upper bound, r_N, (not the lower bound!) of r of the case in which the whole spherical surface of a unit sphere is completely covered with N congruent open spherical caps under the condition, sequentially for $i = 2, ..., N - 1$, that M_i is set on the perimeter of C_{i-1}, and that each area of the set $(\cup_{\nu=1}^{i-1} C_\nu) \cap C_i$ becomes maximum. In this paper, for $N = 10$, 11, and, 12, we found out that the solutions of our sequential covering and the solutions of the Tammes problem were strictly correspondent. Especially, we succeeded to obtain the exact closed form of r_{10} for $N = 10$.

1. Introduction

The circle on the surface of a sphere is called a spherical cap. Among the problems of packing on the spherical surface, the closest packing of congruent spherical caps is the most famous, and is usually known as the Tammes problem (TAMMES, 1930). So far, the mathematically proved solution of Tammes problem were given for $N = 1, ..., 12$, and 24 (SCHÜTTE and VAN DER WAERDEN, 1951; DANZER, 1963; FEJES TÓTH, 1972). The exact closed form of solution in the cases for $N = 1, ..., 9$, 11, 12, and 24 are known, but in the case for $N = 10$, the solution was only known in the rage [1.154479, 1.154480] by DANZER (1963).

In our study, the condition that none of spherical caps contains the center of another one in its interior is called "Minkowski condition" (SUGIMOTO and TANEMURA, 2003, 2004, 2006). Let C_i be the i-th open spherical cap of angular radius r and let M_i be its center under the Minkowski condition (i = 1, ..., N). Then, our problem is as follows; the whole spherical surface of a unit sphere is completely covered with N congruent open spherical caps under the condition, sequentially for $i = 2, ..., N - 1$, that M_i is set on the perimeter of C_{i-1}, and that each area of the set $(\cup_{\nu=1}^{i-1} C_\nu) \cap C_i$ becomes maximum. That is, our problem is to calculate the upper bound of r for our sequential covering. In our previous paper

(SUGIMOTO and TANEMURA, 2006; hereafter, we refer it as I), we calculated the upper bounds for N = 2, ..., 9. Here, we define a "half-cap" as the spherical cap whose angular radius is $r/2$ and which is concentric with the original cap. If the centers of N half-caps are placed on the positions of the centers of C_i (i = 1, ..., N), we get the packing with N congruent half-caps. Then, for N = 2, ..., 9, we found that the upper bound of our problem with N congruent spherical caps and the value of angular diameter of the Tammes problem are equivalent, and that the location of centers of our problem is correspondent to that of the Tammes problem (SUGIMOTO and TANEMURA, 2003, 2004, 2006). Further, it should be said that our method is a systematic and a different approach to the Tammes problem from the works by SCHÜTTE and VAN DER WAERDEN (1951), etc.

In this paper, we calculate the upper bounds of r for N = 10, 11, and 12 theoretically by using our sequential covering procedure. As a result, for N = 10, 11, and 12, we found out that the solutions of our sequential covering and the solutions of the Tammes problem were strictly correspondent as the cases of N = 2, ..., 9. Especially, the exact closed form for N = 10 is obtained (see Eq. (10)). Further, we presented a systematic method which is different from the approach of DANZER (1963).

2. Preparations

2.1. Overlapping area and union W_i

Throughout this paper we assume that the center of the unit sphere is the origin O = (0, 0, 0) and represent the surface of this unit sphere by the symbol S. In the following, open spherical caps are simply written as spherical caps unless otherwise stated. We define the geodesic arc between an arbitrary pair of points $T_1 = (x_1, y_1, z_1)$ and $T_2 = (x_2, y_2, z_2)$ on S as the inferior arc of the great circle determined by T_1 and T_2. Then the spherical distance between T_1 and T_2 is defined by the length of geodesic arc of this pair of points, and we denote $d_s(T_1, T_2) = \cos^{-1}(x_1 \cdot x_2 + y_1 \cdot y_2 + z_1 \cdot z_2)$ as the spherical distance between T_1 and T_2.

In order to solve our problem, we present the overlapping area of congruent spherical caps under the Minkowski condition. Assume C_i and C_j be two congruent spherical caps, of angular radius r, which are mutually overlapping under the Minkowski condition, and let $A_{ij} = A(C_i \cap C_j)$ be the overlapping area where $A(X)$ is the area of X. When $r \le d_s(M_i, M_j) \le 2r$, the overlapping area of C_i and C_j is given by

$$A_{ij} = -2\cos^{-1}\left(\frac{\cos h_{ij} - \cos^2 r}{\sin^2 r}\right)\cos r - 4\cos^{-1}\left(\frac{1-\cos h_{ij}}{\tan r \cdot \sin h_{ij}}\right) + 2\pi. \tag{1}$$

where $h_{ij} = 2\cos^{-1}(\cos r/\cos(d_s(M_i, M_j)/2))$ is the spherical distance between cross points of C_i and C_j. It is obvious that A_{ij} is a monotone increasing function of h_{ij} when r is fixed (SUGIMOTO and TANEMURA, 2003, 2006).

Let N be the number of spherical caps when the whole spherical surface S is completely covered. And, let ∂C_i be the perimeter of C_i (i = 1, ..., N). Then we define:

$$W_i = \left\{ W_{i-1} \cup C_i \;\middle|\; \max_{M_i \in \partial C_{i-1}} A\left(W_{i-1} \cap C_i\right) \right\}, \quad i = 2, \ldots, N-1; \quad W_1 = C_1. \tag{2}$$

In other words, W_i is the union of W_{i-1} and C_i satisfying the condition that the area $A(W_{i-1} \cap C_i)$ is maximum with the restriction $M_i \in \partial C_{i-1}$. Hereafter, we call the case of (2) that "$\cup_{\nu=1}^{i} C_\nu$ is in an extreme state." We are always necessary to examine whether $\cup_{\nu=1}^{i} C_\nu$ is in an extreme state in our sequential covering procedure. Then, we calculate the area $A(W_{i-1} \cap C_i)$ by using (1) and the area formula of spherical triangle. Note that, in order to simplify calculation, we often use the fact that $A(W_{i-1} \cap C_i)$ is maximum with the restriction $M_i \in \partial C_{i-1}$ is the same as that $A((W_{i-1} \cup C_i)^c)$ is maximum with the restriction $M_i \in \partial C_{i-1}$.

2.2. Upper bounds r_N and $\bar{r}_{N-1}$

We define r_N as the upper bound of angular radius r for the case in which N congruent open spherical caps completely cover the whole spherical surface S under the condition, sequentially for $i = 2, ..., N-1$, that M_i is set on ∂C_{i-1}, and that each area of the set $W_{i-1} \cap C_i$ becomes maximum. Next, we define another upper bound of r, $\bar{r}_{N-1}$, such that the set $\cup_{\nu=1}^{N-1} C_\nu$ which contains W_{N-2} cannot cover S under the Minkowski condition. Then $\bar{r}_{N-1}$ should be equal to the spherical distance of the largest interval in the uncovered region $(W_{N-2})^c$ of S. It is because, when the angular radius r is equal to $\bar{r}_{N-1}$, the set $\cup_{\nu=1}^{N-1} C_\nu$ which contains W_{N-2} can cover S except for a finite number of points or a line segment under our sequential covering. Therefore, $\cup_{\nu=1}^{N-1} C_\nu$ is in an extreme state if and only if at least one endpoint of the interval, which has the above mentioned spherical distance $\bar{r}_{N-1}$, comes on the perimeter ∂C_{N-2}. Further, when there are two or more uncovered points, the spherical distance of any pair of these uncovered points is less or equal to $\bar{r}_{N-1}$ since the largest interval is assumed to be $\bar{r}_{N-1}$. Then, we can put the center M_N of C_N at one of the uncoverd points. At this moment, we see that $\cup_{\nu=1}^{N} C_\nu$ which contains W_{N-1} covers S without any gap. Then, we notice the fact that r_N is equal to $\bar{r}_{N-1}$. In this paper, we calculate each solution for $N = 10$, 11, and 12 using above fact.

Refer to I for detailed explanations of r_N and $\bar{r}_{N-1}$.

