Optimal Location of the Terminal Station of Rapid Transit System in a Circular City with Radial-Circular Network

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(Received May 30, 2008; Accepted October 27, 2008)

This paper presents a simple geometric model of rapid transit system which provides a direct access to a particular destination such as an airport located outside the city area. The time to access the airport is the combination of the time to the terminal station using intra-city transportation and the time to the airport using the rapid transit system. Two problems concerning the optimal location of the terminal station are considered: minimization of the average access time and maximization of the number of users accessible to the airport within a given time. In the minimization problem, the optimal location of the terminal station is explicitly derived as a function of the speed ratio of the intra-city transportation and the rapid transit system. **Key words:** Optimal Location, Rapid Transit System, Radial-Circular Network

1. Introduction

In this paper, we present a simple geometric model which describes the optimal location of the terminal station of rapid transit system. As shown in Fig. 1, the model assumes a planar region over which demand points are continuously distributed. There is a particular destination such as an airport located outside the city area. The time required to access the airport is the combination of the time to the terminal station located in the city and the time to the airport by using the rapid transit system. Under these assumptions, we consider the following two problems concerning the optimal location of the terminal station: minimization of the average access time and maximization of the number of users accessible to the airport within a given time.

Our objective is to analyze how the location of the terminal station of rapid transit affects the average access time and the number of users accessible to the airport within a given time, and to find how the optimal location is determined by the speed ratio of intra-city transportation and rapid transit. In this paper, using a continuous modeling approach, we construct a simple circular city model with a radial-circular network. While strong assumptions needed to develop simple analytic models limit their direct applicability to real-world instances, the insights obtained from these simple models can be applicable in a range of contexts. With appropriate modifications, our model can be applied to various situations with a similar structure: sending spatially scattered objects to an important destination by first collecting them to a certain point. For example, in case of disaster, in order to let sufferers effectively evacuate to a safer location, selecting an appropriate pickup point is of vital importance.

The following is an outline of this paper. In Sec. 2, basic assumptions and some important properties of a radialcircular network are discussed. The distribution of distance

and the average distance between demand points spatially distributed over a circular city to a fixed point (terminal station) are introduced which are employed in the following analysis. Then, in Sec. 3, we formulate two optimal location problems of the terminal station: minimization of the average access time and maximization of the number of users accessible to the airport within a given time. In particular, in the minimization problem, the optimal location of the terminal station is explicitly derived and the condition of the optimal location being at the city center is also discussed. Next, in Sec. 4, some numerical examples are presented. Graphs of the average access time and the number of users accessible to the airport are plotted as a function of the location of the terminal station and the speed ratio of the intra-city transportation and the rapid transit. And finally, in Sec. 5, we offer some concluding remarks and future directions of this study.

2. Assumptions and Model Description

A circular city of radius R with a radial-circular network is assumed as shown in Fig. 2. This idealized system of network consists of radial roads running in all directions from the city center and ring roads concentric with the city center. A simple geometric model of rapid transit system shown in Fig. 3 is considered. The distance between a destination and the city center is denoted by h. In the following, we call a destination an airport and demand points homes for simplicity. We make the following assumptions:

(i) Homes are uniformly distributed over the city;

(ii) The city has an infinitely dense radial-circular network;

(iii) Travelers choose the shortest route from home to the station.

The assumption (i) of uniformly distributed homes is widely used in the fields of transportation planning, regional



Fig. 1. Access to the airport by using rapid transit system.

science, location theory, etc. This idealized assumption allows us to treat the problem analytically and to discover geometric and morphological properties of the model under investigation. In addition, a uniform model provides a first approximation of the more "realistic" model having location-dependent densities.

From the assumptions (ii) and (iii), two types of shortest routes between two points are obtained as shown in Fig. 2: route I that uses both ring road and radial road, route II that uses only radial roads passing the city center. It is known that when the angular difference between two points is less than 2 radian the route I is the shortest route and otherwise the route II is the shortest route (Holroyd, 1966; Larson and Odoni, 1981; Vaughan, 1987; Kurita, 2001). This routing system is called *polar routing* as the movement of people in the city is restricted over the dense polar mesh. The metric is also called the *Karlsruhe metric* (Okabe *et al.*, 2000). Some properties of a radial-circular network are explained. We introduce the polar coordinates with its origin at the city center. Let the location of a home be denoted by

$$\mathbf{P} = (\mathbf{y}, \theta) \quad (0 \le \mathbf{y} \le \mathbf{R}, \ 0 \le \theta < 2\pi). \tag{1}$$

