Equilibria for Anisotropic Surface Energies and the Gielis Formula

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Anisotropic surface energies are used to model surface energies which depend on the direction of the surface normal. Equilibria of such energies are characterized as surfaces with *constant anisotropic mean curvature*. The surface of a crystal and certain interfaces of liquid crystals with an isotropic substrate give physical examples of such equilibria. We produce examples of surfaces having constant anisotropic mean curvature for anisotropic energy functionals having a Wulff shape based on the Gielis formula.
Key words: Anisotropic Mean Curvature, Gielis Formula, Wulff Shape, Delaunay Surface

1. Introduction

Thompson (1942) makes the case that since the sphere is the only surface of perfect symmetry, a cause must be sought for the departure from sphericity in the form of any cell. In the absence of external pressures he argues that molecular forces within the cell wall are responsible for the asymmetry and anisotropy of their shapes. The interested reader is directed to the discussion in chapter 5 of Thompson (1942) of how, for example, such an energy determines the cylindrical cell of *spirogyra* and the ellipsoidal shapes of certain yeast cells.

Thompson's expression for the equilibrium equation is

$$\Lambda := T_1/R_1 + T_2/R_2 \equiv \text{constant}, \tag{1}$$

where $1/R_1$, $1/R_2$ are the principal curvatures of the considered smooth surface Σ, of which we will explain the definition in the last section, and T_1, T_2 are orthogonally directed tensions which depend on the material and the normal direction of the surface at each point. Roughly speaking, (1) means that a certain kind of weighted curvature of the surface is constant everywhere on the surface, where this weight depends on the normal direction of the surface. We remark that on the sphere the curvature itself is constant everywhere.

In general, the quantity $\Lambda := T_1/R_2 + T_2/R_2$ depends on each point P of a surface Σ. Λ is called the *anisotropic mean curvature* of Σ. Equation (1) means that the anisotropic mean curvature Λ is constant on the whole of the surface Σ, and we will refer to Σ as being a *surface with constant anisotropic mean curvature* (CAMC surface).

In the special case where $T_1 = T_2 = 1$, Eq. (1) reduces to

$$2H := 1/R_1 + 1/R_2 \equiv \text{constant}. \tag{2}$$

The function H appearing in (2) is called the *mean curvature* of the surface Σ. A surface on which H is constant is called a *surface with constant mean curvature* (CMC surface).

The equation "$H \equiv$ constant" describes equilibria for interfaces between two isotropic media, e.g. the surface of a small drop of water surrounded by air (if we ignore the graviational energy). On the other hand, Eq. (1) describes equilibria for interfaces between two materials when one of them is anisotropic and the surface tension depends on the surface normal.

Over the last twenty years, many new examples of CMC surfaces have been produced. On the other hand, few explicit examples of more general surfaces with constant anisotropic mean curvature can be found in the literature. We will be concerned here with the generation of particularly accessible models for CAMC surfaces which are equilibria for certain anisotropic surface energies. Each energy functional which we will treat has the property that the absolute minimizer (which is called the *Wulff shape* of the energy) among all closed surfaces enclosing a fixed volume is a convex surface whose horizontal slices are all homothetic (similar) to each other. Also, the examples which we will produce have the property that each horizontal slice is homothetic to that of the Wulff shape. We will explain how one can produce examples of such surfaces which will demonstrate how the geometry of an equilibrium shape is effected by the form of the anisotropic energy. Each of the groups of Figs. 3 through 8 shows CAMC surfaces for the same anisotropic energy. And the first figure in each group shows the Wulff shape of the corresponding energy.

The paper is organized as follows: In Sec. 2, we define the anisotropic surface energy in terms of the corresponding Wulff shape, and we define CAMC surfaces as critical surfaces of anisotropic surface energies. We also mention how anisotropic energies and CAMC surfaces can be applied to physical phenomena. In Sec. 3, we introduce the Gielis formula and explain how one can produce examples of CAMC

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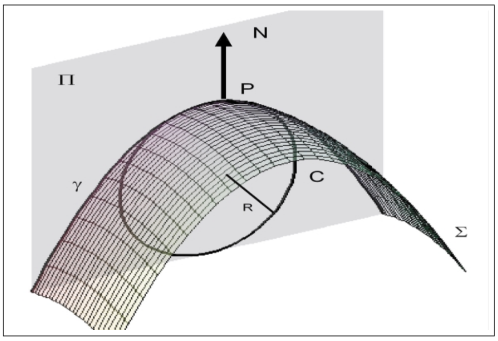


Fig. 1.