3. Results

3.1. $N = 10$

According to the considerations and results for $N = 7$, 8, and 9 of I, the angular radius for $N = 10$ should be shorter than $r_9 = \cos^{-1}(1/3) \approx 1.23096$ rad and should be larger than $\tan^{-1}2 \approx 1.10715$ rad. Therefore, let us consider the case for $N = 10$ under the assumption $\tan^{-1}2 \le r \le r_9$. Even if our answer for $N = 10$ is obtained by this assumption $\tan^{-1}2 \le r$, we will also consider the range of $r < \tan^{-1}2$ later.

If two spherical caps C_a (the coordinates of the center: (a_1, a_2, a_3)) and C_b (the coordinates of the center: (b_1, b_2, b_3)) that satisfy $M_b \in \partial C_a$ intersect, we will have the coordinates (x, y, z) of cross points of the perimeters ∂C_a and ∂C_b by solving the following simultaneous equations:

$$\begin{cases} a_1 x + a_2 y + a_3 z = \cos r, \\ b_1 x + b_2 y + b_3 z = \cos r, \\ x^2 + y^2 + z^2 = 1. \end{cases} \tag{3}$$

Note that there are two cross points of ∂C_a and ∂C_b. When $(a_1, a_2, a_3) = (0, 0, -1)$ and $(b_1, b_2, b_3) = (\sin r, 0, -\cos r)$, we get

$$\begin{cases} -z = \cos r, \\ \sin r \cdot x - \cos r \cdot z = \cos r, \\ x^2 + y^2 + z^2 = 1. \end{cases} \tag{4}$$

When $K_1 = (x_1, y_1, z_1)$ and $K_2 = (x_2, y_2, z_2)$ are the solutions of simultaneous Eq. (4), the coordinates of cross points of ∂C_a and ∂C_b are as follows:

$$(x_1, y_1, z_1) = \left(-\frac{\cos r(\cos r - 1)}{\sin r}, \frac{(\cos r - 1)\sqrt{2\cos r + 1}}{\sin r}, -\cos r \right), \tag{5}$$

$$(x_2, y_2, z_2) = \left(-\frac{\cos r(\cos r - 1)}{\sin r}, -\frac{(\cos r - 1)\sqrt{2\cos r + 1}}{\sin r}, -\cos r \right). \tag{6}$$

First, as in the cases of $N = 7$, 8, and 9 in I, the centers M_1, M_2, and M_3 are placed at $(0, 0, -1)$, K_1, and $(\sin r, 0, -\cos r)$, respectively. At this time, from Theorem 1 in I, the set $\cup_{\nu=1}^{3} C_\nu$ is in an extreme state when the centers M_1, M_2, and M_3 satisfy the relations $d_s(M_1, M_2) = d_s(M_1, M_3) = d_s(M_2, M_3) = r$. Then, let K_3 be the one of the cross points of the perimeters ∂C_1 and ∂C_2, and let it be outside C_3. In addition, let K_4 be one of the cross points of ∂C_2 and ∂C_3, and let it not be the south pole $(0, 0, -1)$. The explicit expressions of cross points K_3 and K_4 are given in Appendix.

Then, from the cases of $N = 7$, 8, and 9 in I, we assume that the allocation of points K_2 for M_4 satisfies the condition that $\cup_{\nu=1}^{4} C_\nu$ is in an extreme state. It is because, in the range $\tan^{-1}2 \le r < \pi/2$, our computations show that the area $A(W_3 \cap C_4)$ is maximum when M_4 is put at K_2 as we will see below. Let us place the center M_4 at K_4 and move it to K_2 along the arc K_4K_2 of C_3. We calculate the area $A(W_3 \cap C_4)$ against the moving point M_4 numerically for several fixed values of r among $\tan^{-1}2 \le r < \pi/2$. The results are shown in Fig. 1. In this figure, the horizontal axis is the position of M_4 on the arc K_4K_2 of C_3 and the vertical axis is the area $A(W_3 \cap C_4)$. In our computation, the arc K_4K_2 is divided into 100 equal intervals and the area $A(W_3 \cap C_4)$ is calculated on 101 end points of the intervals. Hereafter, the similar computations are performed for determination of centers of spherical caps (see Figs. 2, 4, and 7). As to the two values of r in Fig. 1, we find that this curve of $A(W_3 \cap C_4)$ is symmetrical at the center of the arc K_4K_2 (it is evident from the spherical symmetry) and $A(W_3 \cap C_4)$ is maximum at both ends. The same fact as above would hold for every values of r in the range $\tan^{-1}2 \le r < \pi/2$. Therefore, we expect that $\cup_{\nu=1}^{4} C_\nu$ is in an extreme state if and only if M_4 is put at K_2 or K_4 for $\tan^{-1}2 \le r < \pi/2$. To make sure, we shall check that these points K_2 and K_4 satisfy the condition that $\cup_{\nu=1}^{4} C_\nu$ is in an extreme state after obtaining the exact values of the angular radius r at the last paragraph in this subsection.

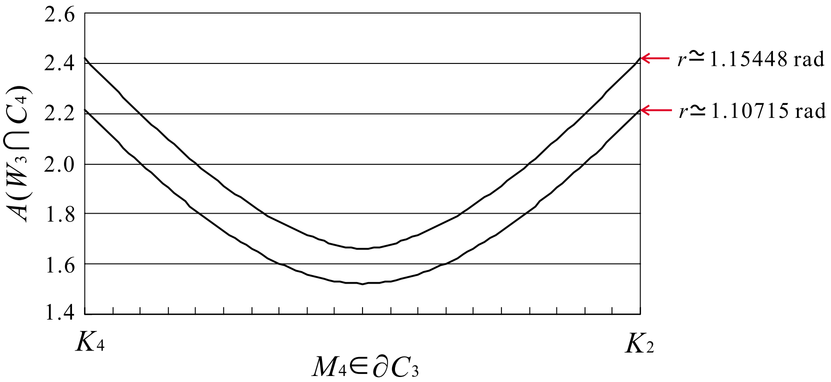

Fig. 1. The curve of $A(W_3 \cap C_4)$ when M_4 is moved on the arc K_4K_2 of C_3. Here, as in the cases of $N = 7$, 8, and 9 in I, the arc K_4K_2 is divided into 100 equal intervals and the area $A(W_3 \cap C_4)$ is calculated on 101 end points of the intervals. Note that the curve of $r \simeq 1.10715$ corresponds to the case of $r = \tan^{-1}2$. The values of r of other curves are described in the text.

Therefore, we choose M_4 on the point K_2. Then, let $K_5 = (x_5, y_5, z_5)$ be one of the cross points of perimeters ∂C_4 and ∂C_1, and let it be outside C_3. Further, let $K_6 = (x_6, y_6, z_6)$ be one of the cross points of ∂C_4 and ∂C_3, and let it not be the south pole (0, 0, –1). The explicit expressions of cross points K_5 and K_6 are given in Appendix.

Next we calculate the area $A(W_4 \cap C_5)$ when M_5 is put at a certain point on the arc K_6K_5 of C_4 and is moved on the arc. Since the fact that $A(W_{i-1} \cap C_i)$ is maximum with the restriction $M_i \in \partial C_{i-1}$ is the same as that $A((W_{i-1} \cup C_i)^c)$ is maximum with the restriction $M_i \in \partial C_{i-1}$, in order to simplify calculation, we calculate $A((W_4 \cup C_5)^c)$ against the moving point M_5 numerically for several fixed values of r among $\tan^{-1}2 \le r < \pi/2$. Here, the computation is performed as in the determination of M_4. Figure 2(a) shows the results. In this figure, the horizontal axis is the position of M_5 on the arc K_6K_5 of C_4 and the vertical axis is the area $A((W_4 \cup C_5)^c)$. As a result, for the two values of r in Fig. 2(a), the curve of $A((W_4 \cup C_5)^c)$ is symmetrical at the center of the arc K_6K_5 (it is evident from the spherical symmetry) and $A((W_4 \cup C_5)^c)$ is maximum when M_5 is placed on K_6 or K_5. The same fact as above would hold for every values of r in the range $\tan^{-1}2 \le r < \pi/2$. Therefore, we expect that $\cup_{\nu=1}^{5} C_\nu$ is in an extreme state if and only if M_5 is put at K_5 or K_6 for $\tan^{-1}2 \le r < \pi/2$. To make sure, we shall check whether the point K_5 and K_6 such points after obtaining the exact values of the angular radius r at the last paragraph in this subsection like the case of M_4. We choose M_5 on the point K_5 as in the cases of $N = 7$, 8, and 9 in I. Then, let $K_7 = (x_7, y_7, z_7)$ be one of the cross points of the perimeters ∂C_5 and ∂C_4, and let it be outside of C_1. Similarly, let $K_8 = (x_8, y_8, z_8)$ be one of the cross points of ∂C_5 and ∂C_2, and let it be outside of C_1. The exact coordinates of K_7 and K_8 are also given in Appendix.