Let *s* be the Karlsruhe metric between P and the fixed terminal station located at Q(x, 0) as shown in Fig. 4. Notice that we can fix the location of Q(x, 0) at any arbitrary angle from the fixed reference line because of radial symmetry. Then *s* is given as follows:

$$s = \begin{cases} \min\{x, y\}\omega + |x - y| & (0 \le \omega < 2) \\ x + y & (2 \le \omega < \pi), \end{cases}$$
(2)

where ω is the angular difference of P and Q:

$$\omega = \min\{\theta, 2\pi - \theta\}.$$
 (3)

We introduce the distribution of distance and the average distance between points uniformly distributed over a circular city to a fixed point obtained in Kurita (2001). The derivation procedure for the cdf of s, denoted by $\Psi(s|x)$, is briefly explained. It should be noted that the following method can also be applied to the non-uniform case. $\Psi(s|x)$ is the proportion of homes from which Karlsruhe distance to the fixed point Q(x, 0) is less than or equal to s. To calculate this measure, the proportion of homes within a given equi-distant contour from Q is specified. Figure 5 shows equi-distant contours of s from various fixed locations of Q. As can be seen from Fig. 5, the shape of the contours changes with the relative positions of the location of the fixed point x and the distance s. Therefore, $\Psi(s|x)$ is expressed differently depending on x and s. When homes are uniformly distributed in a circular city, $\Psi(s|x)$ can be derived as follows:

$$\begin{split} \Psi(s|x) \\ \Psi(s|x) \\ = \begin{cases} \frac{s^3 + 6xs^2}{3\pi R^2 x} & (0 \le s \le x) \\ \frac{1}{3\pi R^2 x} \{s^3 + (3\pi - 6)xs^2 + (24 - 6\pi)x^2s \\ -(12 - 3\pi)x^3\} & (x < s \le 2x) \end{cases} \\ \frac{1}{3\pi R^2} \{3\pi s^2 - (6\pi - 12)xs + (3\pi - 4)x^2\} & (2x < s \le R - x) \\ \frac{1}{3\pi R^2 x} [-s^3 + (3\pi - 3)xs^2 + \{3R^2 - (6\pi - 9)x^2\}s \\ +(3\pi - 5)x^3 + 3R^2 x - 2R^3] & (R - x < s \le R + x) \end{cases}$$

(ii)
$$R/3 < x \le R/2$$

$$\Psi(s|x) = \begin{cases} \frac{s^3 + 6xs^2}{3\pi R^2 x} & (0 \le s \le x) \\ \frac{1}{3\pi R^2 x} \{s^3 + (3\pi - 6)xs^2 + (24 - 6\pi)x^2s \\ -(12 - 3\pi)x^3\} & (x < s \le R - x) \end{cases}$$

$$= \begin{cases} \frac{1}{3\pi R^2 x} [(3\pi - 9)xs^2 + \{3R^2 + (21 - 6\pi)x^2\}s \\ -(13 - 3\pi)x^3 + 3R^2 x - 2R^3] & (R - x < s \le 2x) \end{cases}$$

$$= \begin{cases} \frac{1}{3\pi R^2 x} [-s^3 + (3\pi - 3)xs^2 + \{3R^2 - (6\pi - 9)x^2\}s \\ +(3\pi - 5)x^3 + 3R^2 x - 2R^3] & (2x < s \le R + x) \end{cases}$$
(5)

(iii)
$$R/2 < x \le R$$

$$\Psi(s|x) = \begin{cases} \frac{s^3 + 6xs^2}{3\pi R^2 x} & (0 \le s \le R - x) \\ \frac{1}{3\pi R^2 x} \{3xs^2 + (3R^2 - 3x^2)s - (R - x)^2(2R + x)\} \\ & (R - x < s \le x) \end{cases}$$

$$= \begin{cases} \frac{1}{3\pi R^2 x} [(3\pi - 9)xs^2 + \{3R^2 + (21 - 6\pi)x^2\}s \\ -(13 - 3\pi)x^3 + 3R^2 x - 2R^3] \\ & (x < s \le 2x) \end{cases}$$

$$\frac{1}{3\pi R^2 x} [-s^3 + (3\pi - 3)xs^2 + \{3R^2 - (6\pi - 9)x^2\}s \\ +(3\pi - 5)x^3 + 3R^2 x - 2R^3] \\ & (2x < s \le R + x) \end{cases}$$
(6)

The corresponding pdf of *s*, denoted by $\psi(s|x)$, is obtained by differentiating the above cdf with respect to *s* as follows:



Fig. 2. A circular city with radial-circular network and the shortest routes between two points.

