

Analysis of Facility Location Using Ordered Rectilinear Distance in Regular Point Patterns

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This paper deals with the k th nearest rectilinear distance of two regular point patterns: square and diamond lattices. The probability density functions of the k th nearest rectilinear distance are theoretically derived for $k = 1, 2, \dots, 8$. Upper and lower bounds of the k th nearest distance are also derived. As an application of the k th nearest distance, we consider a facility location problem with closing of facilities. The objective is to find the best configuration of facilities that minimizes the average rectilinear distance from residents to their nearest open facility when some existing facilities are closed. Assuming that facilities are closed independently and at random, we show that the diamond lattice is the best if at least 73% of facilities are open.

Key words: k th Nearest Distance, Regular Point Patterns, Facility Location, Average Distance

1. Introduction

Facility location problems aim to find the optimal location of facilities. The most frequently used objective is minimization of the sum of distances from residents to their nearest facility. However, when some existing facilities are closed, distances to the k th nearest facility are also important. For example, when locating emergency facilities such as hospitals, fire stations, and refuges, the second and third nearest distances should be taken into consideration. This paper analyses facility locations using the k th nearest distance.

Analytical expressions of the k th nearest distance have been obtained for several patterns. The probability density function of the nearest distance was derived in Clark and Evans (1954) for the random pattern, Persson (1964) for the square lattice, and Holgate (1965) for the triangular lattice. The probability density function of the k th nearest distance was derived in Dacey (1968) for the random pattern, Koshizuka (1985) for $k = 1, 2, 3$ for the square lattice, and Miyagawa (2009) for $k = 1, 2, \dots, 7$ for the square, triangular, and hexagonal lattices. These distances are measured as Euclidean distances. Although Euclidean distances are good approximations for actual travel distances, rectilinear distances are more suitable for cities with a grid road network. In fact, rectilinear distances have been frequently used in facility location models (Wesolowsky and Love, 1972; Drezner, 1987; Francis *et al.*, 1992). The probability density function of the nearest rectilinear distance was derived in Larson and Odoni (1981) for the random pattern. The higher order rectilinear distances of regular patterns have not been derived previously.

We focus on the k th nearest rectilinear distance of regular and random point patterns shown in Fig. 1. The regular patterns that we consider are the square and diamond lattices. The diamond lattice is constructed by rotating the square lattice at angle $\pi/4$. These two regular patterns are identi-

cal for Euclidean distances, but not for rectilinear distances. The actual distribution of public facilities can be regarded as the intermediate pattern between regular and random patterns. The theoretical results of the extremes of regular and random will give useful information to empirical studies. We assume that these patterns continue infinitely in an unbounded region. This assumption allows us to derive the k th nearest distance without taking into account the effect of the boundary.

There have been some studies concerning a facility location problem with closing of facilities. Gregg *et al.* (1988) presented a stochastic programming approach for siting and closing public facilities. Wang *et al.* (2003) studied a budget constrained location problem to consider opening new facilities and closing existing facilities simultaneously. ReVelle *et al.* (2007) investigated facility closing in a competitive environment and in the situation of financial exigency. Diamond and Wright (1987) and Church and Murray (1993) developed a multiobjective model for school closing. The present paper differs from these works in employing an analytical approach based on the k th nearest rectilinear distance of the regular and random patterns.

The remainder of this paper is organized as follows. The next section derives the probability density functions of the k th nearest rectilinear distance for the regular and random patterns. Section 3 gives the upper and lower bounds of the k th nearest distance. Section 4 provides an application of the k th nearest distance to a facility location problem. The final section presents the conclusion of this paper.

