Systematic Study of Convex Pentagonal Tilings, II: Tilings by Convex Pentagons with Four Equal-length Edges

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We derived 14 types of tiling cases under a restricted condition in our previous report, which studied plane tilings with congruent convex pentagons. That condition is referred to as the category of the simplest set of node (vertex of edge-to-edge tiling) conditions when the tile is a convex pentagon with four equal-length edges. This paper shows the detailed properties of convex pentagonal tiles with four equal-length edges and tiling patterns. Furthermore, we present the relationship between the idiomatic expression in various overviews and our results.
Key words: Convex Pentagon, Tiling, Tile, Node, Pattern

1. Introduction

In the typical review of the convex pentagonal tiling problem, convex pentagonal tiles[*1] are numbered and classified almost in the order that corresponds to the publications (Schattschneider, 1978; Grünbaum and Shephard, 1987; Sugimoto and Ogawa, 2005, 2006b). This classification shows few systematic properties because it consists of not only the edge-to-edge[*2] tiling (EE tiling), but also non-edge-to-edge tiling (NEE tiling). However, they shift from common patterns with a higher degree of freedom to patterns with many restrictions and little remaining freedom. Still, from a consistent viewpoint this classification has no logic.

In our previous report (Sugimoto and Ogawa, 2005; hereafter, we refer it as Report I), we discussed the existence of pentagonal tiles with four equal-length edges by applying "the simplest node condition." Note that, in our study, a point that is the common vertex of k polygons (tiles) in an edge-to-edge tiling is called a *node* of valence k (Sugimoto and Ogawa, 2005, 2006b). Then, if the number and kind of nodes in EE tiling are restricted to one kind of 4-valent node and two kinds of 3-valent nodes (including the case when the two kinds are identical), we call that the condition is *the simplest node condition*. In Report I, we detailed the logic to screen pentagons by restricting candidates, and explained and exemplified the complicated process throughout. As a result, we derived 14 kinds of concentration methods that allow the EE tiling in accordance with the node restriction (the relations of angles among three nodes which are given from the simplest node condition). However, we could not describe the actual tiling patterns[*3], etc. Therefore, there are two primary purposes of this report: finding the relationship between our results and the convex pentagonal tiles (or tiling patterns) in past studies and summarizing how much we have achieved in deriving previously discovered tiles in a consistent manner. Throughout the report, we will consider only EE tiling. Hereafter, unless noted otherwise, an EE tiling is written simply as "tiling."

Section 2 explains the terms used in this report. Section 3 introduces tilings under the node restriction (the relations of angles among three nodes which are given from the simplest node condition). In Sec. 4, we stop applying the node restriction to tiling and introduce possible tilings that are formed of only 3- and 4-valent nodes with convex pentagonal tiles with four equal-length edges of Sec. 3. In Sec. 5, as to the convex pentagons eliminated by the topological judgment but confirmed by the geometric judgment in Report I, we analyze whether tiling is possible when the node restriction is not applied and discuss the properties of each pentagon. In Sec. 6, we classify and describe the equilateral convex pentagonal tiles in relationship to the convex pentagonal tiles with four equal-length edges that are mentioned in Secs. 3 and 5. Finally, Sec. 7 compares and summarizes the convex pentagonal tiles mentioned in both past reports and our study.

2. Preparation: Terms, etc.

The vertices and edges of pentagons are referred to by the nomenclature described in Fig. 1(b). In addition, as to pentagons with four equal-length edges, these four edges are defined as a, b, c, and d, and the edge e between the vertices D and E of the pentagon is the sole edge of

[*1]A single congruent polygon that tiles the Euclidean plane is called a *polygonal tile*.

[*2]The tiling by convex polygon is called *edge-to-edge* if any two convex polygons either do intersect or share only one vertex or only one edge.

[*3]In this study, the tiled arrangement of polygons on a plane is referred to as a *tiling pattern*.

different length. Therefore, for the pentagonal tilings, the edges DE of adjacent polygons may be joined in a regular (mirror-reflected) or reversed (point-symmetric) pattern. Note that in this study, the mirror-reflected case is called a DE-*regular* pattern, and the point-symmetric case is called a DE-*reversed* pattern. We can consider that the pentagonal pairs that are of DE-regular or DE-reversed patterns are the units of tiling (Sugimoto and Ogawa, 2005).

The most important objective of this report is to collect all convex pentagonal tiles. For these convex pentagonal tiles, there are some possible tilings other than those with simple periodic structures under the node restriction. Therefore, another purpose of this study is to mention these properties. The terms used in this study are then explained in order to depict the tiling properties described below.