When each M_1, M_2, M_3, M_4, and M_5 is placed at (0, 0, –1), K_1, $(\sin r, 0, -\cos r)$, K_2, and K_5, respectively, the shape of the uncovered region $(W_5)^c$ is formed as a kite $K_8K_4K_6K_7$ on the unit sphere (see Fig. 3). We note the sides of the kite $K_8K_4K_6K_7$ are not geodesic arcs

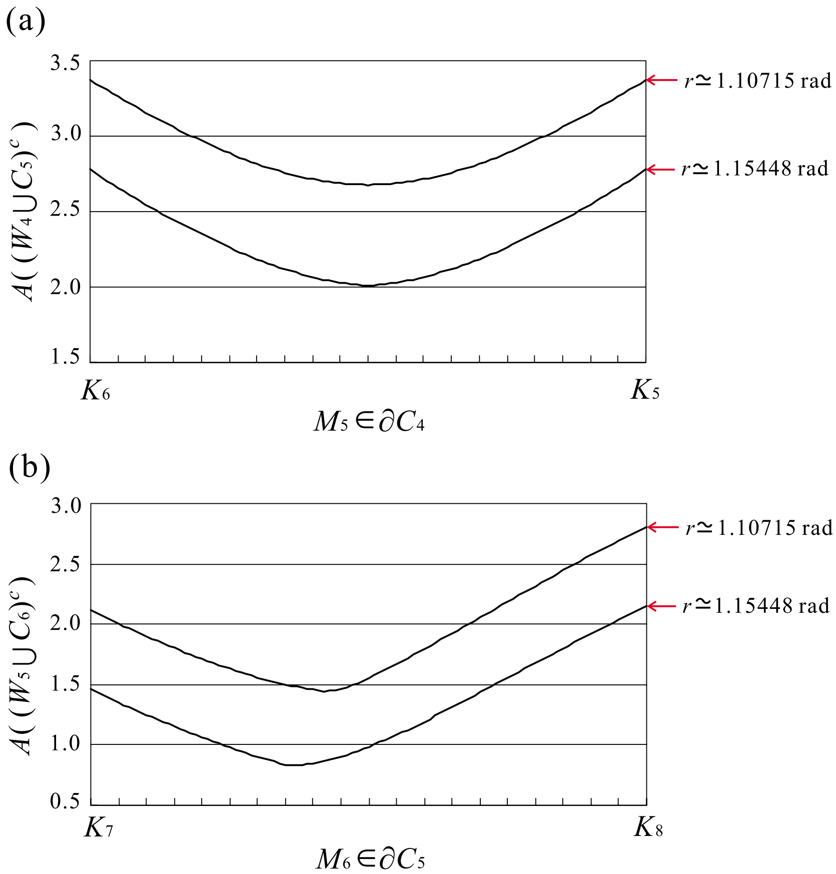

Fig. 2. (a) The curve of $A((W_4 \cup C_5)^c)$ when M_5 is moved on the arc K_6K_5 of C_4. (b) The curve of $A((W_5 \cup C_6)^c)$ when M_6 is moved on the arc K_7K_8 of C_5. The similar computation method as in Fig. 1 is taken. See the legend in Fig. 1.

but are perimeters of spherical caps. From the configuration of the vertices K_8, K_4, K_6, and K_7 of the kite $K_8K_4K_6K_7$, for the range $\tan^{-1}2 \le r < \pi/2$, the relations of spherical distance between each vertices always hold as follows:

$$\begin{cases} d_s(K_8, K_4) = d_s(K_8, K_7), \\ d_s(K_6, K_4) = d_s(K_6, K_7), \\ d_s(K_6, K_4) < d_s(K_8, K_4) < d_s(K_6, K_8), \\ d_s(K_6, K_4) < d_s(K_4, K_7) < d_s(K_6, K_8). \end{cases} \tag{7}$$

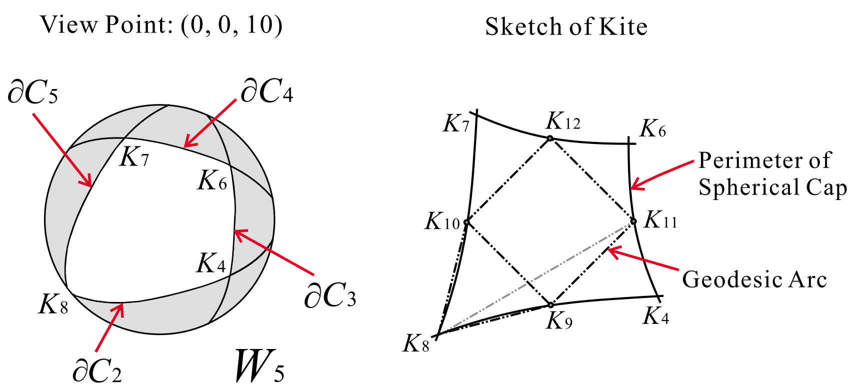

Fig. 3. The kite $K_8K_4K_6K_7$ on unit sphere.

In addition, for $\tan^{-1}2 \le r < \cos^{-1}((2\sqrt{2} - 1)/7)$, there hold the following relations of spherical distance between each vertices of the kite $K_8K_4K_6K_7$.

$$d_s(K_6, K_4) \le r < d_s(K_4, K_7) < d_s(K_8, K_4)\,. \tag{8}$$

Note that we find numerically that these relations (7) and (8) hold by using mathematical software. In passing, $\cos^{-1}((2\sqrt{2} - 1)/7) \approx 1.30653$ rad is equal to the upper bound r_8 for $N = 8$ (Sugimoto and Tanemura, 2003, 2004, 2006).

Now, we want to place five centers of C_6, C_7, C_8, C_9, and C_{10} in the uncovered kite $K_8K_4K_6K_7$ under the Minkowski condition. In this kite, we need to consider five points which keep spherical distance r with each other.

Here, we search the position of M_6 which satisfies the condition that $A((W_5 \cup C_6)^c)$ is maximum with the restriction $M_6 \in \partial C_5$ (i.e. $\cup_{\nu=1}^{6} C_\nu$ is in an extreme state). From the restriction $M_6 \in \partial C_5$, M_6 is put at a certain point on the arc K_7K_8 of C_5 and, during M_6 is moved on the arc K_7K_8 of C_5, we calculate $A((W_5 \cup C_6)^c)$ against the moving point M_6 numerically for several fixed values of r among $\tan^{-1}2 \le r < r_7 = \cos^{-1}(1 - (4/\sqrt{3}) \times \cos(7\pi/18)) \approx 1.35908$ rad. Figure 2(b) shows the graph of computational results. In this figure, the horizontal axis is the position of M_6 on the arc K_7K_8 of C_5 and the vertical axis is the area $A((W_5 \cup C_6)^c)$. As a result, we find that $A((W_5 \cup C_6)^c)$ is maximum when M_6 is put on K_8. Therefore, we expect that $\cup_{\nu=1}^{6} C_\nu$ is in an extreme state if and only if M_6 is put at K_8 in the range $\tan^{-1}2 \le r < r_7$. We shall check that K_8 is such point after obtaining the exact values of the angular radius r as in the cases of M_4 and M_5. We choose here M_6 on the point K_8. Let $K_9 = (x_9, y_9, z_9)$ be one of the cross points of the perimeters ∂C_6 and ∂C_2, and let it be outside of C_1. Similarly, let $K_{10} = (x_{10}, y_{10}, z_{10})$ be one of the cross points of ∂C_6 and ∂C_5, and let it be outside of C_1. The exact coordinates of K_9 and K_{10} are also given in Appendix. Here, let us assume two points $K_{11} \in \partial C_3$, and $K_{12} \in \partial C_4$. In addition, we suppose K_9, K_{10}, K_{11}, and K_{12} satisfy the relations

$$\begin{aligned} & d_s(K_8, K_9) = d_s(K_8, K_{10}) = d_s(K_9, K_{10}) \\ & = d_s(K_9, K_{11}) = d_s(K_{10}, K_{12}) = d_s(K_{11}, K_{12}). \end{aligned} \tag{9}$$