2. k th Nearest Distance Distribution

Let R_k be the rectilinear distance from an arbitrary location in a study region to the k th nearest point and $f_k(r)$ be the probability density function of R_k . We call R_k and $f_k(r)$ the k th nearest distance and the k th nearest distance distribution, respectively. In this section, we derive the k th

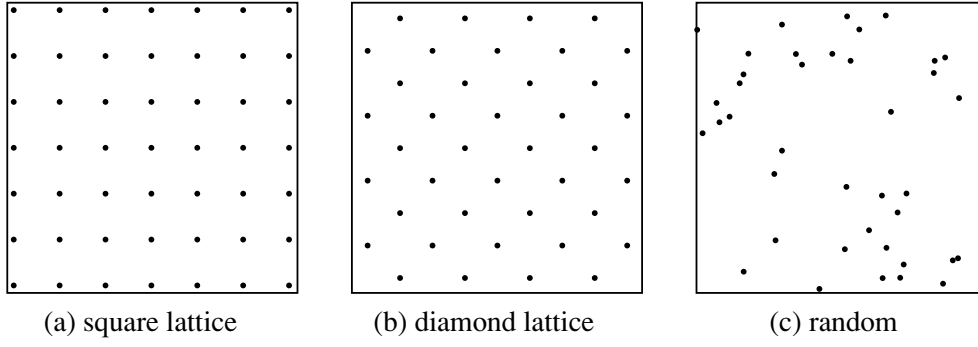


Fig. 1. Regular and random point patterns.

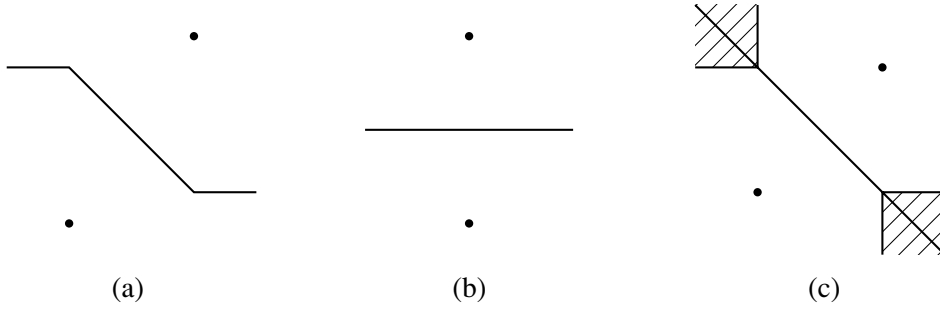
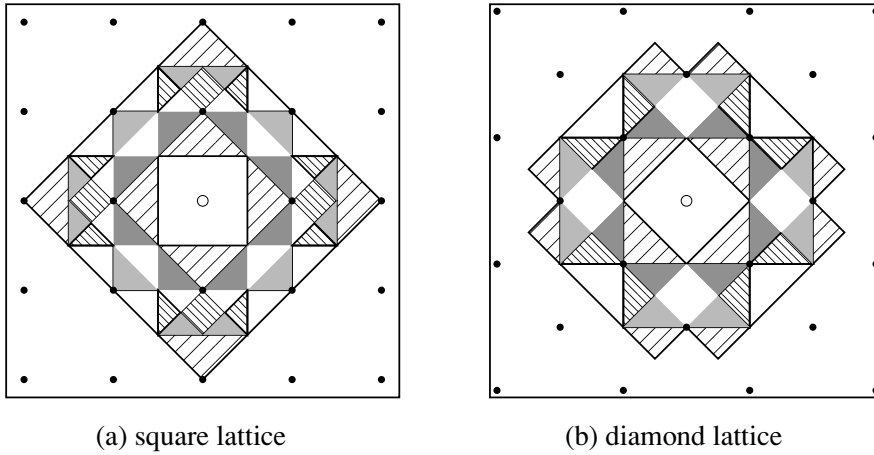


Fig. 2. Bisector defined with rectilinear distances.

Fig. 3. k th nearest regions.

nearest distance distributions $f_k(r)$ ($k = 1, 2, \dots, 8$) for the two regular patterns.