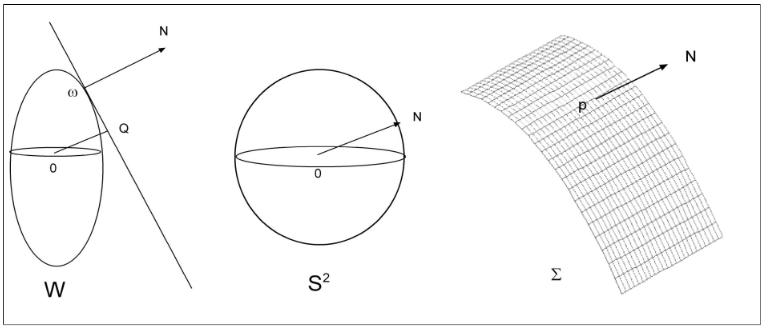


Fig. 2. The point $p \in \Sigma$ is assigned the value $\overline{0Q}$.

surfaces having a Wulff shape based on this formula. In Sec. 4, we will explain the derivation of equations which are used to produce CAMC surfaces in Sec. 3. Only in this section, mathematical calculations appear. Finally in Appendix, we explain the definition of the principal curvatures of surfaces.

2. Anisotropic Surface Energy and Wulff Shape

A smooth, closed, convex surface W defines a functional on the set of oriented smooth surfaces in the three-dimentional Euclidean space $\mathbf{R}^3$ in the following way.

Recall that the Gauss map of a surface Σ, assigns a unit vector $N(p)$ to each point p in Σ in such a way that $N(p)$ is perpendicular to the surface at the point p (see Fig. 2). The Gauss map of a convex surface W is always a diffeomorphism $N : W \to S^2$ from W onto the two-dimensional sphere S^2, that is, N determines a smooth one-to-one correspondence between points on W and points on S^2. Let F denote the support function of W; for $\omega \in W$, $F(\omega)$ is the distance from the origin 0 of $\mathbf{R}^3$ to the tangent plane to W at ω. Take the origin inside of the domain bounded by W. Then F is a positive function on W. In Fig. 2, $F(\omega)$ is the distance $\overline{0Q}$.

Let Σ be an oriented smooth surface. For any point p in Σ, denote by $\omega(p)$ the unique point in W where the normal to W agrees with the normal to Σ at p. See Fig. 2. The mapping $\omega : \Sigma \to W$ which maps a point p in Σ to a point $\omega(p)$ in W is called the anisotropic Gauss map of Σ. In Fig. 2, the distance $\overline{0Q}$ represents the value $F(\omega(p))$ which is assigned to the point $p \in \Sigma$. Now we define the anisotropic surface energy of Σ as

$$\mathcal{F}[X] := \int_\Sigma F(\omega(p))d\Sigma, \tag{3}$$

where $d\Sigma$ is the area element of Σ. This means that the quantity $F(\omega(p))\ d\Sigma$ is added up over all points in the surface to obtain the energy $\mathcal{F}[X]$. $\mathcal{F}[X]$ is a mathematical model of an anisotropic surface energy in the following sense. $F(\omega(p))$ represents a surface free energy per unit area near p which depends on the normal direction of the surface. The sum $\mathcal{F}[X]$ of the surface energy $F(\omega(p))\ d\Sigma$ of small pieces of the surface gives the total energy of Σ.

The surface W is called the *Wulff shape* of the functional $\mathcal{F}$. Wulff's theorem states that W *minimizes the functional* $\mathcal{F}$ *among all closed surfaces enclosing the same three-dimensional volume* as W. In the case where the surface energy is the usual surface tension (that is, a constant multiple of the surface area), this theorem reduces to the well known property of a sphere; it has the least surface area among all surfaces enclosing a fixed volume.