Then, we see that $K_8K_9K_{10}$ is a spherical equilateral triangle and that $K_9K_{11}K_{12}K_{10}$ is a spherical square (see Fig. 3). If such an arrangement of vertices K_9, K_{10}, K_{11}, and K_{12} is actually possible, the upper bound r_{10} ($\bar{r}_9$) is considered to be equal to a side-length (e.g., the spherical distance between the points K_9 and K_{11}) of the spherical square $K_9K_{11}K_{12}K_{10}$. Then, if M_7, M_8, and M_9 are placed on the points K_9, K_{10}, and K_{12}, respectively, the surface S is covered by the set $\cup_{\nu=1}^{9} C_\nu$ except for the point K_{11}. That is, our sequential covering is realized. At this time, K_{11} and K_{12} are just cross points of ∂C_7 and ∂C_3, and ∂C_8 and ∂C_4, respectively. Note that the points K_9 and K_{10} are the positions such that $\cup_{\nu=1}^{7} C_\nu$ and $\cup_{\nu=1}^{8} C_\nu$ are in an extreme state, respectively. However, it is not checked yet that K_9 and K_{10} are such positions. We shall check these facts after obtaining the exact values of the angular radius r and the coordinates of K_{11} and K_{12}.

From the above consideration that our r is equal to a side-length of spherical square $K_9K_{11}K_{12}K_{10}$, the equation $r = d_s(K_9, K_{11})$, for instance, should be satisfied. Then, we get the exact value of r which satisfies this equation. Therefore, to begin with, we calculate the coordinates of the points K_9 and K_{11} which satisfy (9). Since the point K_9 is a cross point of the perimeters ∂C_6 and ∂C_2, the coordinates of K_9 are calculable by using (3), (5), and the coordinates of K_8 in Appendix like the case of $N = 9$ in I. The result is given in Appendix. Next, we need to calculate the coordinates of K_{11} without using the coordinates of K_9. Otherwise, we cannot get an equation of r in a closed form. So we pay attention to the spherical distance $\ell = d_s(K_8, K_{11})$. Then, by applying the spherical cosine theorem to the spherical isosceles triangle $K_8K_9K_{11}$ of legs $d_s(K_8, K_9) = d_s(K_9, K_{11}) = r$, $\cos\ell$ may be written as follows:

$$\cos\ell = \frac{3\cos^3 r + 2\cos^2 r - \cos r - 2\left(1 - \cos^2 r\right)\sqrt{\cos r + 2\cos^2 r}}{(1 + \cos r)^2}.$$

It is because the inner angle at K_9 of the spherical isosceles triangle $K_8K_9K_{11}$ is the sum of the interior angles of spherical equilateral triangle $K_8K_9K_{10}$ and spherical square $K_9K_{11}K_{12}K_{10}$. In this connection, the inner angle of spherical equilateral triangle of side-length r is

$$\cos^{-1}\left(\frac{\cos r}{\cos r + 1}\right),$$

and the inner angle of spherical square of side-length r is

$$\cos^{-1}\left(\frac{\cos r - 1}{\cos r + 1}\right).$$

As a result, we can obtain the coordinates of $K_{11} = (x_{11}, y_{11}, z_{11})$ by using simultaneous equations $d_s(K_{11}, M_3) = r$, $d_s(K_8, K_{11}) = \ell$, and $x_{11}^2 + y_{11}^2 + z_{11}^2 = 1$. Here, $M_3 = (\sin r, 0, -\cos r)$ and refer to Appendix for the coordinates of K_8. The explicit coordinates of cross point K_{11} are shown in Appendix.

From $d_s(K_i, K_j) = \cos^{-1}(x_i{\cdot}x_j + y_i{\cdot}y_j + z_i{\cdot}z_j)$ and the coordinates of K_9 and K_{11} in Appendix, we get the equation of the following type

$$r = d_s(K_9(r), K_{11}(r))\,.$$

Then, the equation is solved against r by using mathematical software. The form of the solution and its value is obtained as

$$\begin{aligned} r_{10} = \bar{r}_9 &= \tan^{-1}\left(\left(\frac{4}{\sqrt{3}}\cos\left(\frac{1}{3}\tan^{-1}\left(\frac{\sqrt{3}\sqrt{229}}{9}\right)\right)+3\right)^{\frac{1}{2}}\right) \\ &\approx 1.15447983341927073783196184042 30\cdots \quad \text{rad}. \end{aligned} \tag{10}$$

By using (10), we find numerically at an arbitrary precision that the relation (9) is attained as we expected. Note that we get also another solution $r \approx 1.192753\cdots$ rad (the exact equation for this value is omitted since it is complicated) in the range of $\tan^{-1}2 \le r \le \cos^{-1}(1/3) \approx 1.23096$ rad when $r = d_s(K_9, K_{11})$ is solved against r. But, for $r \approx 1.192753\cdots$ rad, we find $d_s(K_{11}, K_{12}) < r$ numerically. Namely, K_{11} and K_{12} do not satisfy the relation (9). Therefore, when M_9 is put at K_{12}, C_9 will cover K_{11}. Thus, we can exclude this answer $r \approx 1.192753\cdots$ rad as the solution of inadequacy.

As we have noted, we check here whether the allocations of the points K_2, K_5, K_8, K_9, and K_{10} for M_4, M_5, M_6, M_7, and M_8, respectively, satisfy the condition that $\cup_{\nu=1}^{4} C_\nu$, $\cup_{\nu=1}^{5} C_\nu$, $\cup_{\nu=1}^{6} C_\nu$, $\cup_{\nu=1}^{7} C_\nu$, and $\cup_{\nu=1}^{8} C_\nu$ are in an extreme state, each, by using the value of (10). First, from the considerations for determinations of M_i mentioned above, we checked numerically that the area $A(W_{i-1} \cap C_i)$ (or $A((W_{i-1} \cup C_i)^c)$) are maximum with the restriction $M_i \in \partial C_{i-1}$ ($\cup_{\nu=1}^{i} C_\nu$ are in an extreme state) when M_4, M_5, and M_6 are put at K_2, K_5, and K_8, respectively. Then, these results are graphically presented by the curve of $r \simeq 1.15448$ corresponding to the value of (10) in Figs. 1 and 2. Therefore, for $N = 10$, our choice of M_4, M_5, and M_6 are justified. Next, we examine the position of M_7 where $A((W_6 \cup C_7)^c)$ is maximum. Then, we calculate $A((W_6 \cup C_7)^c)$ against the moving point M_7 on the arc $K_{10}K_9$ of C_6 numerically for several fixed values of r among $\tan^{-1}2 \le r < r_8 = \cos^{-1}((2\sqrt{2} - 1)/7) \approx 1.30653$ rad. As a result, the curve of $A((W_6 \cup C_7)^c)$ is symmetrical at the center of the arc $K_{10}K_9$ (it is evident from the shape of uncovered region $(W_6)^c$) and $A((W_6 \cup C_7)^c)$ is maximum at both end for the two values of r in Fig. 4(a). The same fact as above would hold for every values of r in the range $\tan^{-1}2 \le r < r_8$. Figure 4(a) illustrates the result of computation for M_7. In this figure, the horizontal axis is the position of M_7 on the arc $K_{10}K_9$ of C_6 and the vertical axis is $A((W_6 \cup C_7)^c)$. Then, the result is graphically presented by the curve of $r \simeq 1.15448$ corresponding to the value of (10) in Fig. 4(a).