Let $S_k(r)$ be the area of the region such that $R_k \leq r$ in the study region. Then the cumulative distribution function of R_k , denoted by $F_k(r)$, which is the probability that $R_k \leq r$, is

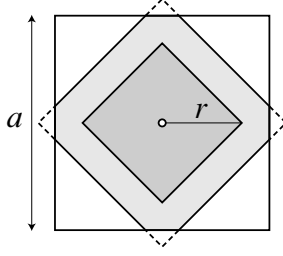
$$F_k(r) = \frac{S_k(r)}{S} \quad (1)$$

where S is the area of the study region. Differentiating Eq. (1) with respect to r gives the k th nearest distance distribution $f_k(r)$ as

$$f_k(r) = \frac{1}{S} \frac{dS_k(r)}{dr}. \quad (2)$$

To obtain $S_k(r)$, we first define the bisector with rectilinear distances. The shape of the bisector is classified into three types as shown in Fig. 2. If the line through two points has angle $\pi/4$ or $3\pi/4$ with the x -axis, the bisector consists of not only a straight line but also an area as shown in Fig. 2(c); see Lee (1980). To avoid the indeterminacy, we define the bisector as the straight perpendicular line, as suggested by Okabe *et al.* (2000).

Since we assume that regular patterns continue infinitely, $S_k(r)$ can be calculated by considering only one point. Figure 3 shows the regions where the white point is the k th nearest. We call these regions the k th nearest regions. The innermost square is the nearest region, and the outside of it is the second nearest region, followed by third, fourth, $\dots$, eighth nearest regions. These k th nearest regions cor-

Fig. 4. $S_1(r)$ of the square lattice.

respond to order- k Voronoi polygons with the Manhattan metric (Okabe *et al.*, 2000).

Then $S_k(r)$ is the area of the rectilinear circle, which is a square rotated at angle $\pi/4$, centred at the white point with radius r in the k th nearest region. For example, $S_1(r)$ of the square lattice is obtained by calculating the area of the rectilinear circle in the square as shown in Fig. 4. Let a be the side length of the square. Then,

$$S_1(r) = \begin{cases} 2r^2 & (0 < r \leq \frac{a}{2}) \\ a^2 - 2(a-r)^2 & (\frac{a}{2} < r \leq a) \end{cases} \quad (3)$$

Substituting Eq. (3) and $S = a^2$ into Eq. (2), we obtain the nearest distance distribution $f_1(r)$ as

$$f_1(r) = \begin{cases} 4\rho r & (0 < r \leq \frac{1}{2\sqrt{\rho}}) \\ -4\rho r + 4\sqrt{\rho} & (\frac{1}{2\sqrt{\rho}} < r \leq \frac{1}{\sqrt{\rho}}) \end{cases} \quad (4)$$

where $\rho (= 1/a^2)$ is the density of points. The k th nearest distance distributions $f_k(r)$ are similarly obtained by calculating $S_k(r)$ in each k th nearest region. The average k th nearest distance $E(R_k)$ is given by

$$E(R_k) = \int_0^\infty r f_k(r) dr. \quad (5)$$

The details of $f_k(r)$ and $E(R_k)$ are provided in Appendix A.

$f_k(r)$ and $E(R_k)$ of the random pattern are obtained by the same way as with Euclidean distances in Dacey (1968) as

$$f_k(r) = \frac{4\rho r (2\rho r^2)^{k-1}}{(k-1)!} \exp(-2\rho r^2) \quad (6)$$

$$E(R_k) = \frac{(2k-1)!!}{(2k-2)!!} \frac{\sqrt{\pi}}{2\sqrt{2\rho}}. \quad (7)$$

$f_k(r)$ of the regular and random patterns are illustrated in Fig. 5, where the density of points is $\rho = 1$. Notice that $f_7(r) = f_8(r)$ for the square lattice, $f_3(r) = f_4(r)$, $f_5(r) = f_6(r)$, and $f_7(r) = f_8(r)$ for the diamond lattice. The average k th nearest distances $E(R_k)$ of the regular and random patterns are given in Table 1. Note that $E(R_1)$ of the diamond lattice is the smallest. This makes intuitive sense, because the shape of the first nearest region of the diamond lattice is a rectilinear circle (see Fig. 3(b)). However, $E(R_2)$, $E(R_3)$, $E(R_5)$, $E(R_7)$, $E(R_8)$ of the square lattice are smaller than those of the diamond lattice.