Besides the absolute minimizer, surfaces which are critica for such functionals, with or without a constraint on the volume, with prescribed fixed or free boundary conditions may occur as equilibria and are of interest. The critica are characterized in terms of the *anisotropic mean curvature* of the surface. The anisotropic mean curvature Λ of the surface Σ is defined as the trace of the differential of the anisotropic Gauss map $\omega : \Sigma \to W$. This means that, at each point p in Σ, $\Lambda(p)$ is the sum of the stretch rates of ω for any two orthogonal directions. This Λ coincides with

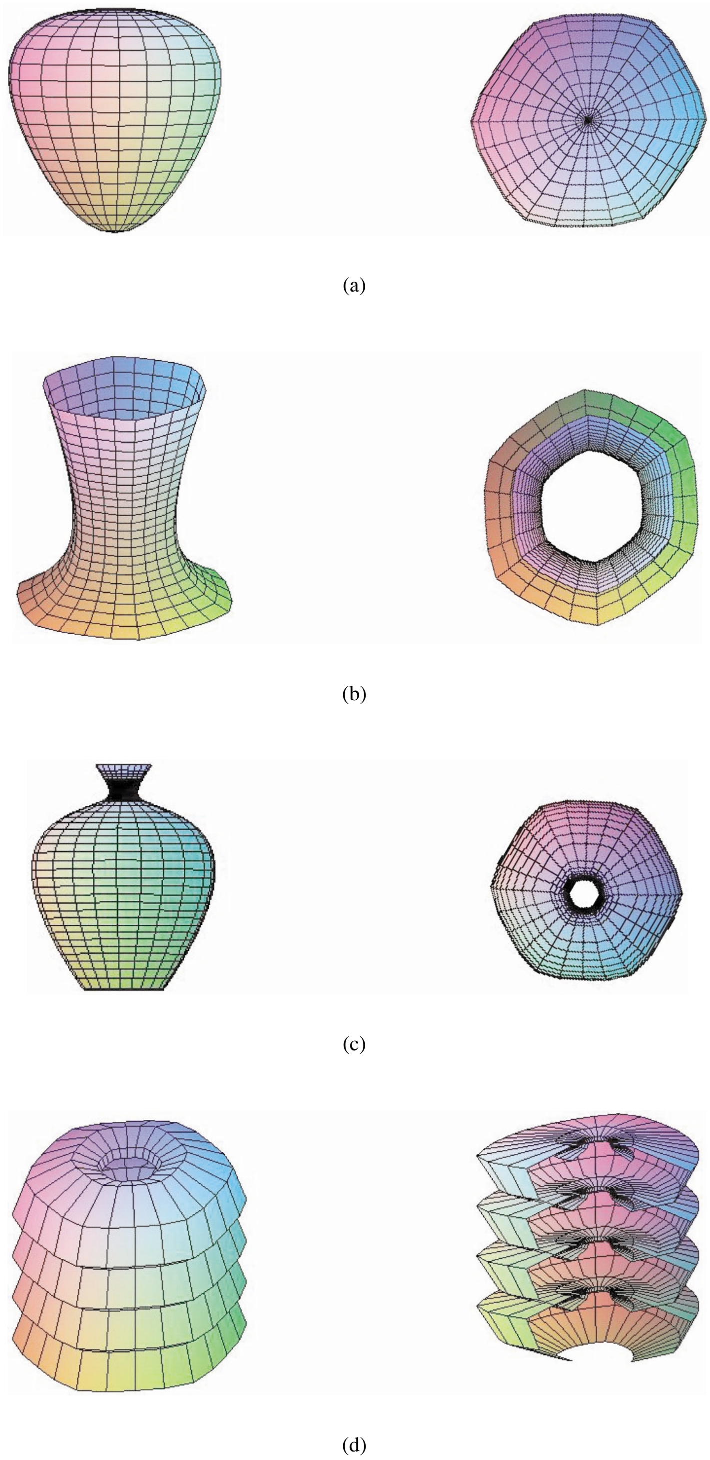
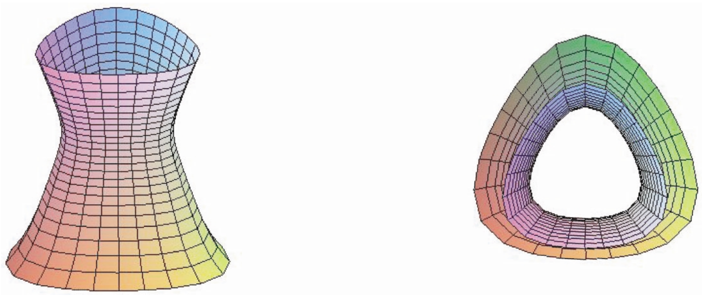