(i)
$$0 \le x \le R/3$$

$$\psi(s|x) = \begin{cases} \frac{s^2 + 4xs}{\pi R^2 x} & (0 \le s \le x) \\ \frac{1}{\pi R^2 x} \{s^2 + (2\pi - 4)xs + (8 - 2\pi)x^2\} \\ (x < s \le 2x) \end{cases}$$

$$= \begin{cases} \frac{1}{\pi R^2} \{2\pi s - (2\pi - 4)x\} \\ (2x < s \le R - x) \\ \frac{1}{\pi R^2 x} \{-s^2 + (2\pi - 2)xs - (2\pi - 3)x^2 + R^2\} \\ (R - x < s \le R + x) \end{cases}$$

(ii)
$$R/3 < x \le R/2$$

$$\begin{aligned} \psi(s|x) \\ &= \begin{cases} \frac{s^2 + 4xs}{\pi R^2 x} & (0 \le s \le x) \\ \frac{1}{\pi R^2 x} \{s^2 + (2\pi - 4)xs + (8 - 2\pi)x^2\} \\ & (x < s \le R - x) \end{cases} \\ \frac{1}{\pi R^2 x} \{(2\pi - 6)xs + (7 - 2\pi)x^2 + R^2\} \\ & (R - x < s \le 2x) \\ \frac{1}{\pi R^2 x} \{-s^2 + (2\pi - 2)xs - (2\pi - 3)x^2 + R^2\} \\ & (2x < s \le R + x) \end{cases}$$

(iii)
$$R/2 < x \le R$$

$$\begin{split} \psi(s|x) \\ &= \begin{cases} \frac{s^2 + 4xs}{\pi R^2 x} & (0 \le s \le R - x) \\ \frac{1}{\pi R^2 x} \{2xs - x^2 + R^2\} \\ & (R - x < s \le x) \\ \frac{1}{\pi R^2 x} \{(2\pi - 6)xs + (7 - 2\pi)x^2 + R^2\} \\ & (x < s \le 2x) \\ \frac{1}{\pi R^2 x} \{-s^2 + (2\pi - 2)xs - (2\pi - 3)x^2 + R^2\} \\ & (2x < s \le R + x) \end{cases} \end{split}$$



Fig. 3. A simple geometric model of rapid transit system.



Fig. 4. Access to the airport by three different origins over a radial-circular network.

The average value of the distance s between homes distributed over a circular city to the fixed terminal station is obtained by the following integration:

$$E(s|x) = \int_0^{R+x} s\psi(s|x) \mathrm{d}s. \tag{10}$$

By calculating Eq. (10) using Eqs. (7), (8) and (9), we obtain the same value of E(s|x) which is the cubic function of the location of the terminal station *x*:

$$E(s|x) = \frac{2}{3\pi R^2} x^3 + \left(1 - \frac{2}{\pi}\right) x + \frac{2R}{3}.$$
 (11)

3. Formulation of Problems (8)

3.1 Minimization of the average access time

Let us denote the average access time by f(x) as a function of the location of the terminal station x. The average access time from uniformly distributed points in the city to the airport is the combination of the average access time to the terminal station and the time to the airport by using the rapid transit. Therefore, f(x) is given as follows:

$$f(x) = \frac{1}{w} \left\{ \frac{2}{3\pi R^2} x^3 + \left(1 - \frac{2}{\pi}\right) x + \frac{2R}{3} \right\} + \frac{h - x}{v}.$$
(12)

Our aim here is to find $x = x^*$ which minimizes the average access time to the airport:

$$minimize f(x) \tag{13}$$

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(9)

(7)



Fig. 5. Equi-distance contours to various locations of fixed points Q(x, 0): (a) x = 0; (b) x = 0.2R; (c) x = 0.4R; (d) x = 0.6R; (e) x = 0.8R; (f) x = R.