3. Upper and Lower Bounds of the k th Nearest Distance

In the previous section, we have theoretically derived the k th nearest distance distributions $f_k(r)$ for $k = 1, 2, \dots, 8$. Since calculating $f_k(r)$ for all k is practically impossible, we obtain the upper and lower bounds of the k th nearest distance R_k instead.

Consider a rectilinear circle with radius R_k . The rectilinear circle has exactly one point in the circumference and $k-1$ points in the inside. Hence the upper (lower) bound of the k th nearest distance R_k is the upper (lower) bound of the radius of the rectilinear circle which contains k points.

The upper and lower bounds of R_k of the square lattice are obtained as follows. Consider the case where k points form a rectilinear circle, that is $k = 2n^2 - 2n + 1$ (n : natural number), as indicated by white points in Fig. 6(a). Let a be the distance between two adjacent points. To obtain the upper bound of R_k , let us consider a rectilinear circle which contains more than k points. As shown in Fig. 6(a), the outer rectilinear circle always contains more than k points. The radius of the circle is na . Thus we have the upper bound of R_k as

$$R_1 < a, \quad R_5 < 2a, \quad R_{13} < 3a, \dots, R_{2n^2-2n+1} < na. \quad (8)$$

Next, to obtain the lower bound of R_k , let us consider a rectilinear circle which contains less than k points. As shown in Fig. 6(a), the inner rectilinear circle always contains less than k points. The radius of the circle is $(n-1)a$. Thus we have the lower bound of R_k as

$$R_1 > 0, \quad R_5 > a, \quad R_{13} > 2a, \dots, R_{2n^2-2n+1} > (n-1)a. \quad (9)$$

From Eqs. (8) and (9), and $a = 1/\sqrt{\rho}$, the upper and lower bounds of the k th nearest distance R_k for general k are given by

$$\left\{ \left\lfloor \frac{1}{2}(\sqrt{2k-1} + 1) \right\rfloor - 1 \right\} \frac{1}{\sqrt{\rho}} < R_k < \left\lceil \frac{1}{2}(\sqrt{2k-1} + 1) \right\rceil \frac{1}{\sqrt{\rho}}. \quad (10)$$

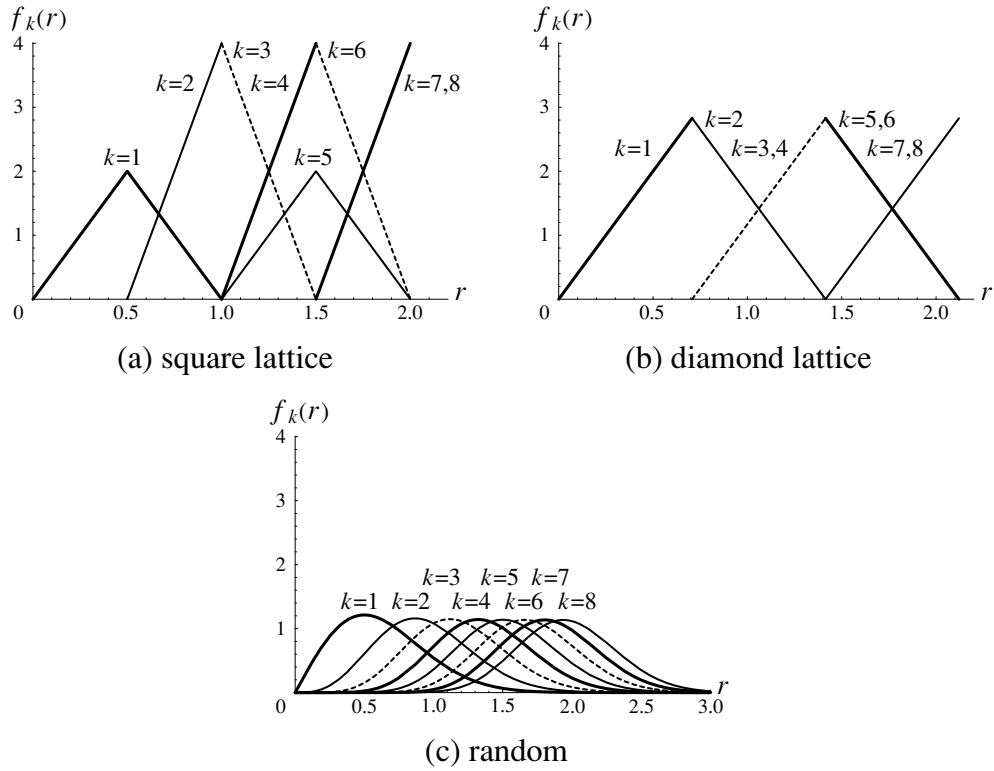
The upper and lower bounds of R_k of the diamond lattice are similarly obtained as