Fig. 3. (a) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3) = (3, 3.2, 4, 4, 6, 10, 4, 4)$, side view (left) and top view (right) of Wulff shape. (b) Side view (left) and top view (right) of a generalized anisotropic catenoid for the Wulff shape in (a). (c) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3, \Lambda, c) = (3, 3.2, 4, 4, 6, 10, 4, 4, 0.5, 1)$, side view (left) and top view (right) of a generalized anisotropic unduloid for the Wulff shape in (a). (d) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3, \Lambda, c) = (3, 3.2, 4, 4, 6, 10, 4, 4, 1, -2)$, a generalized anisotropic nodoid for the Wulff shape in (a).

the one which was given above in (1). The critica of the anisotropic surface energy is characterized by the property that Λ is constant, when a volume constraint is imposed, or is zero, when no volume constraint is imposed (for the proof, see Koiso and Palmer (2005)). We should remark that in the case where both of the Wulff shape W and the surface Σ are surfaces of revolution, $1/T_1$ and $1/T_2$ are the principal curvatures of W. Also, this is true in the case which we will treat in Secs. 3 and 4.

Anisotropic surface energies are used to model the inter-

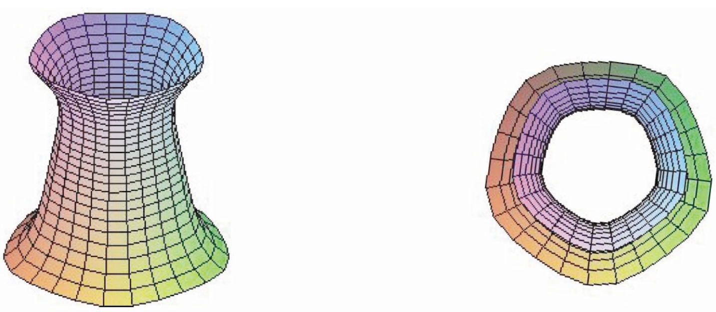
(a)



(b)

Fig. 4. (a) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3) = (6, 10, 4, 4, 3, 3.2, 4, 4)$, side view (left) and top view (right) of Wulff shape. (b) Side view (left) and top view (right) of a generalized anisotropic catenoid for the Wulff shape in (a).

(a)



(b)

Fig. 5. (a) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3) = (5, 6, 4, 4, 5, 10, 4, 4)$, side view (left) and top view (right) of Wulff shape. (b) Side view (left) and top view (right) of a generalized anisotropic catenoid for the Wulff shape in (a).

