

# Space Division—View Points of Pathology and Physics

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Some basic properties of the space division are explained. As elements of periodic division of the 3-dimensional space, the Fedorov's parallel polyhedra are introduced, including the cube, the rhombic dodecahedron and Kelvin's 14-hedron. The concept of stability in space division is explained, which is connected to the mechanical stability of the elements. The high frequency of pentagons as faces of elements is noted, and its relation to the stability of space division is suggested. Finally, some examples of divisions in spaces with higher dimension are given.

**Key words:** Space Division, Kelvin's 14-hedron, Fedorov's Parallel Polyhedra, Stable Division, Pentagon

## 1. Introduction

When we observe natural objects, we often have a question how they have acquired these shapes. One good example of it might be the snow crystal. Although its shape is determined owing to the hexagonal crystal structure of ice, its mechanism cannot be understood simply by packing of hexagons, and we must look for precise mechanism to create real shapes. This question of "how" is the motivation for the science on form.

This question of "how" exists also in medical sciences, where construction of tissue in normal and pathological cases is understood in terms of cell arrangement. The problem is what the principle is to produce this arrangement. This aspect is especially important when cell arrangements are observed in cell diagnosis. Theoretical approach to this problem has been made in various cases. A monograph and a paper by a pathologist Suwa (1981a, b) should be introduced here, which give a unified view on the problem of cell arrangement based on a scientist's eye and a mind of poet. The monograph will be often cited later.

It is worth noting here that we have a lot of cases where the space divisions in both biological and non-biological objects are often attained as packings of polyhedra, such as the seeds of pomegranate, honeycomb and soap bubbles. Moreover, it is known that many of polyhedra in these cases have fourteen faces (denoted by 14-hedron in the following), and that their faces are mostly pentagons. This review article focuses on these properties of space division, and discusses on the following three topics, the space division by 14-hedron and introduction of Fedorov's parallel polyhedra as its extension, the stability of space division and the space division with pentagonal faces. These contents owe much to Suwa (1981a), and are supplemented by some new findings.

## 2. Space Division by 14-hedron and Fedorov's Parallel Polyhedra

Before discussing on the space division, we mention a little on the arrangement of square bricks on flat plane. If we arrange them so that an edge of a brick touches only two other bricks, one brick touches six other bricks around itself (see Fig. 1(a)). This kind of arrangement can be seen in many examples, such as the stone construction of castles and the cell arrangement in onion (Fig. 1(b)). These examples suggest that the basic form of elements in plane division is the hexagon. If two edges of the hexagon are contracted to two points respectively, i.e. two ends of each edge approach each other and coincide, the hexagon becomes a square and the space division changes to that of square meshes. Therefore, the popular square mesh pattern is looked upon as derived from the basic space division by hexagons.

The space division can be obtained in the similar way by piling up plane layers of cubic brick arrangements. Place the second layer of bricks on the first layer so that every vertex of bricks in the first layer touches an inner point of a brick in the second layer, and place the third layer in the same way on the second layer (see Fig. 1(c)). Then, one brick touches 14 bricks, 4 in the lower layer, 4 in the upper layer and 6 in the same layer. Therefore, the basic shape of elements in the space division is considered to be 14-hedron, and the cubic brick is looked upon as a shape contracted from the 14-hedron.

In natural space divisions it is observed that the number of faces  $f$  of elements has a distribution with a peak at about 14. For example, according to a research of plant tissues, 74% of cells had faces within the range (12–16), 56% had those within (13–15) and the average number of faces was 13.9 (Thompson, 1968).

Next, the problem of space division is extended to periodic arrangements of a certain kind of polyhedra, called Fedorov's parallel polyhedra. They are defined as polyhe-

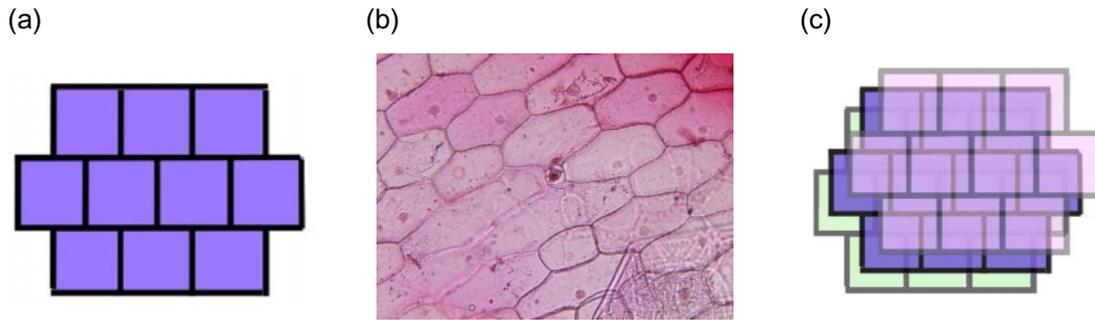


Fig. 1. (a) Two-dimensional arrangement of square bricks, where each brick has 6 neighbors, (b) cell arrangement of onion cells (pictures taken by one of the authors (Sato)), (c) three dimensional arrangement of cubic bricks, where each brick has 14 neighbors.