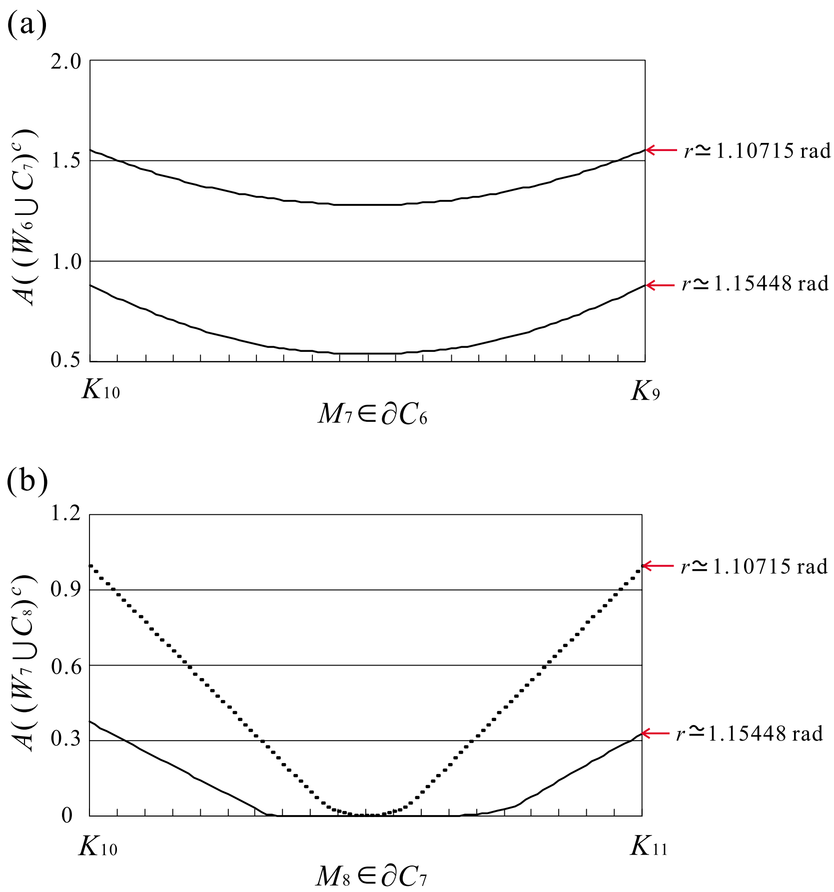

Fig. 4. (a) The curve of $A((W_6 \cup C_7)^c)$ when M_7 is moved on the arc $K_{10}K_9$ of C_6. (b) The curve of $A((W_7 \cup C_8)^c)$ when M_8 is moved on the arc $K_{10}K_{11}$ of C_7. The similar computation method as in Fig. 1 is taken. See the legend in Fig. 1.

Therefore, for the value of (10), we check numerically that $\cup_{v=1}^{7} C_v$ are in an extreme state if and only if M_7 is put at K_9 or K_{10}. In this paper, we choose M_7 on the point K_9. Then, in order to find the optimal position of M_8, let us place M_8 at K_{10} and move it to K_{11} along the arc $K_{10}K_{11}$ of C_7. Therefore, we calculate $A((W_7 \cup C_8)^c)$ against the moving point M_8 on the arc $K_{10}K_{11}$. When r is equal to (10), we find numerically that $A((W_7 \cup C_8)^c)$ is maximum with the restriction $M_8 \in \partial C_7$ ($\cup_{v=1}^{8} C_v$ is in an extreme state) if and only if M_8 is put at K_{10}. The curve of $r \simeq 1.15448$ in Fig. 4(b) presents this result. In Fig. 4(b), the horizontal axis is the position of M_8 on the arc $K_{10}K_{11}$ of C_7 and the vertical axis is $A((W_7 \cup C_8)^c)$. Hence, for $N = 10$, our choices for M_7 and M_8 are justified.

As mentioned above, when eight spherical caps whose angular radius is the value of (10) are placed on S according to our sequential covering, the uncovered region $(W_8)^c$ is a

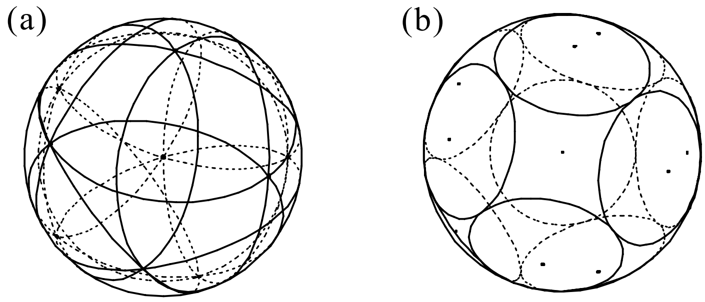

Fig. 5. (a) Our sequential covering for $N = 10$. (b) Our solution of Tammes problem for $N = 10$. Both viewpoints are (0, 0, 10). In this example, the coordinates of the centers are respectively (0, 0, –1), (0.26335, –0.87585, –0.40439), (0.91458,0, –0.40439), (0.26335, 0.87585, –0.40439), (–0.76292, 0.50440, –0.40439), (–0.77575, –0.57681, –0.25593), (–0.13883, –0.78326, 0.60599), (–0.79006, 0.092588, 0.60599), (0.084546, 0.74290, 0.66405), and (0.735778, –0.13295, 0.66405).

quadrangle on S. Then, from Corollary of Theorem 2 in I and the relations for four vertices of $(W_8)^c$, we find that $d_s(K_{11}, K_{12})$ is equal to the spherical distance of the largest interval in the uncovered region $(W_8)^c$. Therefore, $\bar{r}_9$ is equal to (10). In addition, we find that $\cup_{\nu=1}^{9} C_\nu$ is in an extreme state if and only if M_9 is put on the point $K_{12} \in \partial C_8$. Therefore, we choose M_9 on K_{12}, and then K_{11} is the unique uncovered point on S.

For $N = 10$, we initially assumed that r should be in the range $(\tan^{-1}2, \cos^{-1}(1/3)]$. Then, as a result of the investigation, our upper bound r_{10}, (10), has fallen within the range $(\tan^{-1}2, r_9]$. However, one might suspect that the fact is due to the assumption. So, if r is in the range $(0, \tan^{-1}2]$, we examine whether W_9 is able to cover S except for finite points. When r is assumed to be equal to $\tan^{-1}2$, we find that the set W_9 leave an uncovered region on S. For its detail, refer to the consideration of Subsec. 3.2. Furthermore, for $0 < r < \tan^{-1}2$, the uncovered region would become still bigger. Hence, the upper bounds r_{10} cannot be in the range $(0, \tan^{-1}2]$ as in the cases of $N = 8$ and 9 in I. Thus, we note that our initial assumption $\tan^{-1}2 < r \le r_9$ is also confirmed ($r = \tan^{-1}2$ is excluded from the above consideration).

As a result of consideration above, the set W_9 covers S except for K_{11}. Therefore, when M_{10} is put at the point K_{11}, the whole of S is covered by $\cup_{\nu=1}^{10} C_\nu$ which contains W_9 (see Fig. 5(a)). Thus, our consideration that our r is equal to the upper bound r_{10} (a side-length of the spherical square $K_9K_{11}K_{12}K_{10}$ which satisfies (9)) is confirmed and (10) is certainly the upper bound for $N = 10$.

3.2. N = 11 and 12

It is expected that the solution r_{11} for $N = 11$ should not be larger than r_{10}. Then, we assume $r_{11} \le r \le r_{10}$. Further, we assume $\tan^{-1}2 \le r_{11}$. Hence, the relations (7) and (8) hold because of the assumption $\tan^{-1}2 \le r_{11} \le r \le r_{10}$. Then, we can use the same configuration of the first five spherical caps of the case $N = 10$. When the fifth spherical cap C_5 is put on the sphere, in the same way as the foregoing subsection, a quadrilateral kite $K_8K_4K_6K_7$ on the sphere might be formed as the uncovered region. Further, because of the condition that

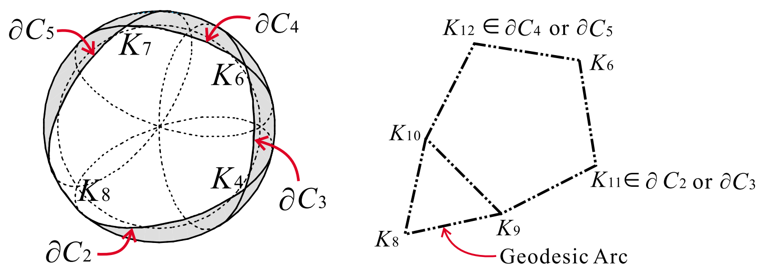

Fig. 6. The kite $K_8K_4K_6K_7$, the spherical equilateral triangle $K_8K_9K_{10}$, and the spherical regular pentagon $K_9K_{11}K_6K_{12}K_{10}$.