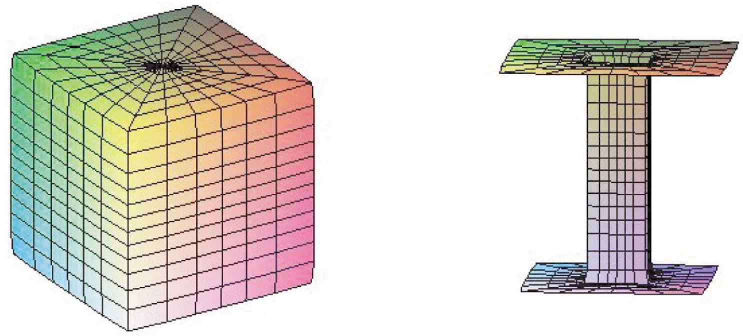
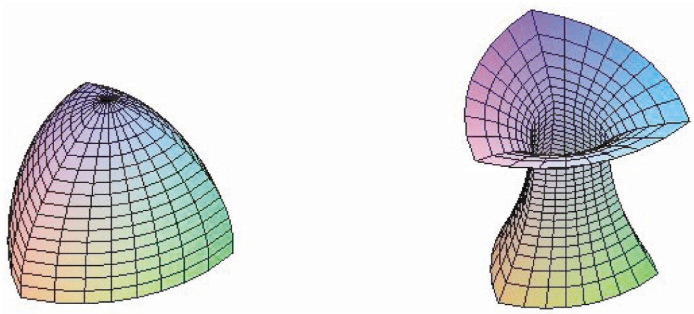
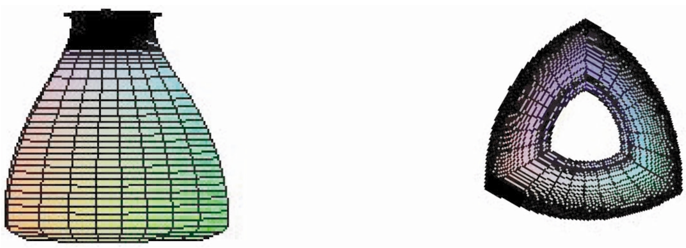


Fig. 6. left: $(m, n_1, n_2, n_3, M, N_1, N_2, N_3) = (4, 40, 40, 40, 4, 40, 40, 40)$, Wulff shape. right: a generalized anisotropic catenoid.



(a)



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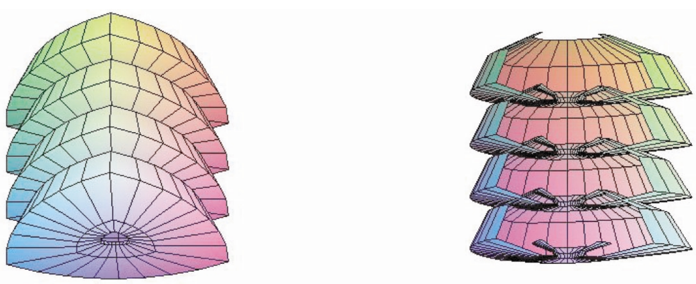


(c)

Fig. 7. (a) left: $(m, n_1, n_2, n_3, M, N_1, N_2, N_3) = (3, 10/9, 10/9, 10/9, 3, 10/9, 10/9, 10/9)$, Wulff shape. right: a generalized anisotropic catenoid. (b) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3, \Lambda, c) = (3, 10/9, 10/9, 10/9, 3, 10/9, 10/9, 10/9, 0.5, 1)$, side view (left) and top view (right) of a generalized anisotropic unduloid for the Wulff shape in (a). (c) $(m, n_1, n_2, n_3, M, N_1, N_2, N_3, \Lambda, c) = (3, 10/9, 10/9, 10/9, 3, 10/9, 10/9, 10/9, 1, -0.5)$, a generalized anisotropic nodoid for the Wulff shape in (a).

face between two materials when one of them has an ordered, internal structure. If we consider a material in which the constituent molecules are, for examples, aligned in a certain direction, then at the surface interface between this material and its surrounding environment, the surface tension will depend on the relation between the surface normal and this direction. The surface of a crystal and certain interfaces of liquid crystals with an isotropic substrate give physical examples of such equilibria.

3. Gielis Formula and Generation of CAMC Surfaces

We take three orthogonal axes in the space $\mathbf{R}^3$ which we will call x, y, z-axis as usual. Each point in $\mathbf{R}^3$ is represented as, for example, (a, b, c) by using its coordinates. We may assume that x, y-axis are horizontal and z-axis is vertical.

