Dynamics in Co-evolving Networks of Active Elements

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We investigate the co-evolving dynamics in a weighted network of various dynamical elements, in which the state of the elements at the nodes and the weights of the links interact with each other. First, we examine the network of phase oscillators with various local, bottom-up rules for the weight of the link, and next investigate the recurrent networks of neurons with the global, top-down learning rule of an extension of the infomax principle. In both cases, some interesting properties of emergent dynamical patterns are found and characterized by mutual information.

Key words: Complex Network, Phase Oscillator, Nonlinear Dynamics, Recurrent Infomax

1. A Co-evolving Network of Phase Oscillators and Coupling Weights

Beautiful ordered pattern observed in nature has been attracting the interest of the scientific researchers, because it indicates the existence of an undelying mechanism which generates such an ordered pattern from the disturbed and randomized state. In general, many things in nature consist of the aggregated elements, and then the formed pattern represents a kind of a coherent state of the elements, in which the interaction among the elements plays a critical role in generating the order. In this context, we believe that coupled dynamical system will provide a theoretical framework to elucidate the underlying mechanism of the emergence of the ordered collective behaviors.

What is essential in the coupled dynamical system is the interaction among the dynamical elements. In this paper, we focus on the coupling connections among the elements, and address a question: how the structure of the connections are determined in the coupled systems in nature? One possibility is that the connections are fixed by the constraints due to the physical properties of the real world. For example, if the interaction is limited in a short range, the system is regarded as the case in which the dynamical elements occupy the site of a lattice and they interact only with the nearest neighborhoods. In contrast to the system with such a regular and static connections, there are other types of systems in which they are coupled with more flexible and complicated connections, which is often seen in the broad fields of sciences: physics, chemistry, biology, and the social sciences. This type of system can be regarded as dynamical networks of active elements. It has been reported that there exist some topological properties in their coupling structures, which suggests that their connections are not random, but have a certain kind of structure. It is, however, still unclear how the structures of connections are determined. The significant feature observed in the type of system is that the connection is flexible and can be changed depending on the dynamical behavior of the elements. For example, in neural networks, it is known that the synaptic coupling between neurons is altered depending on the activities of the neurons, and believed that this activity-dependent change in the synaptic connections causes the changes of neuronal circuits, resulting in the emergence of functionality, such as learning or memory. This activity-dependent changes of the network were observed in a number of biological and social networks, and would determine the connections in the coupled dynamical systems, generating the interesting collective behaviors. The distinctive characteristics of this dynamical network are that both the coupling between the nodes and the states of the active elements at the nodes interact and evolve together. Hence, it is important to elucidate the essential features of such co-evolving dynamics: what types of connections are organized? what types of pattern emerge?

Among the typical types of coupled dynamical systems, the coupled limit-cycle system has been studied most intensively. Limit-cycle oscillation is widely observed in real dissipative systems, and the coupled limit-cycle oscillators often generate a rich variety of collective behaviors. The limit-cycle oscillation is structurally stable, and it can be described by a simple model of phase oscillator that is mathematically tractable (Kuramoto, 1984; Acebron *et al.*, 2005). Thus, the coupled oscillator system has served one of the fundamental models to examine the essential nature of the collective behaviors which emerge in coupled dynamical systems. For this reason, we consider the coupled oscillators to extend it to the co-evolving system by inducing the evolution dynamics of the coupling connections. Using a standard reduction technique, a network of N limit-cycle oscillators are described by the coupled phase equation

$$\frac{d\phi_i}{dt} = \omega - \frac{1}{N}\sum_j k_{ij}\sin(\phi_i - \phi_j + \alpha), \qquad (1)$$

where ϕ_i denotes the phase of the limit-cycle oscillation at the i-th node of the network ($i = 1, \ldots, N$), and ω is its natural frequency. The last term in the equation represents the interaction between the oscillators, in which k_{ij} is the coupling weight from the j-th to the i-th oscillator. The parameter α denotes the phase difference due to a short transmission delay of the interaction.