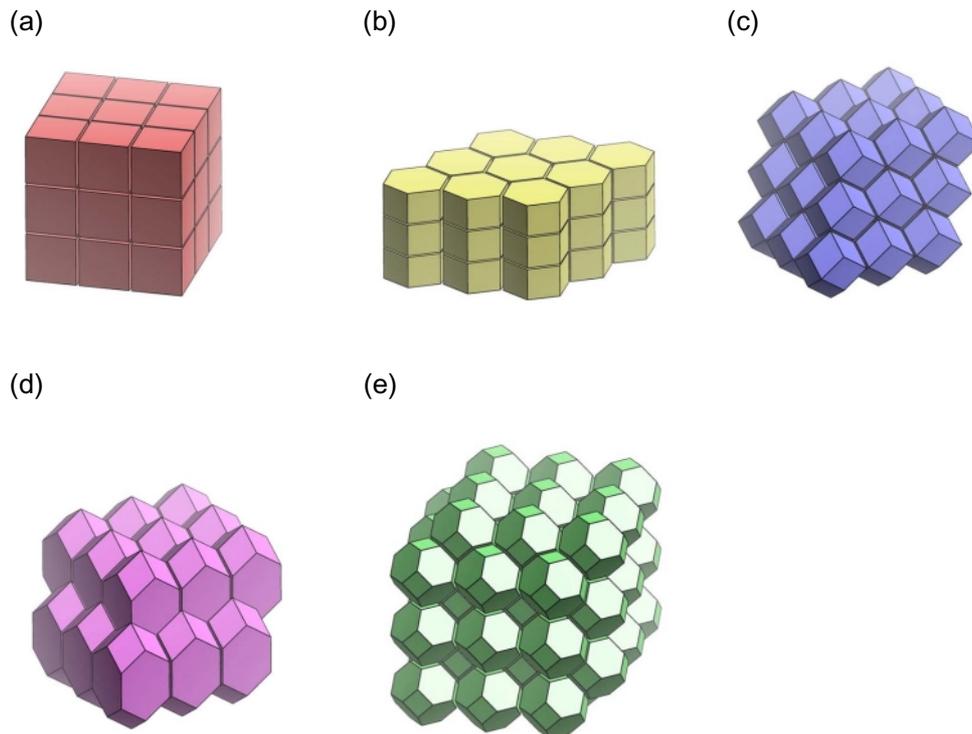


Fig. 2. Fedorov's parallel polyhedra. (a) Cube, (b) hexagonal prism, (c) rhombic dodecahedron, (d) elongated rhombic dodecahedron, (e) truncated octahedron (Kelvin's 14-hedron). (Provided by Ishii (2008, private communication).)

dra with the following conditions:

- a. each face has a parallel counterpart,
- b. each edge belongs to a group of parallel edges,
- c. the space must be filled by repeated translation of the same polyhedron only.

These polyhedra are limited to the five members consisting of truncated octahedron (Kelvin's 14-hedron), elongated rhombic dodecahedron, rhombic dodecahedron, hexagonal prism and parallelepiped. Their forms are shown in Fig. 2 as closely packed groups. Their faces are limited to parallelograms or hexagons with opposite parallel edges. These polyhedra are less popular than the Platonic bodies, but are considered equally important because they play a role of space packing.

Fedorov himself found that the truncated octahedron is a basic form and the other four members can be produced from it by contracting some of edges to points. By choosing

four hexagons in the truncated octahedron and contracting a pair of parallel edges in each of chosen hexagons we obtain the elongated rhombic dodecahedron (readers are suggested to confirm it themselves). Furthermore, by contracting a pair of parallel edges in the remaining four hexagons in the elongated rhombic dodecahedron, we have the rhombic dodecahedron. By another way of contraction we obtain the hexagonal prism from the elongated rhombic dodecahedron. Finally, from either of rhombic dodecahedron or the hexagonal prism we obtain the cube. These processes are shown in Fig. 3. The values of  $d$  indicated in this figure show the numbers of dimensions of hyper-cubes whose projections on the 3-dimensional space give respective polyhedra (for precise see Sec. 6).

An aspect of hyper-dimensional body for understanding Fedorov's parallel polyhedra is mentioned in the later section.