The Gielis formula, Gielis (2003),

$$r(\theta) = \left\{ \left|(1/a)\cos\left(\frac{m\theta}{4}\right)\right|^{n_2} + \left|(1/b)\sin\left(\frac{m\theta}{4}\right)\right|^{n_3} \right\}^{-n_1} \tag{4}$$

can be used to generate a large number of interesting closed planer curves $(\xi(\theta), \eta(\theta)) = (r(\theta)\sin\theta, r(\theta)\cos\theta)$ with symmetries. We will only use the case $a = 1 = b$ here so we will omit these parameters. When such a curve is represented parametrically, we will denote it by $G(\theta, m, n_1, n_2, n_3)$. Two such curves $G(\sigma, m, n_1, n_2, n_3) = (u(\sigma), v(\sigma))$, $G(t, M, N_1, N_2, N_3) = (\alpha(t), \beta(t))$ can be used in the formula

$$\chi(\sigma, t) = (u(\sigma)\alpha(t), u(\sigma)\beta(t), v(\sigma)), \quad u(\sigma) \geq 0, \tag{5}$$

to produce a closed surface. We remark that each curve is symmetric with respect to the second axis. When each of the curves is convex, the surface χ will be a closed convex surface without self-intersection (see Figs. 3a, 4a, 5a, 6-left, and 7a-left). We remark that all the curves obtained by intersecting χ with horizontal planes are homothetic (similar) to the curve $(\alpha(t), \beta(t))$ (see Figs. 3a, 4a, and 5a). In the special case where the curve $(\alpha(t), \beta(t))$ is a circle with radius 1, the surface χ is rotationally symmetric and the curve $(u(\sigma), v(\sigma))$ is its profile curve, that is, we obtain the surface χ by rotating the curve $(u(\sigma), 0, v(\sigma))$ along the third axis. In the general case, the curve $(u(\sigma), 0, v(\sigma))$ is also the profile curve in the following sense: Consider the vertical half plane containing the point $(\alpha(t), \beta(t), 0)$ with the vertical axis as its boundary. The curve obtained by intersecting χ with this half plane is a curve which is obtained from the curve $(u(\sigma), 0, v(\sigma))$ by stretching along the u direction. The stretch rate is $\sqrt{(\alpha(t))^2 + (\beta(t))^2}$.

We will regard the closed surface χ as the Wulff shape W of the functional (3). A recently developed construction, Koiso and Palmer (2008), will be applied with this functional to give examples of surfaces with constant anisotropic mean curvature.

The generalized anisotropic catenoid is a surface with anisotropic mean curvature $\Lambda \equiv 0$. It can be parameterized as follows. Its profile curve $(x(\sigma), z(\sigma))$ is given by

$$x = \frac{c}{2u}, \quad z = \frac{-c}{2}\int^{\sigma} \frac{dv(\sigma)}{u^2},$$

where $c \neq 0$ is an arbitrary nonzero constant. The generalized anisotropic catenoid can then be parameterized as

$$X(\sigma, t) = (x(\sigma)\alpha(t), x(\sigma)\beta(t), z(\sigma)). \tag{6}$$

The generalized anisotropic catenoid has the property that *sufficiently small pieces of it minimize the anisotropic energy (3) defined by W among all surfaces having the same boundary*. Below, we will display some Wulff shapes generated in this way together with the corresponding generalized anisotropic catenoids with $c = 2$ (Figs. 3a, 3b, 4a, 4b, 5a, 5b, 6, 7a, and 8a).

The generalized anisotropic unduloid is a surface with constant anisotropic mean curvature $\Lambda \equiv$ constant $\neq 0$. It can be obtained by using (6) for constants $c > 0, \Lambda > 0$ with

$$x := \frac{u \pm \sqrt{u^2 - \Lambda c}}{\Lambda}, \tag{7}$$

$$z := \frac{1}{\Lambda}\int^{\sigma}\left(1 \pm \frac{u(\sigma)}{\sqrt{u^2(\sigma) - \Lambda c}}\right) dv(\sigma). \tag{8}$$

The branches for the two signs glue together smoothly if σ is varied over the interval for which $u^2(\sigma) - \Lambda c \geq 0$ holds. The surface can be extended periodically in the sense that the extended surface is invariant under a vertical translation, and we have a complete surface without self-intersection. Figures 3c, 7b, and 8b-left will show a part of generalized anisotropic unduloids. Equations (7) and (8) with the plus sign produce the positively curved parts of the surface, while (7) and (8) with the minus sign produce the negatively curved parts.