We introduce the dynamics which describes the evolution rule of the coupling weight, k_{ij} (Aoki and Aoyagi, 2009). This dynamics depends on the states of the oscillators at the nodes. In this model, the state is given by the phase of the oscillation, and it is natural that this dynamics depends on the relative phases between the oscillators, not on the absolute value of the phases. In this way, the dynamics is given by

$$\frac{dk_{ij}}{dt} = \epsilon\Lambda(\phi_i - \phi_j), \quad |k_{ij}| \le 1, \qquad (2)$$

where ϵ determines the time scale of the dynamics of the coupling weight k_{ij}. Because the evolution of the coupling weight tends to be very slow compared with that of the dynamical elements at the nodes, we assume that $\epsilon \ll 1$. The function $\Lambda(\phi)$ defines the evolution rule of the coupling weight as a function of the relative phase between oscillators. In general, this function is a 2π periodic function, and then we assume the form $\Lambda(\phi) = -\sin(\phi + \beta)$, considering the lowest-order Fourier mode. The condition, $|k_{ij}| \le 1$, gives a constraint to the range of k_{ij}, because the coupling weight can not grow indefinitely in practical cases.

The mathematical model given above has two parameters α and β. In particular, β controls the characteristics of the evolution rule of the coupling weight. By varying the value of β, the system drastically changes its pattern of collective behavior and the coupling weights. We found that this model exhibits three distinct types of the dynamical patterns depending on the parameters: a two-cluster state, a coherent state with a fixed phase relation, and a chaotic state with frustration. We show an example of these states in the case of the seven-oscillator system (Figs. 2, 3 and 4). In these figures, the time development of the state of the system, $\{\phi_i(t), k_{ij}(t)\}$, is displayed by the sequence of matrix graphs. As illustrated in Fig. 1, the value of ϕ_i is represented by a circle graph at the diagonal position, and the value of k_{ij} is expressed by the color at the off-diagonal position in the matrix graph.

In the two-cluster state shown in Fig. 2, the oscillators organize the two synchronized groups ($\alpha = 0.1\pi$, $\beta = -0.6\pi$). The initial phases $\phi_i(0)$ and coupling weights $k_{ij}(0)$ are chosen randomly from uniform distributions on $[0, 2\pi)$ and $[-1, 1]$, respectively. Then, the system realizes two groups within which the oscillators of the group are synchronized and the phase difference between the groups is π. The coupling weights between the oscillators belonging to the same cluster take $k_{ij} = k_{ji} = 1$, while those between the oscillators belonging to the different clusters take $k_{ij} = k_{ji} = -1$. It is a result of the evolution rule of the coupling weight. For the parameters $\beta \sim -\pi/2$, the coupling weights between the oscillators of similar phases are

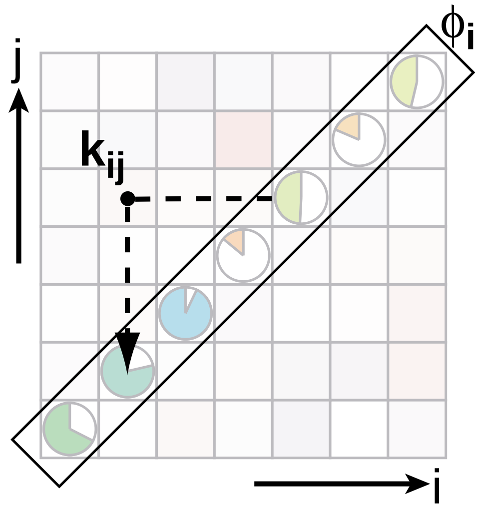

Fig. 1. An illustration of the matrix graph shown in Figs. 2, 3 and 4. It represents the state of the dynamical system, $\{\phi_i(t), k_{ij}(t)\}$ of the model which is given by Eqs. (1) and (2).

increased, whereas those between the oscillators of quite different phases are decreased, owing to the form of the function $\Lambda(\phi)$. This like-and-like rule leads the system to the emergence of the clustered state.

When the parameter β is set to -0.1π, a coherent state emerges (Fig. 3), in which oscillators rotate coherently, maintaining a fixed phase relationship among the oscillators. For this sequential phase-pattern, the coupling weights are organized depending on the order of the phases. From the leading oscillator to the succeeding one, a positive coupling weight, $k_{ij} = 1$, is formed, while the connection to the opposite direction, k_{ji} take a negative weight, -1. In contrast to the case of two-cluster state, the form of $\Lambda(\phi)$ for $\beta \sim 0$ is asymmetric with respect to the sign of the phase difference. For this reason, the growth of the coupling weight is strongly affected by the sequential order of the oscillators, and it results in the emergence of this coherent state.