$$\frac{\lfloor \sqrt{k} \rfloor - 1}{\sqrt{2\rho}} < R_k < \frac{\lceil \sqrt{k} \rceil}{\sqrt{2\rho}}. \quad (11)$$

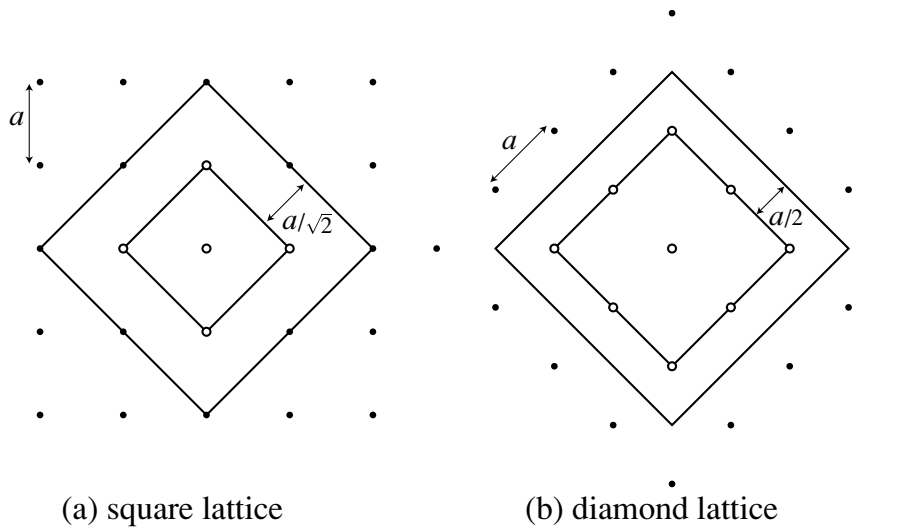
The upper and lower bounds of R_k of the two regular patterns are shown in Fig. 7. The ranges of $f_k(r)$ and $E(R_k)$ for $k = 1, 2, \dots, 8$ are also shown in the figure. It can be seen that the bounds of the diamond lattice are tighter than those of the square lattice.

4. Application to a Facility Location Problem

In this section, we provide an application of the k th nearest distance to a facility location problem. Assume that residents are uniformly distributed. Although this assumption is unrealistic, the method is applicable to realistic situations if the study region can be divided into subregions with

Fig. 5. k th nearest distance distribution.Table 1. Average k th nearest distance.

	$E(R_1)$	$E(R_2)$	$E(R_3)$	$E(R_4)$	$E(R_5)$	$E(R_6)$	$E(R_7)$	$E(R_8)$
square	0.500	0.833	1.167	1.333	1.500	1.667	1.833	1.833
diamond	0.471	0.943	1.179	1.179	1.650	1.650	1.886	1.886
random	0.627	0.940	1.175	1.371	1.542	1.696	1.838	1.969

Fig. 6. Rectilinear circle which can (cannot) contain k points.

nearly uniform distributions. Under the uniformity assumption, the optimal facility location is the diamond lattice as shown in Larson and Odoni (1981). This is because the average distance to the nearest facility $E(R_1)$ of the diamond

lattice is the smallest (see Table 1). However, when some existing facilities are closed, it is uncertain whether or not the diamond lattice is still optimal.