The generalized anisotropic nodoids can be obtained from (7) and (8) with only the $+$ sign being used and with constants $\Lambda > 0$ and $c < 0$. They are complete periodic surfaces with self-intersections (Figs. 3d, 7c, and 8b-right). $u > 0$ gives the positively curved parts of the surface, while $u < 0$ gives the negatively curved parts.

Sufficiently small parts of generalized anisotropic unduloids and nodoids *minimize the anisotropic energy defined by W among all surfaces having the same boundary and enclosing the same three-dimensional volume*.

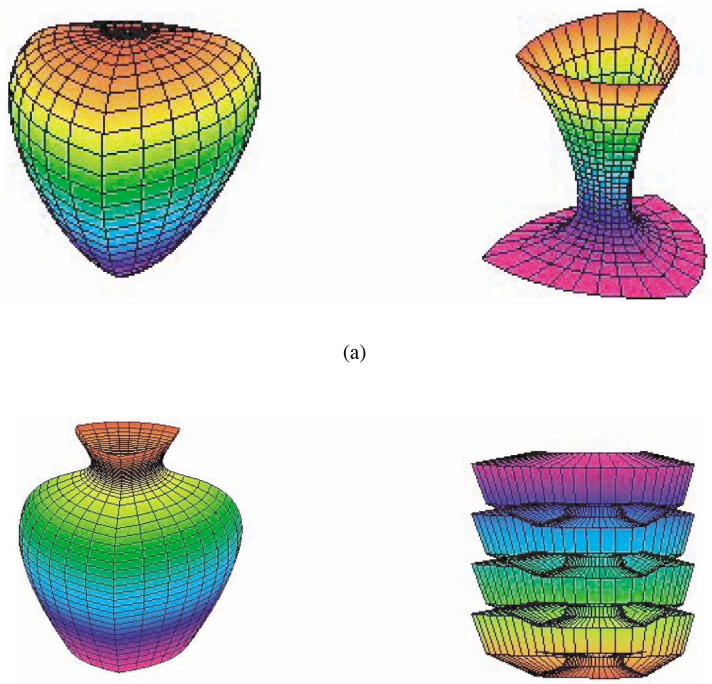
The surfaces discussed in this section are called *generalized anisotropic Delaunay surfaces*. They are a generalization of anisotropic Delaunay surfaces. Anisotropic Delaunay surfaces are surfaces of revolution with constant anisotropic mean curvature, and they were extensively studied in Koiso and Palmer (2005).

4. Derivation of Equations of Generalized Anisotropic Delaunay Surfaces

In this section we derive the formulas which we mentioned in the last section.

Assume that the Wulff shape W is given by

$$\chi(\sigma, t) = (u(\sigma)\alpha(t), u(\sigma)\beta(t), v(\sigma)), \ u(\sigma) \geq 0, \tag{9}$$



(a)

(b)

Fig. 8. (a) left: $(m, n_1, n_2, n_3, M, N_1, N_2, N_3) = (3, 3.2, 4, 4, 3, 10/9, 10/9, 10/9)$, Wulff shape. right: a generalized anisotropic catenoid. (b) left: $(m, n_1, n_2, n_3, M, N_1, N_2, N_3, \Lambda, c) = (3, 3.2, 4, 4, 3, 10/9, 10/9, 10/9, 0.7, 1)$, a generalized anisotropic unduloid for the Wulff shape in (a). right: $(m, n_1, n_2, n_3, M, N_1, N_2, N_3, \Lambda, c) = (3, 3.2, 4, 4, 3, 10/9, 10/9, 10/9, 1, -2)$, a generalized anisotropic nodoid for the Wulff shape in (a).

where $(\alpha(t), \beta(t))$ is a convex curve, and $\Gamma_W : (u(\sigma), v(\sigma))$ is a convex curve which is symmetric with respect to the v-axis. We assume here that σ is arc length parameter of $(u(\sigma), v(\sigma))$. We remark that all the curves obtained by intersecting W with horizontal planes are homothetic (similar) to the curve $(\alpha(t), \beta(t))$, so they are homothetic to each other.