If β=0.4π, the system does not settle into a fixed state, in contrast to the above types of emergent patterns. In this state, the phase relationship among the oscillators and the coupling weights continue changing in time. Moreover, we found that Lyapunov exponents take positive values, and then we refer to it as a chaotic state (Fig. 4). In this state, $\Lambda(\phi)$ takes the opposite form to that of the two-cluster state, which has the opposite effect on the evolution of the coupling weight: the coupling weight between the synchronized oscillators is decreased, while that between the oscillators of quite different phases is increased. According to this evolution rule, a reciprocal destabilization of the phase pattern and the network structure is observed.

In summary, we have investigated co-evolving dynamics in a weighted network of phase oscillators in which phase oscillators at the nodes and the weights of their links interact and co-evolve. We found that this system exhibits three distinct types of dynamical patterns: a two-cluster state, a coherent state with a fixed phase relation, and a chaotic state

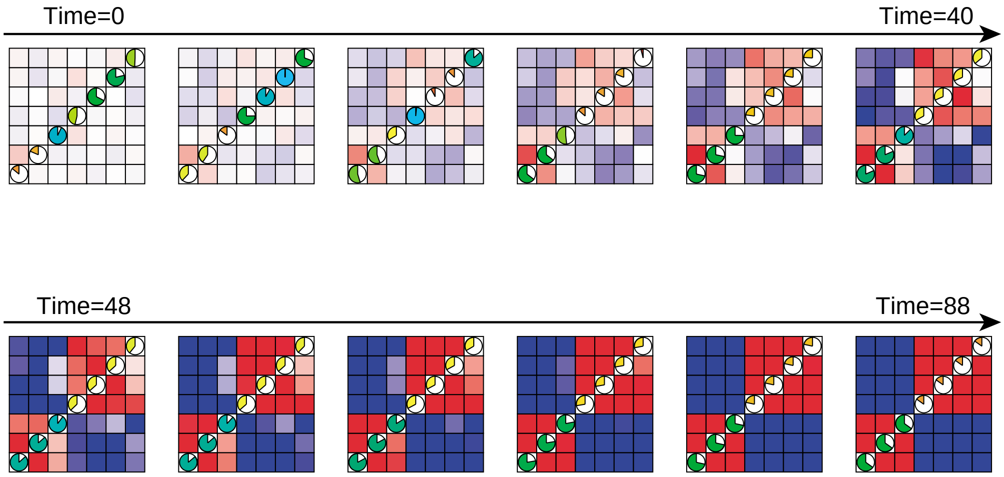

Fig. 2. The emergence of a two-cluster state. The sequence of the matrix graphs displays the time development of the state of the dynamical system, $\{\phi_i(t), k_{ij}(t)\}$. In this state, the phases of the oscillators, $\phi_i(t)$, are organized into two synchronized groups as shown in the diagonal blocks in the matrix graphs. The coupling weights among the oscillators, $k_{ij}(t)$, shown in the off-diagonal blocks, take positive couplings (red) within a synchronized group and negative couplings (blue) between the different groups. This two-cluster state emerges in the case of $\beta \sim -\pi/2$, with which the evolution rule of the coupling weight has a similar effect of like-and-like rule.