$\cup_{v=1}^{6} C_v$ is in an extreme state, the center M_6 should be placed again on the point K_8. Then, let K_9 be one of the cross points of the perimeters ∂C_6 and ∂C_2, and let it be outside C_1. Similarly, let K_{10} be one of the cross points of ∂C_6 and ∂C_5, and let it be outside C_1. At this time, the uncovered region $(W_6)^c$ is reduced to the pentagon $K_9K_4K_6K_7K_{10}$ which is bounded by perimeters of spherical caps.

Before considering the case of $N = 11$, we look back upon the cases of $N = 9$ and 10. For $N = 9$, we considered two spherical equilateral triangles $K_8K_9K_{10}$ and $K_9K_6K_{10}$ in the kite $K_8K_4K_6K_7$ on the sphere. At that time, we placed the centers of four caps on each vertices of two spherical equilateral triangles and obtained the upper bound r_9 by using the relation that the angular radius r is equal to a side-length of the spherical equilateral triangle (SUGIMOTO and TANEMURA, 2003, 2006). For $N = 10$, in Subsec. 3.1, we considered the spherical equilateral triangle $K_8K_9K_{10}$ and the spherical square $K_9K_{11}K_{12}K_{10}$ (see Fig. 3). Then, the centers of five caps are put on each vertices and the upper bound r_{10} is obtained by using the relation that r is equal to a side-length of the spherical square $K_9K_{11}K_{12}K_{10}$.

Now, for $N = 11$, we assume the spherical equilateral triangle $K_8K_9K_{10}$ and the spherical regular pentagon of side-length r on the kite $K_8K_4K_6K_7$, in order to place six more caps under the Minkowski condition. Therefore, here, let us assume $K_9, K_{10}, K_{11} \in \partial C_2$ or ∂C_3, and $K_{12} \in \partial C_4$ or ∂C_5 satisfy

$$\begin{aligned} d_s(K_8,K_9) &= d_s(K_8,K_{10}) = d_s(K_9,K_{10}) = d_s(K_9,K_{11}) \\ &= d_s(K_{11},K_6) = d_s(K_6,K_{12}) = d_s(K_{12},K_{10}). \end{aligned} \tag{11}$$

If the arrangement of vertices K_9, K_{10}, K_{11}, and K_{12} are actually possible, we obtain the upper bound r_{11} ($\bar{r}_{10}$) by using the relation that the angular radius r is equal to a side-length of the spherical regular pentagon $K_9K_{11}K_6K_{12}K_{10}$ as the cases of $N = 9$ and 10 (see Fig. 6). Then, for example, when M_i ($i = 6, \ldots, 10$) are placed on the points K_8, K_9, K_{10}, K_{12}, and K_6, respectively, we guess that $\cup_{v=1}^{i} C_v$ is in an extreme state and that K_{11} is an uncovered point on S. Therefore, finally, M_{11} can be placed at K_{11}. However, it is not checked yet that K_9, K_{10}, K_{12}, and K_6 satisfy such a condition that $\cup_{v=1}^{i} C_v$ is in an extreme state for $i = 7, \ldots, 10$, respectively. We shall check this fact after obtaining the exact values of the angular radius

r and the coordinates of K_{11} and K_{12}.

Now, by applying the spherical cosine and sine theorems, the inner angle of spherical regular pentagon of side-length r is given as follows:

$$\cos^{-1}\left(\frac{2\cos r - 1 - \sqrt{5}}{2(\cos r + 1)}\right).$$

Here, we pay attention to the spherical isosceles triangle $K_9K_{11}K_6$ whose legs satisfy $d_s(K_{11}, K_9) = d_s(K_{11}, K_6) = r$. Then, by applying the spherical cosine theorem to this isosceles triangle $K_9K_{11}K_6$, we have

$$d_s(K_6, K_9) = \cos^{-1}\left(\cos^2 r + \frac{(1-\cos r)\left(2\cos r - 1 - \sqrt{5}\right)}{2}\right). \tag{12}$$

Note that the left hand side of (12) is presented as the function of r by using the coordinates of K_6 and K_9. Refer to Appendix for the explicit coordinates of K_6 and K_9. Equation (12) is solved against r by using mathematical software. As a result, the value of r is obtained as

$$r = \tan^{-1}2. \tag{13}$$

Therefore, the side-length of spherical regular pentagon $K_9K_{11}K_6K_{12}K_{10}$ which satisfies the relation (11) is $\tan^{-1}2$.

Here, from the relation (8), let us consider the special case $r = d_s(K_6, K_4)$. When $r = d_s(K_6, K_4)$ is solved against r by using mathematical software, we get again the solution $r = \tan^{-1}2$. Therefore, $r = d_s(K_6, K_4) = d_s(K_6, K_7) = \tan^{-1}2$ from the relation (7). On the other hand, from the fact that the side-length of spherical regular pentagon $K_9K_{11}K_6K_{12}K_{10}$ which satisfies (11) is $\tan^{-1}2$, $d_s(K_6, K_{11}) = d_s(K_6, K_{12}) = \tan^{-1}2$. Thus, we find the result as follows:

$$K_{11} \equiv K_4 \text{ and } K_{12} \equiv K_7$$

if and only if $r = \tan^{-1}2$. It follows that the spherical regular pentagon $K_9K_{11}K_6K_{12}K_{10}$ which satisfies (11) is identical with the spherical regular pentagon $K_9K_4K_6K_7K_{10}$ of side-length $\tan^{-1}2$. We note that the vertex K_8 of kite $K_8K_4K_6K_7$ is on the perimeter of the first spherical cap C_1 then. In addition, we have shown that $\tan^{-1}2$ is equal to the upper bound of the range of angular diameter for the kissing number $k = 5$ (SUGIMOTO and TANEMURA, 2003, 2006). Therefore, as a result, we find that the spherical regular pentagon $K_9K_4K_6K_7K_{10}$ satisfies the relation as follows:

$$\begin{aligned} d_s(K_8, K_9) &= d_s(K_8, K_{10}) = d_s(K_9, K_{10}) = d_s(K_9, K_4) \\ &= d_s(K_4, K_6) = d_s(K_6, K_7) = d_s(K_7, K_{10}) = d_s(P, K_9) \\ &= d_s(P, K_4) = d_s(P, K_6) = d_s(P, K_7) = d_s(P, K_{10}) = \tan^{-1}2, \end{aligned} \tag{14}$$

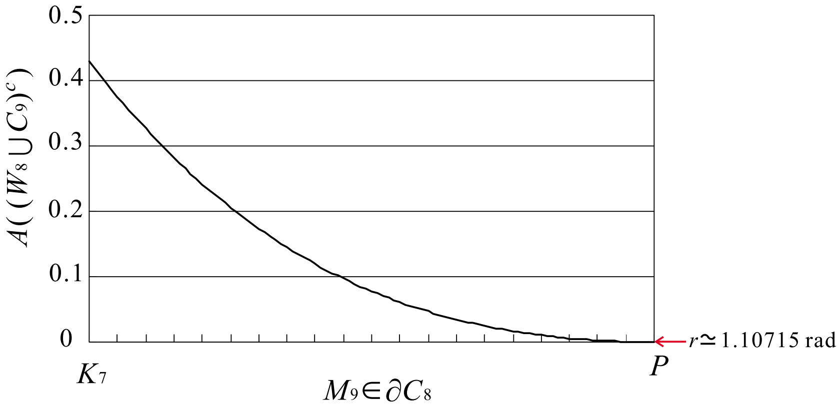

Fig. 7. The curve of $A((W_8 \cup C_9)^c)$ when M_9 is moved on the arc K_7P of C_8. The similar computation method as in Fig. 3 is taken. See the legend in Fig. 3.

where P is the center point of spherical regular pentagon $K_9K_4K_6K_7K_{10}$. From the above considerations, for $\tan^{-1}2 \le r \le r_{10}$, we see that the maximal spherical regular pentagon of side-length r on the uncovered region $(W_6)^c$ is coincident with the spherical regular pentagon $K_9K_4K_6K_7K_{10}$ which satisfies the relation (14).