Let Σ be a smooth surface such that all the curves obtained by intersecting Σ with horizontal planes are homothetic to the curve $(\alpha(t), \beta(t))$. Then Σ is given by

$$X(s,t) = (x(s)\alpha(t), x(s)\beta(t), z(s)), x(s) \geq 0, \qquad (10)$$

using a smooth curve $\Gamma_\Sigma : (x(s), z(s))$ with arc length s.

The anisotropic Gauss map $\omega : \Sigma \to W$ can be regarded as a mapping which maps a point $(x\alpha(t), x\beta(t), z)$ in Σ to a point $(u\alpha(t), u\beta(t), v)$ in W. This means that u can be regarded locally as a function of x through ω.

Recall that, at each point p in Σ, $\Lambda(p)$ is the sum of the stretch rates of the anisotropic Gauss map $\omega : \Sigma \to W$ for any two orthogonal directions. In the present case, we can take the vertical direction and the horizontal direction as these two directions. Recall that σ, s are arc lengths of the profile curves Γ_W, Γ_Σ, respectively. Because of this, the stretch rate of ω for the vertical direction is $d\sigma/ds$, which is the rate of σ with respect to s. On the other hand, it is clear that the stretch rate of ω for the horizontal direction is u/x. Therefore, Σ is a surface with constant anisotropic mean curvature if and only if

$$\Lambda = d\sigma/ds + u/x \equiv \text{constant} \qquad (11)$$

on the whole surface. Because of the definition of ω, the tangent to Γ_Σ at a point $(x(s), z(s))$ coincides with the tangent to Γ_W at the corresponding point $(u(\sigma), v(\sigma))$. This means that $dx(s)/ds = du(\sigma)/d\sigma$, $dz(s)/ds = dv(\sigma)/d\sigma$ holds. Therefore,

$$du/dx = d\sigma/ds = dv/dz \qquad (12)$$

holds. Hence, Eq. (11) can be written as

$$du/dx + u/x = \Lambda \equiv \text{constant}, \qquad (13)$$

which is equivalent to

$$x(du/dx) + u = \Lambda x. \qquad (14)$$

Integrating (14) with respect to x, we obtain

$$ux = \Lambda(x^2/2) + c/2, \qquad (15)$$

where c is any constant. In the case where $\Lambda \neq 0$, we obtain

$$x = (1/\Lambda)\left(u \pm \sqrt{u^2 - \Lambda c}\right), \qquad (16)$$

while if $\Lambda = 0$, we obtain

$$x = c/(2u). \qquad (17)$$

In both cases, z is given by the integral of dx/du with respect to v as follows.

$$z = \int (dx/du)dv. \tag{18}$$

For $dz = (dz/dv)dv = (ds/d\sigma)dv = (dx/du)dv$, using (12).

The surface (10) with (17) and (18) is called a generalized anisotropic catenoid (Figs. 3b, 4b, 5b, 6-right, 7a-right, and 8a-right). The surfaces (10) with (16) and (18) are divided into two classes: One class includes only periodic surfaces without self-intersection, and each surface is called a generalized anisotropic unduloid (Figs. 3c, 7b, and 8b-left). The other class includes only periodic surfaces with self-intersection, and each surface is called a generalized anisotropic nodoid (Figs. 3d, 7c, and 8b-right). Here periodic means that the surface is invariant under a certain vertical translation.

Appendix A. Definition of Principal Curvatures

The values R_1, R_2 in (1) are defined at each point P on the surface Σ in the following way. Let N be a unit normal vector to Σ at P and let Π be a plane which includes N (see Fig. 1). Then, the intersection γ of Π with the surface Σ is a smooth curve. We denote by C the circle which is in second order contact to γ at P. Intuitively C is the circle which best approximates γ at the point P. It is called the *curvature circle* of γ at P, and its radius $R(\Pi)$ is called the *radius of curvature* of γ at P. Now, we take all planes Π which include N and we consider $R(\Pi)$. The maximum $R_1 := R(\Pi_1)$ and minimum $R_2 := R(\Pi_2)$ of $R(\Pi)$ are *called radii of curvature* of the surface Σ at P, and $1/R_1$ and $1/R_2$ are called *principal curvatures* of Σ at P. It is known that Π_1 and Π_2 are orthogonal to each other.

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