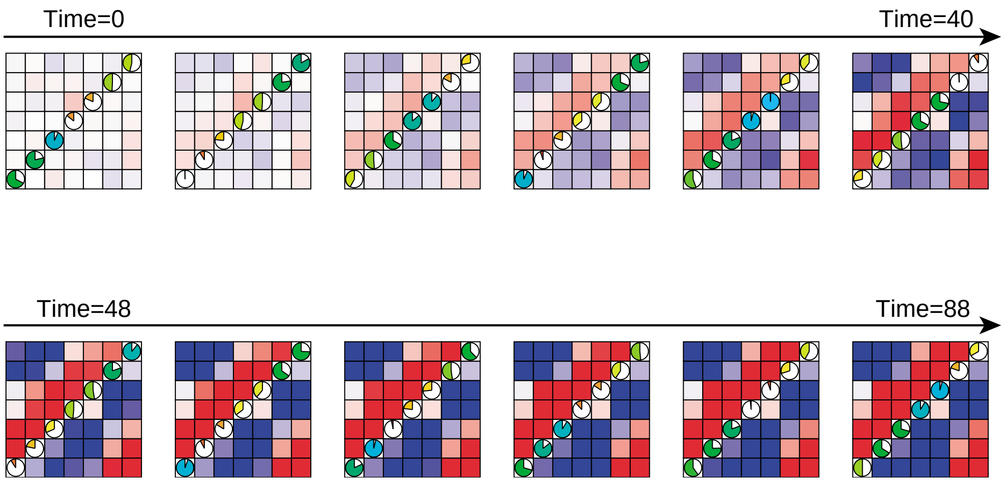

Fig. 3. The emergence of a coherent state with a fixed phase relation. A sequential pattern of the oscillators are organized, in which a fixed phase relationship is maintained over a long period. This state is observed for the parameter $\beta \sim 0$, with which the evolution rule of the coupling weight is strongly dependent on the order of the phases of oscillators, in a similar way as the spike-timing dependent plasticity in neural networks.

with frustration. These distinct dynamical behaviors can be characterized by mutual information between the initial and final phase patterns, and by entropy of the final phase pattern. In Fig. 5, the mutual information is largest for the coherent state. Since mutual information measures the information that the initial and final states share, the initial phase pattern can be most easily inferred from the final one in the coherent state. This suggests that the coherent state preserves a phase pattern through the co-evolving dynamics. A similar situation is observed for the case of the two-cluster state, except that the entropy is much smaller than that for the coherent state. This is because the possible phase patterns are restricted for the two-clustered state. For the chaotic state, the mutual information takes almost zero and the entropy is almost maximum. This fact implies that the information of the initial state is lost with time and the system is wandering over all possible phase patterns.

2. Firing Activity of Optimized Neuronal Networks

In the previous section, we descibed a simplified model of co-evolving dynamics. In this simple model, the behavior of the system is reduced to a few essential parameters. Thus this model allowed us to understand the behavior of the co-evolving systems without knowing the details of the systems. However, because this model is an abstract model, more specific models are needed to understand the detailed behavior of individual systems, such as our brain. As an example, here we describe a specific neuronal network model to explain the activity of the neuronal networks in the brain. This neuronal network model is a top-down model, whose dynamics we derived to maximize an objective function, which is the mutual information in this case. It is in a sharp contrast to the model in the previous section, which explained the behavior of the co-evolving systems by using two bottom-up rules (Eqs. (1) and (2)). The

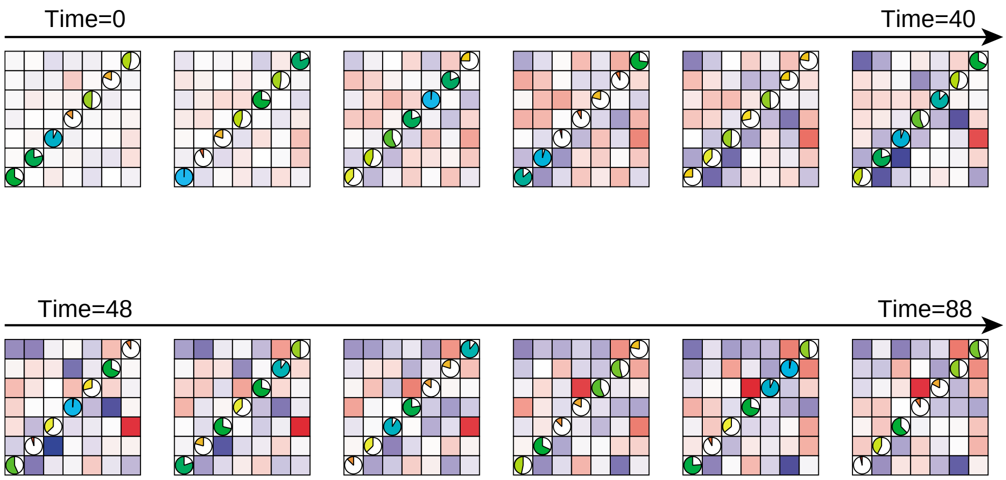

Fig. 4. The emergence of a chaotic state with frustration. In contrast to the above two steady states, the structure of the network and the phase relationship among the oscillators continue changing in time, and do not converge to a fixed state. This result is caused by the evolution rule specified by $\beta \sim \pi/2$, which has the opposite effect of the like-and-like rule in the two-cluster state. According to this anti-like-and-like rule, the system exhibits a reciprocal destabilization of the phase pattern and the coupling weights.

neuronal network model in this section explains the activity observed in the experiments of the neuronal networks from an information-theoretic point of view.