Suppose that facilities are closed independently and at

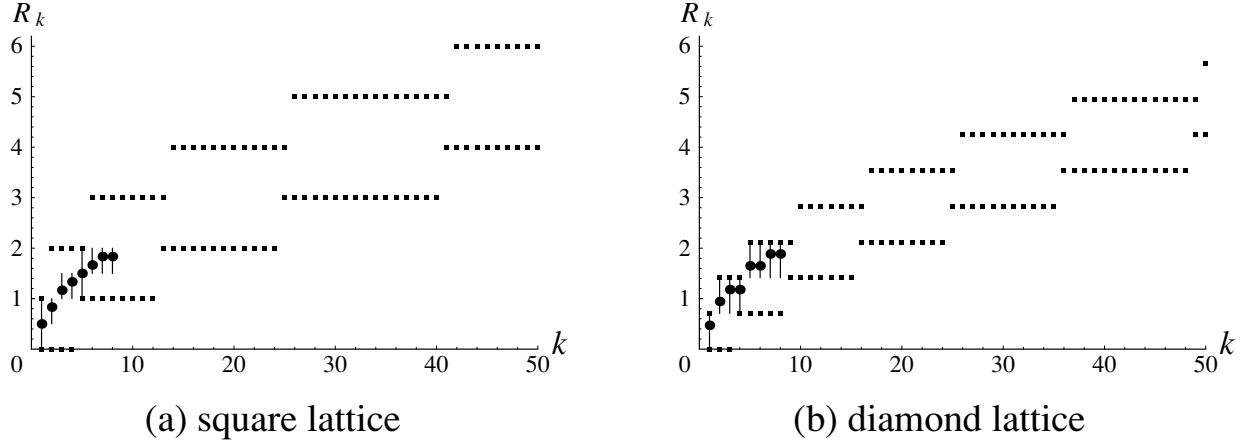
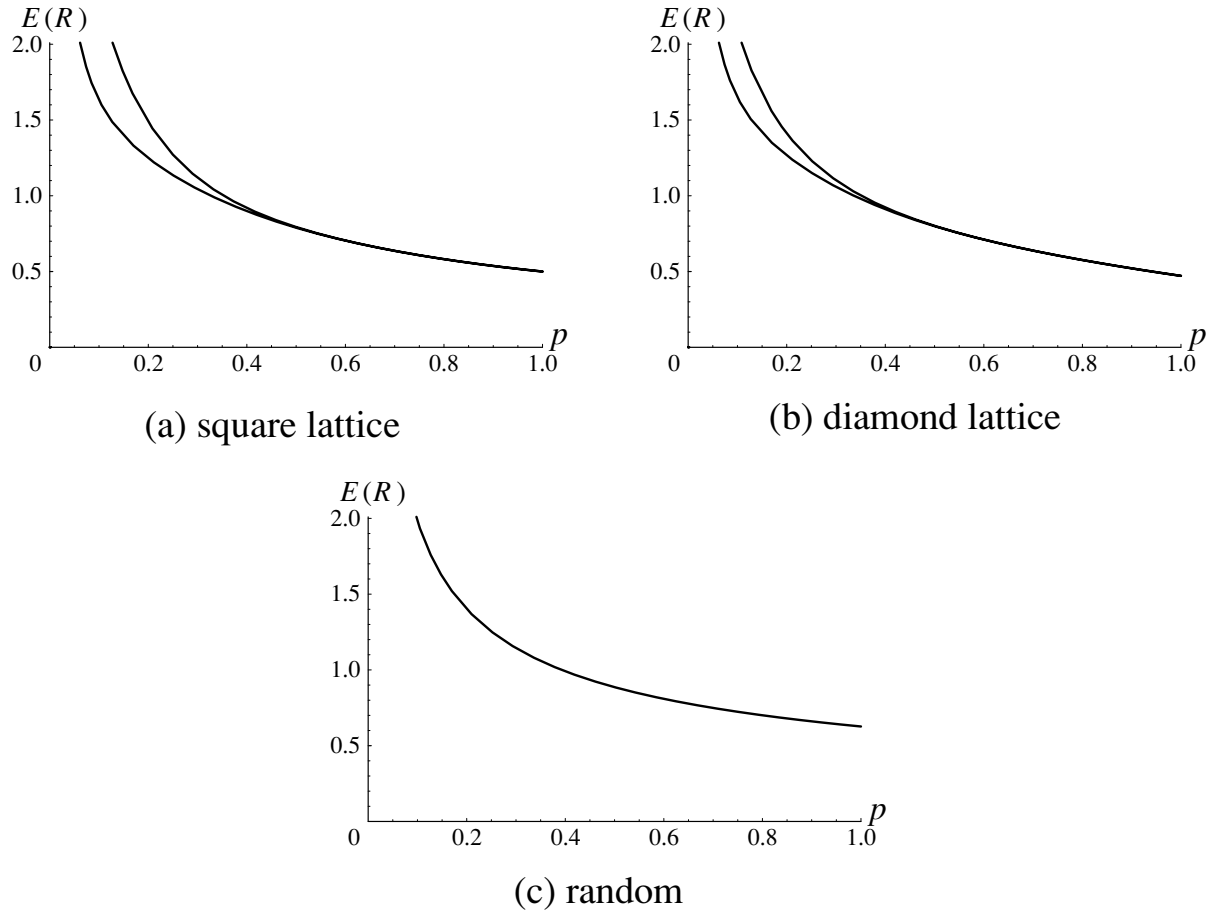
Fig. 7. Upper and lower bounds of the k th nearest distance.