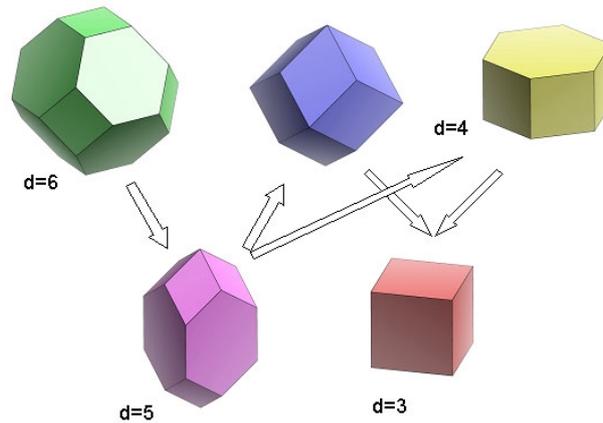


Fig. 3. Processes to reproduce four members of Fedorov's parallel polyhedra from the truncated octahedron. These figures are provided by Ishii (2008, private communication).

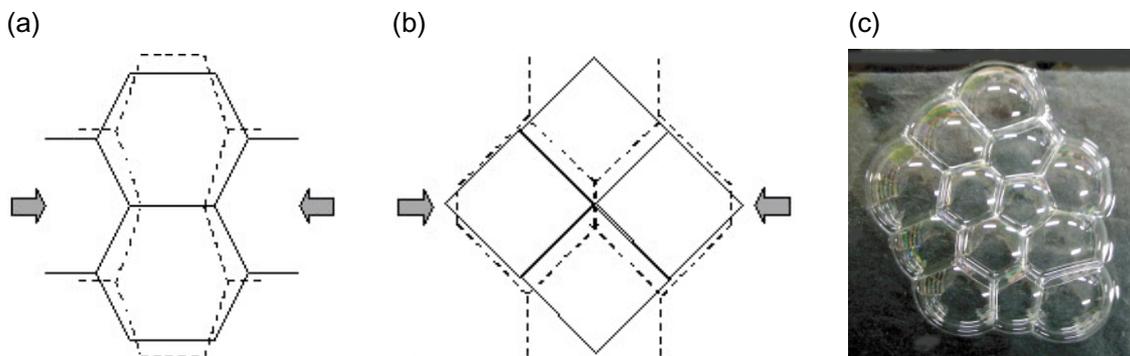


Fig. 4. Examples of stability of 2-dimensional arrangements. (a) Hexagon packing (stable), (b) square packing (unstable), where solid and dashed lines show arrangements before and after pressure, respectively. (c) Soap bubbles arranged on a horizontal plate.

### 3. Stability of Space Division

The usual meaning of stability is a property of construction of objects, where a small deformation of construction causes responses either restoring to the original one (stable case) or leading to larger deformation (unstable case). The concept of stability discussed in this section is similar to this definition, while some notes are necessary for better understanding. The deformation here means change of positions of elements (for example, due to an external pressure), while its response is the change of types of element shapes, i.e. whether the square remains square, or the hexagon remains hexagon.

We begin with 2-dimensional cases. It is well known that the regular polygons filling the plane with equal members are limited to the triangle, the square and the hexagon. Among these arrangements the triangle and square cases have different property from the hexagonal case. In the latter case each hexagon touches neighbors through edges, while in the former cases each triangle or square has neighbors which it touches only through a vertex. In this case a small deviation of positions of polygons leads to change of types of shapes. A simple example of this situation is shown in Fig. 4, where the hexagons pressed horizontally remain hexagons (Fig. 4(a)) while the squares change to hexagons (Fig. 4(b)). It should be noted here that in the stable case always three edges meet at a vertex. Ensemble of soap bub-

bles arranged on a horizontal plate has this kind of stability (see Fig. 4(c)).

In the 3-dimensional case the stable division must satisfy the following conditions, that 4 vertices of 4 polyhedra touching each other gather at one point, and that 3 edges of 3 polyhedra touching each other gather at one line segment. These conditions are satisfied by most space divisions, both in biological and non-biological systems.

The idea of stable division can be understood in terms of Voronoi division. In the most point arrangements in the plane or in the space their Voronoi divisions are stable. Rare cases such as the point arrangement in the square lattice in the plane or the cubic lattice in the space produce unstable division.