Fig. 6. The optimal location of the terminal station as a function of the speed ratio of intra-city transportation and rapid transit system.

The first and second derivatives of f(x) is given as follows:

$$f'(x) = \frac{1}{w} \left(\frac{2}{\pi R^2} x^2 + 1 - \frac{2}{\pi} \right) - \frac{1}{v}, \qquad (14)$$

$$f''(x) = \frac{1}{w} \cdot \frac{4x}{\pi R^2}.$$
 (15)

These equations show f(x) is a strictly convex function.

Let $x = \tilde{x}$ be the solution of the equation of f'(x) = 0. Then \tilde{x} is obtained as follows:

$$\tilde{x} = R \sqrt{\frac{\pi}{2} \left(\frac{w}{v} - \frac{\pi - 2}{\pi}\right)}.$$
(16)

The value of the first derivative at the peripheral of the city, x = R, is given by

$$f'(R) = \frac{1}{w} - \frac{1}{v}.$$
 (17)

This indicates f'(R) > 0 by the assumption of w < v.



Fig. 7. The parameter setting for the following numerical example.

Consequently, the optimal location of the station is given by $x^* = \tilde{x}$ when $0 \le \tilde{x}$ and $x^* = 0$ otherwise. From the above argument, the minimizer of f(x) is given as follows:

$$x^* = \begin{cases} 0 & \text{when } \frac{v}{w} \ge \frac{\pi}{\pi - 2}, \\ R\sqrt{\frac{\pi}{2}\left(\frac{w}{v} - \frac{\pi - 2}{\pi}\right)} & \text{when } \frac{v}{w} < \frac{\pi}{\pi - 2}. \end{cases}$$
(18)

Equation (18) shows that if the rapid transit system has sufficiently high rapid, that is the speed ratio c = v/w is more than $\pi/(\pi - 2) \approx 2.752$, the desirable location of the terminal station is at the city center. Figure 6 shows the optimal location x^*/R as a function of the speed ratio c = v/w.

3.2 Maximization of the number of users accessible to the airport within a given time

The number of people accessible to a fixed point (such as an important facility) within a given time is an important measure for evaluating the accessibility of the point under study. In this section, we consider the problem of maximiz-



Fig. 8. 3D plot of the average access time as a function of the location of the terminal station and the speed ratio.



Fig. 10. The average access time as a function of the location of the terminal station for various speed ratios for $c < \pi/(\pi - 2) \approx 2.752$.

ing the number of users who can access the airport within a given time *u*.

Let us denote the proportion of users accessible to the airport within a given time by p(x) as a function of the location of the terminal station. To derive this function, the cdf of access time t, $\Phi(t|x)$, should be obtained since

$$p(x) = \Phi(t = u|x). \tag{19}$$

By considering that the access time t to the airport is given by

$$t = \frac{s}{w} + \frac{h - x}{v},\tag{20}$$

 $\Phi(t|x)$ can be related to $\Psi(s|x)$, the cdf of the Karlsruhe distance *s* to the station from points uniformly distributed over the circular city considered in the proceeding section. It should be noted that the minimum access time is given by (h - x)/v and the maximum access time by (R + x)/w + (h - x)/v. The second term of Eq. (20) can be treated as a fixed constant τ , so that $s = w(t - \tau)$. Using this



Fig. 9. Contour plot of the average access time as a function of the location of the terminal station and the speed ratio.



Fig. 11. The average access time as a function of the location of the terminal station for various speed ratios for $c \ge \pi/(\pi - 2) \approx 2.752$.

relationship, the cdf of the access time t, $\Phi(t|x)$, can be related to $\Psi(s|x)$ as follows:

$$\Phi(t|x) = \Pr\left\{\frac{s}{w} + \tau \le t\right\}$$

= $\Pr\{s \le w(t - \tau)\} = \Psi(w(t - \tau)|x)$
 $\left(\frac{h - x}{v} \le t \le \frac{R + x}{w} + \frac{h - x}{v}\right).$ (21)

From the above discussions, the problem of maximizing the proportion of users accessible to the airport within a given time u can be formulated as follows:

$$\max_{x} \min_{x} p(x) = \Psi(w(u-\tau)|x).$$
(22)

The pdf of access time t, $\varphi(t|x)$, is in itself an important index which describes the accessibility measure for the airport on a city-wide basis. By differentiating $\Phi(t|x)$ with respect to t, we obtain the probability density function of t



Fig. 12. 3D plot of the users accessible to the airport as a function of the location of the terminal station and the speed ratio.