Fig. 8. Upper and lower bounds of the average distance to the nearest open facility.

random. Let p be the probability that facilities are open. Residents are assumed to use their nearest open facility. The residents whose nearest facility is closed have to use the second nearest facility, if the second nearest is open. The probability that the nearest facility is closed and the second nearest is open is $(1 - p)p$. In general, the probability that residents have to use the k th nearest facility is $(1 - p)^{k-1}p$. Using this probability and the average k th nearest distance

$E(R_k)$, we can express the average distance to the nearest open facility $E(R)$ as

$$E(R) = p \sum_{k=1}^{\infty} (1 - p)^{k-1} E(R_k). \quad (12)$$

We regard the facility configuration that minimizes $E(R)$ as the best.

Substituting $E(R_k)$ for $k = 1, 2, \dots, 8$ and the upper and

lower bounds of R_k for $k \geq 9$ into Eq. (12) yields the upper and lower bounds of $E(R)$. Figure 8 depicts the upper and lower bounds of the average distance to the nearest open facility $E(R)$ as a function of probability p . The upper and lower bounds of $E(R)$ are equivalent at $p = 1$ where all facilities are open, and the difference increases with decreasing p . By comparing the upper and lower bounds of $E(R)$, we can show the range of p that one pattern outperforms the other. The diamond lattice outperforms the square lattice, that is the upper bound of $E(R)$ of the diamond lattice is smaller than the lower bounds of the square lattice, when

$$0.7257 < p \leq 1. \quad (13)$$

It follows that the diamond lattice is the best if at least 73% of facilities are open.

$E(R)$ of the random pattern is obtained from Eq. (7) as

$$E(R) = \frac{\sqrt{\pi}}{2\sqrt{2p\rho}}. \quad (14)$$

The ratio of the average distance of the square (diamond) lattice to that of the random pattern is 79.8% (75.2%) at $p = 1$, and 89.0–89.6% (90.2–90.4%) at $p = 0.5$.

5. Conclusion

This paper has theoretically derived the k th nearest distance distributions with rectilinear distances for $k = 1, 2, \dots, 8$ for the square and diamond lattices. These analytical expressions of higher order distances have not been found in previous studies. As an application of the k th nearest distance, we also have considered a facility location problem with closing of facilities.

The following implications are worth noting. First, we have revealed that some of the average k th nearest distances of the square lattice are smaller than those of the diamond lattice. This means that the diamond lattice is not always optimal if the distances to the k th nearest facility are also considered. Second, we have obtained the average distances to the nearest open facility when facilities are closed independently and at random. These average distances of regular and random patterns give an estimate for the distance for actual facility patterns. Finally, we have proved that the diamond lattice is the best if at least 73% of facilities are open. This finding gives an insight into the further studies on facility location problem with closing of facilities.

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Appendix A.