Stability of space division is closely connected to the mechanical stability in cell arrangements in biological tissues. If a force is applied to a cell in a 2-dimensional stable arrangement, this force balances with restoring forces appearing in the two cells touching the first cell (see Fig. 5(a)). In the same way the force applied on a cell in 3-dimensional stable arrangement balances with the three forces appearing in the three cells touching the first cell (see Fig. 5(b)). When an external force is applied, to a cell in unstable division, the cell and neighboring cells change its type of shapes. If the cells are made of rigid material, the balance of forces cannot be determined uniquely. This situation is similar to that

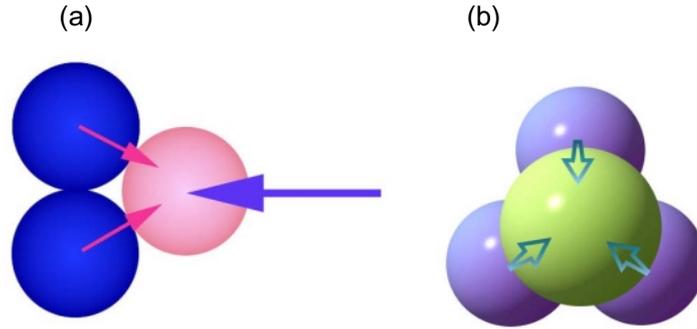


Fig. 5. Stable arrangements of cells in (a) 2D and (b) 3D cases. A force acted on a sphere with light color is balanced by the resistive forces by the spheres with dark color.

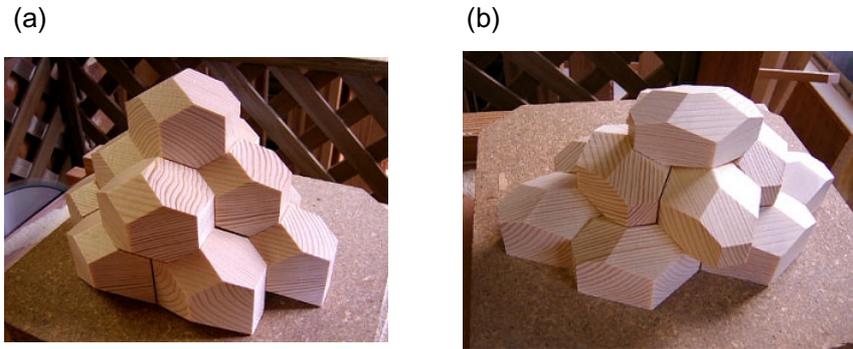


Fig. 6. Models of two 14-hedra for stable space division hand-made by Hiroshi Nakagawa. (a) Kelvin's 14-hedron, (b) Williams' 14-hedron.

in the four-leg problem in the basic mechanics, where the forces at the bottom of four legs of a table touching the floor cannot be determined uniquely.

As one of polyhedra whose equal groups fill the space, the rhombic dodecahedron and the truncated octahedron is widely known in addition to the cube. The rhombic dodecahedron is a Voronoi polyhedron from the point arrangement with face-centered cubic lattice, while the truncated octahedron is the Voronoi polyhedron from the body-centered cubic lattice. The rhombic dodecahedron has four vertices where four edges meet, and in the space division by rhombic dodecahedra six polyhedra gather at these vertices. Therefore, this space division is not a stable one. On the other hand, the space division by the truncated octahedra is stable, since in this case always four polyhedra gather at each vertex.

#### 4. Pentagon Face in Space Division

In real space divisions the faces are often curved planes. Here, it is assumed that faces and edges of polyhedra in space division can be curved. In the following we denote the numbers of faces, edges and vertices by  $f$ ,  $e$ ,  $v$ , respectively. For a polyhedron taken out from a stable space division the numbers  $e$  and  $v$  satisfy the relation

$$2e = 3v, \quad (1)$$

since three edges gather at a vertex and an edge connects with two vertices. On the other hand, the Euler's theorem gives

$$v - e + f = 2. \quad (2)$$

From these equations we have

$$v = 2(f - 2), e = 3(f - 2). \quad (3)$$

For a polyhedron with  $f = 14$ , we have  $v = 24$  and  $e = 36$ .

Shapes of faces of a 14-hedron cannot be uniquely determined, and it is assumed here that a face has  $p$  edges on the average. Since one edge is shared by two faces, we have  $e = pf/2$ , hence

$$p = 2e/f = 5.14 \dots \quad (4)$$

This result means that the distribution of types of polygons should have a peak around the value 5, and agrees with observations that the highest frequency for pentagons in the space division. For example, according to a research of metallic glasses, the distribution of  $p$  showed  $p = 5$  (40%),  $p = 6$  (30%),  $p = 4$  (20%), and an average number of edges per a face was 5.12 (Lines, 1994).

One of the important space divisions is that by soap bubbles, where the minimal surface principle is working and the total area of faces gives a criterion for selection of space division pattern. Let the surface area and the volume of a polyhedron be denoted by  $S$  and  $V$ , respectively. Then, a good parameter of the criterion is the nondimensional ratio  $S^3/V^2$  or its cubic root,

$$c = (S^3/V^2)^{1/3}. \quad (5)$$

The minimal surface principle requires smaller value of  $c$ . In the following the shapes of faces are examined for some examples based on this principle.