Our brain is the most complex coupled dynamical system. There are $\sim 10^9$ neurons in our cerebral cortices, and each of these neurons has $\sim 10^4$ synaptic connections. In terms of coupled dynamical systems, neurons in the brain are regarded as active elements, and synaptic connections among them correspond to the connections among active elements. As an active element, the activity of a neuron is characterized by the dynamics of the membrane potential. The membrane potential of a neuron fluctuates around the resting potential until the neuron receives a strong excitatory input. An excitatory input makes the membrane potential cross the threshold, and the the neuron generates an action potential. When the action potential reaches synaptic terminals, the excitatory or inhibitory input is given to the neurons connected by the synaptic terminals, and this input, in turn, evokes the generation of action potentials in these postsynaptic neurons. Thus, in our brain, a large number of neurons interact through synaptic connections in a quite complicated manner, and the brain activity consists of the generation and propagation of firings, i.e., action potentials.

Although the number of neurons and complicated connections among them have made researches on the central nervous system (CNS) difficult, recent advances in multineuronal recording have allowed us to observe phenomena in the networks of the CNS that are much more complex than previously thought to exist. The existence of interesting types of neuronal activity, such as patterned firings, synchronization, oscillation, and global state transitions has been revealed by multielectrode recording and calcium imaging (Nadasdy *et al.*, 1999; Cossart *et al.*, 2003; Ikegaya *et al.*, 2004; Fujisawa *et al.*, 2006; Sakurai and Takahashi, 2006). Recently, we have succeeded in explaining and reproducing patterned firings observed in experiments on the basis of the process of “information maximization” (Tanaka *et al.*, 2009). The process of information maximization (infomax (Linsker, 1988)) maximizes the information transmission from the input to the output of a feedforward network. We extended infomax to the case of recurrent networks, in which neurons are interconnected and the input to the neurons at time t consists of their own output at time $t-1$. This algorithm adjusts the connection weights to realize the most efficient information transfer from the input to the output. This learning algorithm optimizes the information retention and transmission in the model network. In other words, the model network self-organizes to an information-efficient one through learning by this algorithm. We found that the network optimized for information retention exhibits (1) stereotyped spontaneous activity (Hebb, 1949; Abeles, 1991) and (2) a critical neuronal avalanche (Beggs and Plenz, 2003).

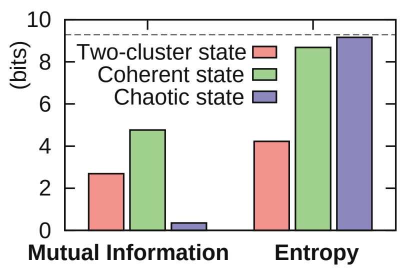

Fig. 5. Mutual information between the initial and final phase patterns and the entropy of the final phase pattern for the three asymptotic states of a small system ($N=5$). The parameter values α, β are the same as in Figs. 2, 3 and 4. The horizontal dashed line represents the maximum attainable entropy for the phase pattern.

2.1 Repeated patterns and sequences

We examined the evolution of the spontaneous activity in a neuronal network without external input. Figure 6(a) shows the raster plot of the firing activity in the model network before optimization. Neuronal firings are indicated by triangles. To identify repeated activity in the model network, we defined a repeated pattern as a spatial pattern of neuronal firings that occurs at least twice in the test block.