Fig. 14. The users accessible to the airport as a function of the location of the terminal station for various speed ratios.



Fig. 16. The pdf of the access time t for c = 2.



Fig. 13. Contour plot of the users accessible to the airport as a function of the location of the terminal station and the speed ratio.



Fig. 15. The users accessible to the airport as a function of the location of the terminal station for various access time threshold.



Fig. 17. The pdf of the access time t for c = 6.

as follows:

$$\varphi(t|x) = \Phi'(t|x) = w\Psi'(w(t-\tau)|x)$$
$$= w\psi(w(t-\tau)|x)$$
$$\left(\frac{h-x}{v} \le t \le \frac{R+x}{w} + \frac{h-x}{v}\right). \quad (23)$$

4. Numerical Examples

In this section, graphs of the average access time and the number of users accessible to the airport derived in the proceeding section are presented. In the following example, parameter values of R = 1, h = 2, and w = 1 are assumed as illustrated in Fig. 7.

Figures 8 and 9 show the 3D plot and the contour plot of the average access time as a function of the location of the terminal station and the speed ratio. In Fig. 9 the optimal location of the terminal station for each speed ratio is also shown. In Figs. 10 and 11, the average access time are shown as a function of the location of the terminal station for $c < \pi(\pi - 2)$ and $c \ge \pi(\pi - 2)$ for various speed ratios.

In Figs. 12 and 13, the 3D plot and the contour plot of the number of users accessible to the airport within a time u = 1.2 are shown as a function of the location of the terminal station and the speed ratio. In Fig. 13 the optimal location of the terminal station numerically calculated for each speed ratio is also shown. In this case, the optimal location is discontinuous around c = 3.784; when c is more than this value, the optimal location coincides with the city center. In Fig. 14, the graph of p(x) for various values of the speed ratio c is shown. In Fig. 15, the graph of p(x) for various values of access time threshold u is shown. As can be seen from Fig. 15, optimal location of the terminal station will be closer to the city center as the value of u increases.

In Figs. 16 and 17, the pdf of the access time t, $\varphi(t|x)$, as formulated in Eq. (23) for c = 2 and c = 6 are shown. The optimal location of the terminal station which minimizes the average access time for Fig. 16 is about $x^* \approx 0.463R$ while that of Fig. 17 is $x^* = 0$. As can be seen from Fig. 17, if the rapid transit has sufficiently high speed, the advantage of the terminal station being at the city center is obvious because of its high accessibility.

5. Conclusion and Future Work

In this paper, we have presented a continuous model of rapid transit system which provides a direct access to a particular destination located outside a city area. Two problems concerning the optimal location of the terminal station have been considered: minimization of the average access time and maximization of the number of users accessible to the airport within a given time. In the minimization problem, the optimal location of the terminal station has been explicitly derived as a function of the speed ratio. We have also shown that the condition of the optimal location being at the city center is given by $c \ge \pi/(\pi - 2) \approx 2.752$. This result suggests that it is desirable to construct the terminal station of a super high-speed transit close to the city center. The proposed model can be a basic model to describe super-rapid transit, such as MagLev transportation (RTRI, 2004).

There are many interesting possible directions for this work.

For example, more than two circular regions can be incorporated in our model. This model can be considered as a basic model for evaluating the location of terminal stations for inter-city rapid transit. In the area of computational geometry, similar problems of bridging convex regions are proposed such as the minimum diameter bridge problem (MDBP) (Cai *et al.*, 1999). It is interesting to introduce speed difference between intra-regional movement and movement on the bridge into these problems as presented in this paper.

The case when the airport is located within a city area should also be considered. It is also interesting to consider the situation in which there are multiple stations on the rapid transit system.

Another possible extension of this work is to consider a non-uniform spatial distribution of homes. Some simple models assuming a decreasing falloff from the city center could provide a closed form optimal location of the terminal station for the minimization problem. An interesting problem is to compare between a non-uniform density and the uniform density the condition of the optimal location being at the city center, and how this condition changes as the value of the deterrence parameter representing the decreasing effect of a density from the city center changes.

To consider other metrics such as Manhattan distance, Euclidean distance can also be an interesting possibility. An analysis using real network data such as railway network and road network should also be explored.

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