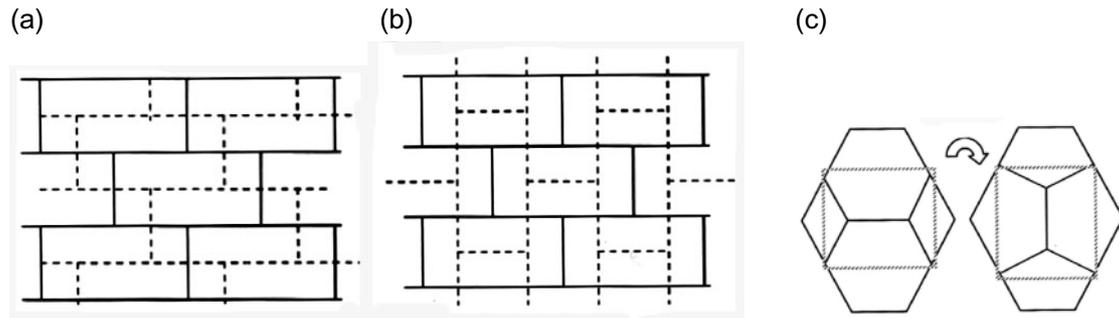


Fig. 7. Construction of Williams' 14-hedron ( $\beta$ 14-hedron). (a) Piling up of brick layers for Kelvin's 14-hedra, (b) piling up of brick layers for Williams' 14-hedra, (c) transformation to produce Williams' 14-hedron from Kelvin's 14-hedron. ((a) and (b) are reproduced from Suwa (1981a) while (c) is a modification of a figure in Williams (1968).)



Fig. 8. Weaire and Phelan's structure. Sketch from a figure in Kusner and Sullivan (1996) by one of the present authors (Takaki).

#### 4.1 Two types of 14-hedron

The rhombic dodecahedron gives a value  $c = 5.345\dots$ , while the truncated octahedron (Kelvin's 14-hedron) gives a value  $c = 5.314\dots$ , which is smaller than the value for the rhombic dodecahedron by 0.5%. The English physicist Lord Kelvin (W. Thomson) found in 1887 (Lord Kelvin, 1887) that the space can be filled by the truncated octahedra and that the interfacial area is smaller than that in the case of the rhombic dodecahedron.

About one century after Kelvin's finding Williams (1968) found another 14-hedron with curved faces which fills the space. This polyhedron has two quadrilaterals, eight pentagons and four hexagons (see Fig. 6), which is called  $\beta$ 14-hedron (see also Suwa, 1981a) and is expressed symbolically as  $(4^25^86^4)$ . On the other hand, Kelvin's 14-hedron is called  $\alpha$ 14-hedron with symbol  $(4^66^8)$ . The Williams' 14-hedron is constructed by piling up layers of arrangements of bricks as shown in Fig. 7. This piling is similar to that for Kelvin's 14-hedron (Fig. 7(a) and Fig. 1(c)), but the direction of the upper layer is rotated by 90 degree from that of the lower. This fact is connected to the transformation from the  $\alpha$ 14-hedron to the  $\beta$ 14-hedron, as shown in Fig. 7(c). If two parts of  $\alpha$ 14-hedron (shown by a quadrilateral with gray edges in Fig. 7(c) and a similar part on the opposite side) are rotated by 90 degree and fitted to the remaining middle part, we have the  $\beta$ 14-hedron.

Since Williams' 14-hedron has a lot of pentagons on its surface, it is considered to represent well a general property

Table 1. Typical structures of clathrate hydrates. The right column shows the precise of structures, where the symbol "+" stands for combination of polyhedra expressed like  $5^{12}6^4$  for example.

Names of structures	Structures
Kelvin 14-hedron (1887)	$4^66^8$
Williams 14-hedron (1968)	$4^25^86^4$
Weaire and Phelan (1994)= I	$5^{12} + 5^{12}6^2$
Structure II	$5^{12} + 5^{12}6^4$
Structure H	$5^{12} + 4^35^66^3 + 5^{12}6^8$

of stable space division, expressed by Eq. (4). In the present stage the polyhedra for stable space division seems to be limited to the two bodies, i.e. Kelvin's and Williams' 14-hedra.

#### 4.2 Space division by Weaire and Phelan

A problem of "What is the shape of bubbles with the same volume filling the space?" is not yet solved. For more than 100 years Kelvin's 14-hedron has been believed to be the answer. On the other hand, a botanist Matzke (1946) made an experiment to produce a space packing of equal size bubbles by the use of an injector, but he could not find the Kelvin's body. This experiment had a technical problem, as is pointed out by Weaire (1996a).