Then, for $N = 11$, let us examine whether our sequential covering is actually possible when the angular radius r of caps is equal to $\tan^{-1}2 \approx 1.10715$. First, we find numerically that the allocations of points K_2, K_5, K_8, and K_9 for M_4, M_5, M_6, and M_7, respectively, satisfy the condition that each $A(W_{i-1} \cap C_i)$ (or $A((W_{i-1} \cup C_i)^c)$) is maximum with restriction $M_i \in \partial C_{i-1}$ ($\cup_{\nu=1}^{i} C_\nu$ are in an extreme state) for $i = 4, 5, 6$ and 7. As mentioned above, the results are indicated by the curve of $r \simeq 1.10715$ rad corresponding to $r = \tan^{-1}2$ in Figs. 1, 2, and 4(a). Then, we see that K_{10} is the cross point of ∂C_6, ∂C_5, and ∂C_7. Furthermore, K_4 is the cross point of ∂C_2, ∂C_3, and ∂C_7. Therefore, from the restriction $M_8 \in \partial C_7$, M_8 is put at a certain point on the arc $K_{10}K_4$ of C_7. Then, we move M_8 on the arc $K_{10}K_4$ of C_7 and search for the position of M_8 where the area $A((W_7 \cup C_8)^c)$ is maximum. For $r = \tan^{-1}2$, we checked numerically that $\cup_{\nu=1}^{8} C_\nu$ are in an extreme state if and only if M_8 is put at K_{10} or K_4. The fact is graphically presented by the curve of $r \simeq 1.10715$ in Fig. 4(b). Note that the arc $K_{10}K_{11}$ in Fig. 4(b) turns out the arc $K_{10}K_4$ since $K_{11} \equiv K_4$ for $r = \tan^{-1}2$. Thus, in this paper, M_8 is put at K_{10}. Then, the uncovered region $(W_8)^c$ is the quadrangle $K_4K_6K_7P$ on S. Next, for $r = \tan^{-1}2$, let us place M_9 at K_7 and move it to P along the arc K_7P of C_8, and then we search for the position of M_9 where the area $A((W_8 \cup C_9)^c)$ is maximum. As a result, we find that $\cup_{\nu=1}^{9} C_\nu$ is in an extreme state when M_9 is put at K_7. The curve of $r \simeq 1.01715$ in Fig. 7 indicates this fact. In this figure, the horizontal axis is the position of M_9 on the arc K_7P of C_8 and the vertical axis is the area $A((W_8 \cup C_9)^c)$. Therefore, we put M_9 at K_7. Then, K_6 and P are the cross points on ∂C_9 and the uncovered triangle K_4K_6P on S is the equilateral triangle of side-length $\tan^{-1}2$. Hence, when M_{10} is put on the point $K_6 \in \partial C_9$, we see that

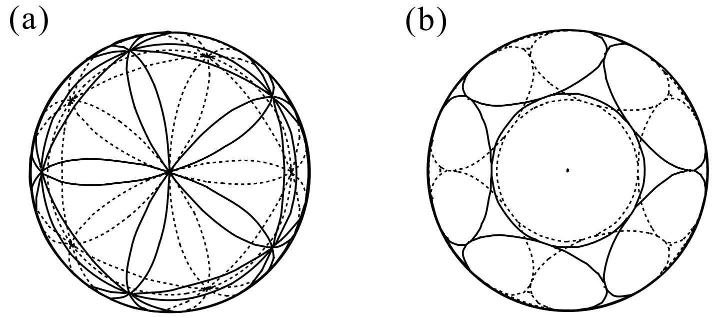

Fig. 8. (a) Our sequential covering for $N = 12$. (b) Our solution of Tammes problem for $N = 12$. Both viewpoints are (0, 0, 10). In this example, the coordinates of the centers are respectively (0, 0, –1), (0.27639, –0.85065, –0.44721), (0.89443,0, –0.44721), (0.27639, 0.85065, –0.44721), (–0.72361, 0.52573, –0.44721), (–0.72361, –0.52573, –0.44721), (–0.27639, –0.85065, 0.44721), (–0.89443,0, 0.44721), (–0.27639, 0.85065, 0.44721), (0.72361, 0.52573, 0.44721), (0.72361, –0.52573, 0.44721), and (0, 0, 1).

$\cup_{\nu=1}^{10} C_\nu$ is in an extreme state automatically and that S is covered by the set W_{10} except for two points K_4 (or K_{11}) and P (the center point of spherical regular pentagon $K_9K_4K_6K_7K_{10}$) from the relation (14). At this time, we see that P is the cross point of perimeters ∂C_7, ∂C_8, and ∂C_9. Furthermore, as a result of calculation by using the coordinates of K_9, K_{10}, and K_7 for $r = \tan^{-1}2$, P is just the north pole (0, 0, 1) certainly. Therefore, if M_{11} is put on $K_4 \in \partial C_{10}$, $\cup_{\nu=1}^{11} C_\nu$ is in an extreme state and P is a unique uncovered point on S. Then, in order to cover whole of S, we have to put one more cap at P. In other words, when $r = \tan^{-1}2$, according to our sequential covering, we can put twelve caps on S under Minkowski covering. Therefore, M_{11} and M_{12} are put on K_4 and P, respectively, then $\cup_{\nu=1}^{12} C_\nu$ which contains W_{11} covers the whole of S. Namely, we are able to see that $d_s(P, K_4) = \tan^{-1}2 = \bar{r}_{11}$. Thus, we consider that $\tan^{-1}2$ is the upper bound r_{12} for $N = 12$. As a result, we find that the positions of caps of our sequential covering for $N = 12$ correspond to the regular icosahedral vertices (see Fig. 8(a)). Therefore, if all spherical caps of our sequential covering for $N = 12$ are replaced by half-caps, all of those half-caps contact with other five half-caps and there is no space for those half-caps to move.

Then, how about r_{11}? From the results of $N = 10$ and 12, it becomes obvious that our initial assumption $r_{12} = \tan^{-1}2 \le r \le r_{10}$ is indispensable. At the time the center M_6 is placed on the point K_8, we had to place five more caps of the angular radius r on the uncovered region $(W_6)^c$ under the Minkowski covering. Then, from the above considerations, we see that the maximal spherical regular pentagon of side-length r ($r_{12} \le r \le r_{10}$) on $(W_6)^c$ is identical with the spherical regular pentagon $K_9K_4K_6K_7K_{10}$ which satisfies the relation (14). So, we conclude that r_{11} is equal to $r_{12} = \tan^{-1}2$ and use the same configuration of the first eleven spherical caps of the case $N = 12$. However, when $r_{11} = \tan^{-1}2$, our sequential covering for $N = 11$ covers S except for the point P. But, if only the eleventh cap C_{11} is replaced by a closed cap, our eleven caps can completely cover the whole of S under the Minkowski condition. Thus, for $N = 11$, we need a special care as above.

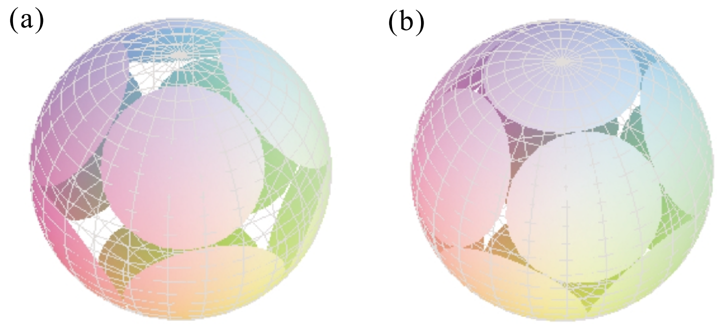

Fig. 9. (a) Our solution of Tammes problem for $N = 10$. (b) Our solution of Tammes problem for $N = 12$. Note that the spheres are drawn with the wireframe.

4. Conclusion

From the results of Sec. 3, if all of those spherical caps are replaced by half-caps, the results of our problem for $N = 10$, 11, and 12 are coincident with those of the Tammes problem (see Figs. 5, 8, and 9).

Especially, we obtained in this paper the exact closed form of r_{10} for $N = 10$ (see (10)), whereas Danzer have obtained the range [1.154479, 1.154480] of angular diameter for $N = 10$ (DANZER, 1963). Further, Danzer have solved the Tammes problem for $N = 10$ and 11 through the consideration on irreducible graphs obtained by connecting those points, among N points, whose spherical distance is exactly the minimal distance. Then he needed the independent considerations for $N = 10$ and 11, respectively. However, we presented a systematic method which is different from the approach of Danzer about the Tammes problem. Namely, as shown in Sec. 3, our method is able to obtain a solution for N by using the results for the case $N - 1$ or $N - 2$. In addition, in this study, we have considered the Tammes (packing) problem from the standpoint of sequential covering. The advantages of our approach are that we only need to observe uncovered region in the process of packing and that this uncovered region decreases step by step as the packing proceeds.