The k th nearest distance distributions $f_k(r)$ and the average k th nearest distances $E(R_k)$ ($k = 1, 2, \dots, 8$) with rectilinear distances of the square and diamond lattices are as follows:

Square lattice

$$\begin{aligned} f_1(r) &= \begin{cases} 4\rho r & \left(0 < r \leq \frac{1}{2\sqrt{\rho}}\right) \\ -4\rho r + 4\sqrt{\rho} & \left(\frac{1}{2\sqrt{\rho}} < r \leq \frac{1}{\sqrt{\rho}}\right) \end{cases} & E(R_1) &= \frac{1}{2\sqrt{\rho}} \\ f_2(r) &= 8\rho r - 4\sqrt{\rho} & \left(\frac{1}{2\sqrt{\rho}} < r \leq \frac{1}{\sqrt{\rho}}\right) & E(R_2) = \frac{5}{6\sqrt{\rho}} \\ f_3(r) &= -8\rho r + 12\sqrt{\rho} & \left(\frac{1}{\sqrt{\rho}} < r \leq \frac{3}{2\sqrt{\rho}}\right) & E(R_3) = \frac{7}{6\sqrt{\rho}} \\ f_4(r) &= 8\rho r - 8\sqrt{\rho} & \left(\frac{1}{\sqrt{\rho}} < r \leq \frac{3}{2\sqrt{\rho}}\right) & E(R_4) = \frac{4}{3\sqrt{\rho}} \\ f_5(r) &= \begin{cases} 4\rho r - 4\sqrt{\rho} & \left(\frac{1}{\sqrt{\rho}} < r \leq \frac{3}{2\sqrt{\rho}}\right) \\ -4\rho r + 8\sqrt{\rho} & \left(\frac{3}{2\sqrt{\rho}} < r \leq \frac{2}{\sqrt{\rho}}\right) \end{cases} & E(R_5) &= \frac{3}{2\sqrt{\rho}} \\ f_6(r) &= -8\rho r + 16\sqrt{\rho} & \left(\frac{3}{2\sqrt{\rho}} < r \leq \frac{2}{\sqrt{\rho}}\right) & E(R_6) = \frac{5}{3\sqrt{\rho}} \\ f_7(r) &= 8\rho r - 12\sqrt{\rho} & \left(\frac{3}{2\sqrt{\rho}} < r \leq \frac{2}{\sqrt{\rho}}\right) & E(R_7) = \frac{11}{6\sqrt{\rho}} \\ f_8(r) &= f_7(r) & & E(R_8) = E(R_7) \end{aligned}$$

Diamond lattice

$$\begin{aligned} f_1(r) &= 4\rho r & \left(0 < r \leq \frac{1}{\sqrt{2\rho}}\right) & E(R_1) = \frac{\sqrt{2}}{3\sqrt{\rho}} \\ f_2(r) &= -4\rho r + 4\sqrt{2\rho} & \left(\frac{1}{\sqrt{2\rho}} < r \leq \sqrt{\frac{2}{\rho}}\right) & E(R_2) = \frac{2\sqrt{2}}{3\sqrt{\rho}} \\ f_3(r) &= 4\rho r - 2\sqrt{2\rho} & \left(\frac{1}{\sqrt{2\rho}} < r \leq \sqrt{\frac{2}{\rho}}\right) & E(R_3) = \frac{5}{3\sqrt{2\rho}} \\ f_4(r) &= f_3(r) & & E(R_4) = E(R_3) \\ f_5(r) &= -4\rho r + 6\sqrt{2\rho} & \left(\sqrt{\frac{2}{\rho}} < r \leq \frac{3}{\sqrt{2\rho}}\right) & E(R_5) = \frac{7}{3\sqrt{2\rho}} \\ f_6(r) &= f_5(r) & & E(R_6) = E(R_5) \\ f_7(r) &= 4\rho r - 4\sqrt{2\rho} & \left(\sqrt{\frac{2}{\rho}} < r \leq \frac{3}{\sqrt{2\rho}}\right) & E(R_7) = \frac{4\sqrt{2}}{3\sqrt{\rho}} \\ f_8(r) &= f_7(r) & & E(R_8) = E(R_7) \end{aligned}$$

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