Recently, Weaire and Phelan (1994) found a structure by the use of computer, which is composed of dodecahedra (with 12 pentagons) and 14-hedra (12 pentagons and 2 hexagons) as shown in Fig. 8. This structure has a value of  $c$  smaller than that for Kelvin's 14-hedra by 0.3%. These polyhedra have curved faces as in the Williams' polyhedron, and application of computer allowed this finding through precise treatment of curved surfaces. Review of this problem is published as a special issue of *FORMA* (Weaire, 1996b) and a monograph (Weaire, 1996c).

It should be noted here that the Weaire and Phelan structure contains larger fraction of pentagons than that of Williams' polyhedron.

#### 4.3 Clathrate hydrate

The clathrate hydrate has a structure, where a basket made of water molecules surrounds a small molecule, a simple example being the methane hydrate. The shape of the basket shows a polyhedron or a combination of polyhedra. Several kinds of structures are known for clathrate

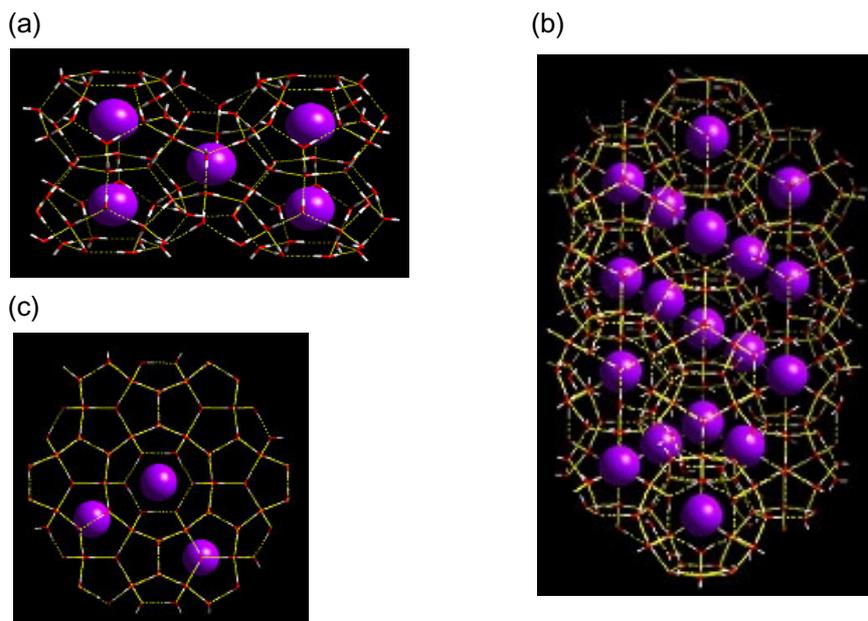


Fig. 9. Clathrate hydrates are formed when water molecules arrange themselves in a cage-like structure around small molecules. (a) Structure I, whose unit cell consists of cage  $5^{12}$  and cage  $5^{12}6^2$ , (b) structure II, (c) structure H. (Provided by Belosludov (Belosludov *et al.*, 2007).)

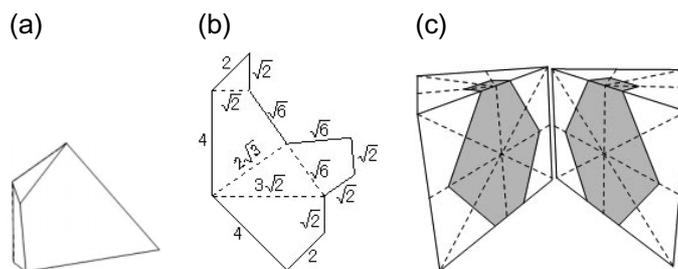


Fig. 10. (a) Pentahedron as an element building Federov's polyhedra, (b) the development of this pentahedron, where edge lengths are shown, (c) construction of a cube with edge length 8 by packing 96 elements of the pentahedron (the gray part indicates the truncated octahedron).

hydrate, as shown in Table 1, among which the last three are found universally and called structures I, II and H, respectively (Fig. 9). The clathrate hydrate corresponding to the Williams' 14-hedron was recently discovered (Manakov, 2002, 2004) and exists in a special physical environment.

It is worth noting that all of the three clathrate hydrates in this table have a high fraction of pentagons. One answer to the question "why the pentagon plays a dominant role in complex polyhedral structure?" is given by the average number of edges in stable packing of included polygons (Eq. (4)). An easy way of convincing ourselves of this answer would be to consider a process to bend a finite plane sheet to a closed surface. Assume that a molecular structure forming a sheet has the graphite structure, which is an arrangement of equal hexagons. In order to form a closed surface from graphite structure we need to exchange some hexagons with simpler polygons, otherwise the Euler's theorem is not satisfied. In order to produce a smooth closed surface, it will be natural to choose pentagons rather than quadrilaterals or triangles. The term "natural" means the less amount of bending energy.