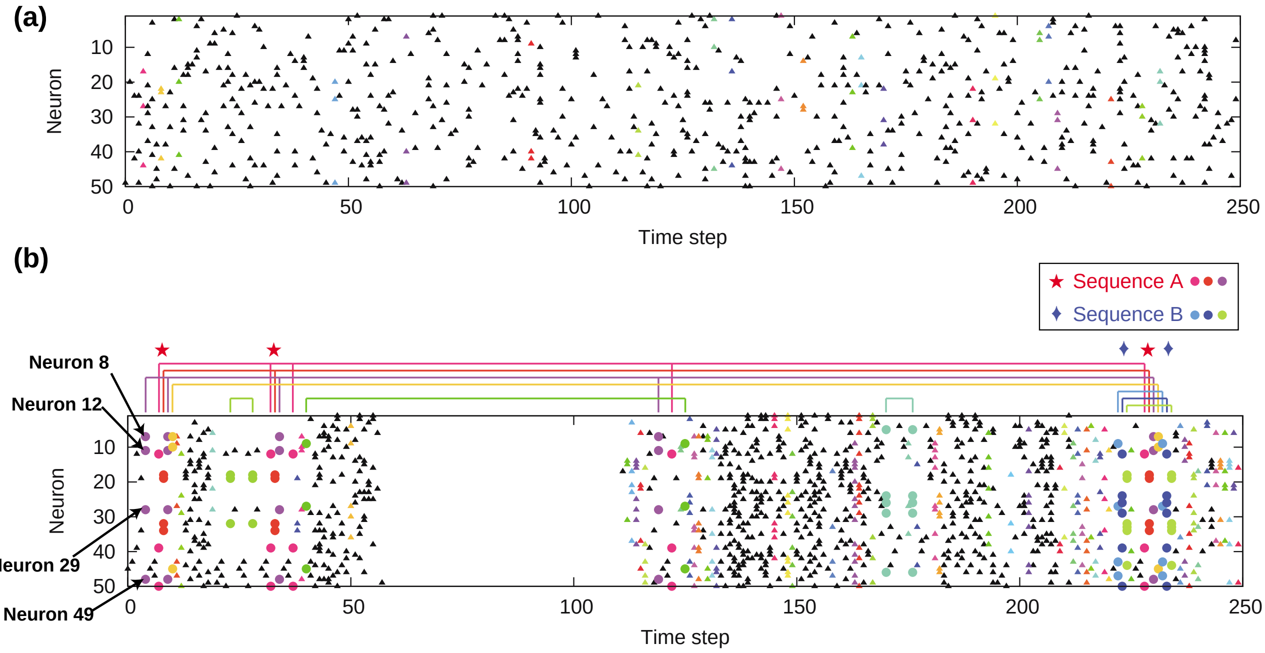

Fig. 6. Repeated spatial patterns and spatiotemporal sequences occurred frequently in the network after learning. (a) Raster plot before learning. When the repeated patterns in a test block of 50,000 steps were colored, it was found that no pattern occurred more than once in this short raster plot before learning. (b) Raster plot after learning. Several patterns appeared multiple times in the raster plot after learning. The repeated patterns are indicated by consistently colored circles and connected by lines.

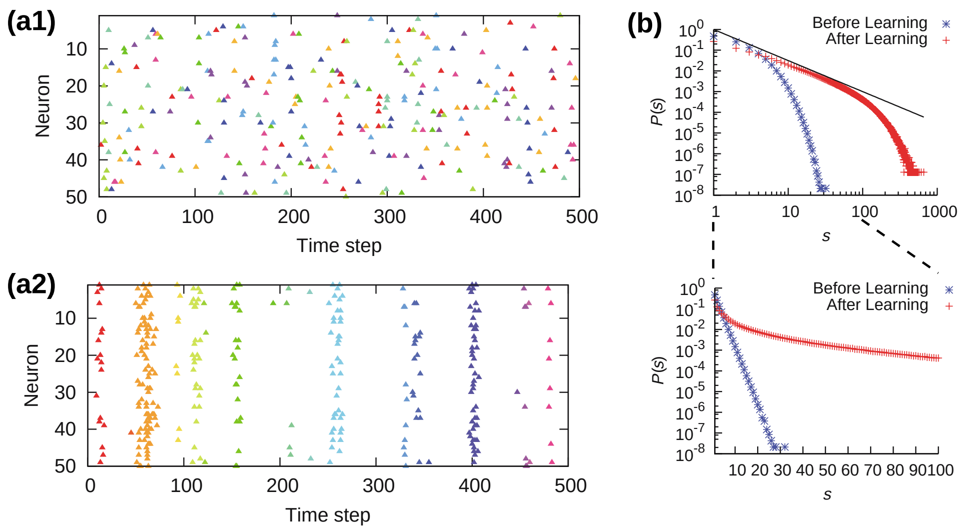

Fig. 7. (a1) Raster plot before learning. Individual bursts in the spontaneous activity are indicated by different colors. The bursts before learning were short and frequently interrupted by steps without firing. (a2) Raster plot after learning. The bursts after learning had much longer durations than before learning. (b) Size distribution of avalanches. The black line corresponds to a slope of -1.5.