A set of spherical caps is said to be a Minkowski set if none of its elements contains in its interior the center of another. In our study, the condition of Minkowski set of centers is called a Minkowski condition, and the covering which satisfies the Minkowski condition is called a Minkowski covering. On the other hand, replacing each spherical cap in a Minkowski set of spherical caps with a concentric caps of radius half as big as the original, FEJES TÓTH (1999) called a Minkowski packing (i.e., the Minkowski packing is our packing of half-caps) and demonstrated the upper bounds for densities of a Minkowski packing and a Minkowski set. Note that their densities are sharp for $N = 3$, 4, 6, and 12, and are asymptotically sharp for great values of N. Then, we checked that our solutions for $N = 3$, 4, 6, and 12 are coincident with his results. In I, we defined the efficiency of covering on spherical surface. Our efficiency is the reciprocal of the density of a Minkowski set.

Therefore, our solutions for $N = 3$, 4, 6, and 12 are the worst efficient covering under the Minkowski condition.

It is interesting to point out that our method presented in this paper will be also useful for providing a conjectured value of optimal angular radius for any N by using the already known value for $N - 1$. This will be done as follows: (i) first we set the angular radius of a spherical cap by using the known exact or approximate value for $N - 1$, namely, we put $r = \hat{r}_{N-1}$ where $\hat{r}_{N-1}$ is the exact or approximate value of r for $N - 1$; (ii) then we put N spherical caps with radius r through our method of sequential covering; (iii) in this stage, it is possible that the Minkowski condition will be broken especially when the N-th spherical cap is placed. If this is true, decrease the size of r so far as the Minkowski condition is satisfied and go to Step (ii). Otherwise, check if the whole of S except for a point or a line segment is covered by $N - 1$ spherical caps. If it holds, the procedure ends, else increase r by a prescribed value Δr then go to Step (ii). We will be able to estimate the value of an optimal angular radius for any N by this procedure.

Throughout our paper, we used the mathematical software, Maple 8 and 9.5, which is capable of manipulating complicate algebraic expressions exactly and is also useful for numerical computations.

The authors would like to thank Prof. L. Danzer, Univ. of Dortmund, and Prof. H. Maehara, Univ. of Ryukyu, for their helpful comments.

This research was supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for JSPS Fellows, 15·1832, and Young Scientists (B), 19740061.

Appendix: The Coordinates of K_i ($i = 3$, 4, 5, 6, 7, 8, 9, and 11)

In the following, r ($\tan^{-1}2 \le r < \pi/2$) is the angular radius.

The coordinates of K_i ($i = 3$, 4, 5, 6, 7, and 8) are shown for reader's convenience, although indicated in I.

$K_3 = (x_3, y_3, z_3)$:

$$\left(\frac{\sin r(\cos^2 r - 2\cos r - 1)}{(\cos r + 1)^2}, -\frac{2\cos r \sin r\sqrt{2\cos r + 1}}{(\cos r + 1)^2}, -\cos r\right).$$

$K_4 = (x_4, y_4, z_4)$:

$$\left(\frac{2\cos r \sin r(2\cos r + 1)}{(\cos r + 1)^2}, -\frac{2\cos r \sin r\sqrt{2\cos r + 1}}{(\cos r + 1)^2}, -\frac{4\cos^2 r - \cos r - 1}{\cos r + 1}\right).$$

$K_5 = (x_5, y_5, z_5)$:

$$\left(\frac{\sin r(\cos^2 r - 2\cos r - 1)}{(\cos r + 1)^2}, \frac{2\cos r \sin r\sqrt{2\cos r + 1}}{(\cos r + 1)^2}, -\cos r\right).$$

$K_6 = (x_6, y_6, z_6)$:

$$\left(\frac{2\cos r\sin r(2\cos r+1)}{(\cos r+1)^2}, \frac{2\cos r\sin r\sqrt{2\cos r+1}}{(\cos r+1)^2}, -\frac{4\cos^2 r-\cos r-1}{\cos r+1}\right).$$

$K_7 = (x_7, y_7, z_7)$:

$$\left(\frac{2\cos r\sin r(2\cos r+1)(\cos r-1)}{(\cos r+1)^3}, \frac{2\cos r\sin r(3\cos r+1)\sqrt{2\cos r+1}}{(\cos r+1)^3},\right.$$
$$\left.-\frac{4\cos^2 r-\cos r-1}{\cos r+1}\right).$$

$K_8 = (x_8, y_8, z_8)$:

$$\left(\frac{2\cos r\sin r(\cos r-1)(2\cos r+1)}{9\cos^3 r-\cos^2 r-\cos r+1}, \frac{2\cos r\sin r(\cos r-1)\sqrt{2\cos r+1}}{9\cos^3 r-\cos^2 r-\cos r+1}\right.$$
$$\left.-\frac{4\cos^4 r-\cos^3 r+5\cos^2 r+\cos r-1}{9\cos^3 r-\cos^2 r-\cos r+1}\right).$$

$K_9 = (x_9, y_9, z_9)$:

$$\left(\frac{\sin r\left(\cos^3 r-5\cos^2 r-\cos r+1\right)}{9\cos^3 r-\cos^2 r-\cos r+1}, -\frac{4\sin r\cos^2 r\sqrt{2\cos r+1}}{9\cos^3 r-\cos^2 r-\cos r+1}\right.$$
$$\left.-\frac{\cos r\left(\cos^3 r+11\cos^2 r-\cos r-3\right)}{9\cos^3 r-\cos^2 r-\cos r+1}\right).$$

$K_{11} = (x_{11}, y_{11}, z_{11})$ are shown in cos ℓ (see Subsec. 3.1) and the coordinates of K_8 due to their lengthy expressions.

$$x_{11} = \left(\left(x_8\cos\ell\cos r + y_8^2\sin r + z_8^2\sin r + z_8\cos\ell\sin r + x_8 z_8\cos r\right)\cos r\right.$$
$$+\left(\left(-\cos^2\ell - 2z_8\cos\ell\cos^2 r + 2x_8\cos\ell\sin r\cos r + 2x_8 z_8\sin r\cos r + z_8^2 - 2z_8^2\cos^2 r\right.\right.$$
$$\left.\left.\left.+y_8^2 - y_8^2\cos^2 r\right) y_8^2\cos^2 r\right)^{(1/2)}\right)\Big/\left(y_8^2 + z_8^2 + x_8^2\cos^2 r + 2x_8 z_8\sin r\cos r - z_8^2\cos^2 r\right),$$

$$y_{11} = \frac{\cos\ell + z_8}{y_8} - \frac{x_8 + z_8 \tan r}{y_8} x_{11}\ ,$$

$$z_{11} = -\,1 + \tan r \cdot x_{11}\,.$$

REFERENCES

DANZER, L. (1963) Endliche Punktmengen auf der 2-Sphäre mit möglichst großen Minimalabstand, Universität Göttingen. (Finite Point-Set on S^2 with Minimum Distance as Large as Possible, Discrete Mathematics, **60**, 3–66 (1986).)

FEJES TÓTH, L. (1972) *Lagerungen in der Ebene, auf der Kugel und im Raum*, 2nd ed., Springer-Verlag, Heidelberg.

FEJES TÓTH, L. (1999) Minkowski circle packings on the sphere, *Discrete & Computational Geometry*, **22**, 161–166.

SCHÜTTE, K., and VAN DER WAERDEN, B. L. (1951) Auf welcher Kugel haben 5, 6, 7, 8, oder 9 Punkte mit Mindestabstand Eins Platz?, *Mathematische Annalen*, **123**, 96–124.

SUGIMOTO, T. and TANEMURA, M. (2003) Packing and Minkowski covering of congruent spherical caps on a sphere, *Research Memorandum*, ISM, **901**.

SUGIMOTO, T. and TANEMURA, M. (2004) Packing and covering of congruent spherical caps on a sphere, *Symmetry: Art and Science - 2004*, ISIS-Symmetry, 230–233.

SUGIMOTO, T. and TANEMURA, M. (2006) Packing and Minkowski covering of congruent spherical caps on a sphere for N = 2, ..., 9, *Forma*, **21**, 197–225.

TAMMES, P. M. L. (1930) On the origin of number and arrangement of the places of exit on the surface of pollen grains, *Recueil de Travaux Botaniques Néerlandais*, **27**, 1–84.