## 5. A New Fact about Parallel Polyhedra

Before describing on a new fact, a recent episode suggesting the meaning of this fact is introduced. In June 2008 an international congress was held in Moscow for celebrating the 100 years anniversary of birth of the Russian mathematician L. S. Pontrjagin. In this congress a Japanese mathematician J. Akiyama made a lecture on the set and the element number of regular polyhedra (Akiyama, 2008), where he talked on the development patterns of the regular tetrahedron (Akiyama, 2007). If one cuts a paper-made tetrahedron by scissors to a planar development of any shape (not necessarily cutting along the edges of the tetrahedron), one can fill the plane precisely with this development and its congruent copies. In this sense the tetrahedron is looked upon as a "planar-tessellation producer". Any tetrahedron made of four congruent triangular faces has this property.

Now, let us consider a problem "what is a space-tessellation producer?", i.e. "what is an elementary body producing space filling bodies?". General solution is quite difficult, but simple examples are found as follows (Akiyama, 2009). One can construct all of the parallel polyhedra with one kind of element as shown in Figs. 10(a) and (b), where its mirror image is looked upon as the same one.

Table 2. List of hypercubes with numbers of edges and their projections onto the 2D, 3D, 4D and  $d$ D spaces.

Dimension $d$	$m$ sets of $k$	Parallel edges	Projected bodies (dimension of these bodies)
2	$m = 2$	$k = 2$	square (2D)
3	3	4	cube (3D), hexagon (2D)
4	4	8	rhombic dodecahedron (3D), hexagonal prism (3D)
5	5	16	elongated rhombic dodecahedron (3D)
6	6	32	truncated octahedron (3D)
10	10	$2^9$	primitive 30-tope (4D)
$n(n+1)/2$	$n(n+1)/2$	$2^{(n+2)(n-1)/2}$	primitive $2(2^n - 1)$ -tope ( $d$ D)

Let this element be denoted by  $\sigma$ , then the cube is constructed by 96 elements of one kind ( $\sigma_{96}$ ), which includes a truncated octahedron as an inner structure with half volume of the whole cubic (see Fig. 10(c)). In the same way, the hexagonal prism is constructed by  $\sigma_{144}$ , the rhombic dodecahedron by  $\sigma_{192}$ , the elongated rhombic dodecahedron by  $\sigma_{384}$  and the truncated octahedron by  $\sigma_{48}$ .

## 6. Some Notes on Division in Hyperspace

Since the hyperspace with dimension larger than 3 is not familiar to most of us, the space division in the hyperspace is a difficult problem. However, some notes on polytopes (polyhedra in hyperspace) are helpful in understanding the space division in the 3-dimensional (abbreviated to 3D) space. In the following such notions are given briefly without proof.

It will be interesting to note that the four members of Fedorov's parallel polyhedra are projections of the hypercubes in 4D, 5D and 6D spaces onto the 3D space. In order to understand this fact a list of hypercubes up to 6D is useful, as shown in Table 2.

As an example the case of 4D cube is illustrated in Fig. 11. The existence of 4 sets of 8 parallel edges is understood by observing that the 8 vertices of the 3D cube move to the fourth direction, and also the numbers of parallel edges in 3D cube are doubled. Projection of this cube onto the 3D space corresponds to making an envelope of the 4D cube, as shown in Fig. 11(b). Cases with other dimensions are understood in similar ways. In particular, the case of the truncated octahedron (Kelvin's 14-hedron) is understood by the fact that the 6D cube has 6 sets of 32 parallel edges, which are reduced to 6 through projection. Note that the contraction processes shown in Fig. 3 corresponds to the contraction shown in Table 2.

As for the parallel polyhedra with dimensions higher than 3, we confine ourselves into mentioning some facts. While Fedorov's parallel polyhedra in 3D space have 5 types, the family of parallel polytopes in 4D space consists of 52 types, whose cell number (corresponding to the number of faces for 3D polyhedra) does not exceed 30. The primitive 30-tope has 10 sets of parallel edges and is a projection of the 10D cube onto 4D space (Delaunay, 1929; Stogrin, 1973), as shown in the 2nd lowest line in Table 2 ( $d = 4$ ).