Coloring repeated patterns consisting of ≥ 3 firing neurons in raster plots of the network, we found that there are few repeated patterns in the activity of the network before optimization. Colored patterns in Fig. 6(a) did not occur twice in this short raster plot of 250 steps and were repeated later. Thus, there are few repeated patterns in the network activity before learning. However, the number of repeated patterns increased after learning (Fig. 6(b)). Several patterns were repeated in a sample of 250 steps as seen in Fig. 6(b), where the repeated patterns are indicated by consistently colored circles and connected by lines. For example, the purple pattern consisting of firings of neurons 8, 12, 29, and 49 (indicated by arrows) occurs four times (t =4, 9, 34, 119, and 230) in this raster plot. Moreover, some patterns appeared to constitute repeated sequences. For example, sequence A, composed of the magenta, orange, and purple patterns, appears three times in Fig. 6(b). This indicates that the present algorithm embeds not only repeated patterns but also repeated sequences of firings into the network structure as a result of the optimization. This result is consistent with experimental results suggesting that the neuronal activity in our brain consists not of uncorrelated firings, but of repeated patterns and sequences of firings.

In the optimized network, when a pattern in a sequence is activated at one step, it is highly probable that the next pattern in that sequence will be activated at the next step. This predictability means that the state of the network at one time step shares much information with the state at the next time step. In contrast, when the dynamics of a network is highly stochastic and thereby repeated patterns are rare, we cannot predict which pattern follows a given pattern nor reduce the uncertainty of the next pattern by using the knowledge of the present pattern. Thus, the optimization of information retention and transfer embeds repeated patterns and repeated sequences of firings into the network structure.

2.2 Neuronal avalanches

We next examined the behavior of the same neuronal network model in the case that the reliability of the neuronal firing is low. When the neurons are less reliable, a neuron does not always fire even if it receives a strong enough input. Thus, the number of identically repeated sequences is small, and the network seems to lose structured activity. However, we found characteristic network activity consisting of firing in bursts (Fig. 7(a2)), which are defined as consecutive firing steps that are immediately preceded and followed by "silent" steps, with no firing. We found that after learning, the distribution $P(s)$ of the burst size s, which is the total number of firings in a burst, obeys a power-law distribution $P(s) \propto s^{\gamma}$ with $\gamma \approx -1.5$, whereas, before learning, we have $P(s) \propto \exp(-\alpha s)$ (Fig. 7(b)). This result is consistent with experimental results. Recently, Beggs and Plenz (2003) recorded the spontaneous activity of an organotypic culture from the cortex using multielectrode arrays. Defining an avalanche similarly to our bursts following a period of inactivity, they found that the size distribution of avalanches is accurately fit by a power-law distribution with exponent -1.5. To explain this, they argued that a neuronal network is tuned to minimize the information loss and that this is realized when one firing induces an average of one firing at the next step. They showed that this condition yields the universal exponent $-3/2$, using the self-organized criticality of the sandpile model (Bak *et al.*, 1987; Harris, 1989). This condition also holds for the present network, because, after learning, each neuronal firing evoked one firing in the next step on average. We thus conclude that our neuronal network model reproduces the patterned avalanche activity in the brain.

3. Conclusion

In conclusion, we consider two types of the co-evolving dynamics in weighted network of dynamical elements. One is the network of phase oscillators, in which the phase pattern at the nodes modifies the weights between phase oscillators via a local rule of slow dynamics for the weights. As a result, we found that this system exhibits three types of dynamical behaviors: a two-cluster state, a coherent state with a fixed phase relation, and a chaotic state with frustration. These distinct dynamical behaviors can be characterized by mutual information between the initial and final phase patterns, and by entropy of the final phase pattern. In particular, when the local rule is similar to the spike-timing dependent plasticity observed in recent neuronal experiments, the system exhibits the coherent state, in which the temporal order of all the phases of oscillators is preserved. Interestingly, the mutual information is maximal for the coherent state. This result inspires us to adopt a top-down principle that the system maximizes the mutual information among the dynamical states. The other network we consider is the neuronal networks, in which the synaptic weights are adjusted according to an extended version of the infomax principle. In the case of highly reliable neurons, the maximal information retention results that the system exhibits some specific repeated sequential firing patterns. In the case of less reliable neurons, the neuronal avalanche emerges in the firing pattern of the network. The size of avalanche obeys a power-law distribution. The fact that a power-law dynamical behavior is realized by the infomax principle might bridge the gap between nonlinear dynamics and information theory in various complex networks.

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