In the plane division (2D division) at least 3 polygons must gather at each vertex, and for stable division the number (say, contact number) is limited to 3. Similarly in the space division (3D division) at least 4 polyhedra must

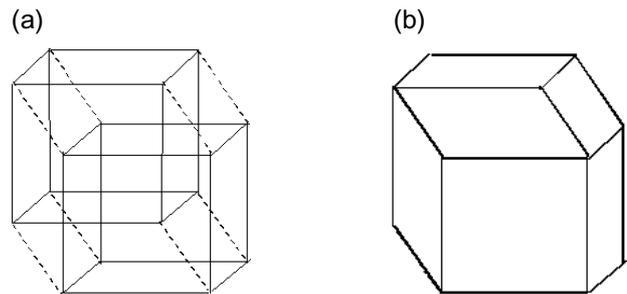


Fig. 11. (a) 4D cube produced by shifting a 3D cube to the fourth direction. The 32 edges of 4D cube are divided to 4 groups with four directions, and each group has 8 members. (b) Projection of 4D cube onto the 3D space, which is an envelope of the 4D cube and equivalent to the rhombic dodecahedron.

gather at each vertex, and for stable division the contact number is limited to 4. In the  $d$ -dimensional space ( $d$ D) a similar theorem exists, i.e. in the  $d$ D space division at least  $d + 1$  polytopes must gather at each vertex (Lebesgue's tessellation theorem). For stable division, the contact number is limited to  $d + 1$ , and then the cell number reaches up to maximum  $2(2^n - 1)$  among the parallel polytopes of  $d$ D space (Minkowski's tessellation theorem; see Appendix).

The hexagon allows a stable division of 2D space, and is produced by truncating three vertices of triangle. The truncated octahedron, allowing a stable space division, is produced also by truncating four vertices and chamfering six edges of a tetrahedron. These facts are generalized as follows. The body allowing a stable division of  $d$ D space can be obtained by truncating and chamfering  $d$ D simplex, and is a  $2(2^n - 1)$ -tope. For example, in 4D space the body for stable space division is a 30-tope. This process of producing truncated body is called Hinton's model (Hinton, 1978). Hinton lived the same time as that of Kelvin, and wrote a manuscript of the monograph "What is the fourth dimension?" in 1880 at his age of 27, which was published much later in 1978.

The same result by a different process is proposed by Conway (1997), and is called Conway model. The hexagon, allowing a stable plane division, is looked upon as a combination of two trapezoids, which are produced by truncating one vertex of a triangle. In the similar way in the 3D case one vertex and three edges of a tetrahedron are truncated (the bottom face is untouched), thus seven faces appear except the bottom face. By combining two of this body (truncated hexagonal pyramid) and matching the two bot-

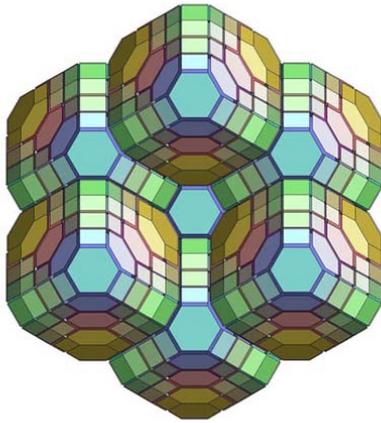


Fig. A.1. Projection of the 126-tope onto 3D space, provided by Ishii (2008, private communication).

tom faces each other, we obtain the truncated octahedron after some adjustment. This process is generalized to  $dD$  case and leads to the  $2(2^n - 1)$ -tope for stable division.

## 7. Concluding Remarks

The problem of space division has been a center of interest of researches in various fields, from basic sciences (mathematics, physics, crystallography, chemistry, etc.) to applied sciences (pathology, geology, etc.). Key concepts and methods of analysis concerned to the space division are common among these fields. Therefore, interdisciplinary activity is essential for future development of this topic.

Motivations of researches of this kind of problems are often a simple curiosity and a sense of play. In addition, a strong motive force is a fact that many of shapes observed in the nature and created in researches are beautiful. It will be proper to claim that we should enjoy studying geometrical shapes.

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## Appendix A.

The Minkowski's tessellation theorem mentioned in Sec. 6 is composed of the following two theorems:

**THEOREM A.** If  $P$  is a  $d$ -dimensional parallel polyhedron, then

- 1)  $P$  is centrally symmetric,
- 2) all faces of  $P$  are centrally symmetric,
- 3) the projection of  $P$  along any of its  $(d - 2)$ -faces onto the complementary 2D plane is either a parallelogram or a centrally symmetric hexagon.

**THEOREM B.** The number of faces  $f_{d-1}$  in a  $dD$  parallel polyhedron  $P$  does not exceed  $2(2^d - 1)$  and there is a parallel polyhedron  $P$  with  $f_{d-1} = 2(2^d - 1)$ .

As an example the  $2(2^6 - 1)$ -tope, i.e. the 126-tope, which makes a stable division in the 6D space, is shown in Fig. A.1. This figure is a projection onto 3D space, and shows how the group of this polytope fills the space.

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