Tiling is inseparably connected to crystallography because both studies use periodic structures. However, the premises behind the idea of the unit cell of the periodic structure are different between the case of crystallography, which is concerned with point location, and the case relevant to tiling. We need considerations for tiling other than the point location. In this study, the criterion of the pentagonal tile is an actual perception that several pentagons certainly form a periodic structure. Our central task is not to classify the tiling by crystalline structures (group of plain surfaces). Therefore, we call the tiling region that can form a periodic structure only with translation of a "*fundamental region*" instead of the term "unit cell." In this study, we adopt the fundamental region consisting of several pairs of regular or reversed patterns mentioned above for all convex pentagons with four equal-length edges. Also in this study, the fundamental region that can be made of the minimum number of tile pieces is particularly called *the smallest fundamental region* (Sugimoto and Ogawa, 2004). The smallest fundamental region is required for considering tiling properties described hereinafter.

As mentioned above, tilings other than periodic structures made only with translations of the smallest fundamental region, are possible for some of our tilings. For example, the tiles in Fig. 1(b), C22-T1E (DE-regular 7), form the smallest fundamental region (the pale gray region) with four pieces and can form a periodic tiling by it. However, the C22-T1E tiles can mix the smallest fundamental region (the dark gray region) rotated by 180° in one tiling without breaking the node restriction as shown in the illustration. From this property, the C22-T1E tile can make the periodic tiling of only the smallest fundamental region, and the many periodic tilings by freely combining the smallest fundamental regions with those rotated by 180° (e.g., periodic tiling of the fundamental region consisting of eight pentagons made by combining the single smallest fundamental region and the single smallest region rotated by 180°). In addition, this combination allows quasi-periodic and random arrangements in one-dimension. We name this kind of tiling *multipatterned tiling* because it allows several kinds of tiling (Sugimoto and Ogawa, 2004). As shown in Fig. 4(b), the fundamental region formed by pentagonal tiles that cannot allow multipatterned tiling is always the smallest fundamental region.

3. Tiling Using Convex Pentagonal Tiles with Four Equal-length Edges in Accordance with the Node Restriction

In this section, as to the 14 kinds of concentration methods deduced in Report I, we clarify the relationship between the classification corresponding to our working process and the idiomatic expressions used in conventional reports. Additionally, the properties of each tiling are explained.

3.1 Classification and properties of convex pentagonal tiles

The 14 kinds of concentration methods are initially classified from the two viewpoints of "C" and "T". "C" indicates the classification focusing on the combinations of five vertices of tiles appearing in three relational expressions (expressions that include only angles) of the node restriction. "T" indicates the classification focusing on the idiomatic expressions of the conditions of angles and edges satisfied by convex pentagons in order to allow tiling by considering the relative relationship of the five vertices. Here, cautions for the idiomatic expressions are described. As discussed above, the 14 kinds of convex pentagonal tiles on a plane mentioned in the reviews include EE and NEE tiling. Some cases that illustrate NEE tiling (i.e., type 1 and type 2 in the conventional expressions) include special cases that allow EE tiling. The tilings in our results that fall under this classification belong to special cases. They are EE tiling.

First, we focus on the combinations of vertices used in one kind of 4-valent node and two kinds of 3-valent nodes. The format of node restriction can be classified into the four kinds as shown in Table 1. The three equations described in the format of the node restriction express the pattern of concentration of one kind of 4-valent node and two kinds of 3-valent nodes, respectively, in terms of tiling. Five vertices (angles) A, B, C, D, E of a pentagon correspond to X_i ($i = 1, \dots, 5$), respectively. For example, the node restriction of the DE-regular 7 in Table 3 of Report I is one kind of 4-valent node $2D + 2C = 360°$ and two kinds of 3-valent nodes $2B + A = 2E + A = 360°$. Assuming $X_1 = D$, $X_2 = C$, $X_3 = B$, $X_4 = E$, $X_5 = A$, the node restriction equation of DE-regular 7 can be expressed as $2X_1 + 2X_2 = 2X_3 + X_5 = 2X_4 + X_5 = 360°$ by using X_i ($i = 1, \dots, 5$). Based on the simplest node condition, each X_i is contained in twice in three equations expressing the node restriction. The four kinds of format of node restriction are called C22, C20, C12, and C11. The numbers following C are corresponding to the category in Table 1 of Report I. That is, the first digit is the number of degeneracy (the number of vertices of the same kind appearing in one node) in the 4-valent node. The second digit indicates the number of degeneracy in the 3-valent nodes. C20 is the case in which two kinds of 3-valent nodes are reduced to one. Therefore, the classification that uses C for 14 kinds of concentration methods corresponds to the figures shown in the classification column in Tables 3 and 4 in Report I. This classification has been completed.

We then classified the tiles in 14 kinds of concentration methods by considering the position of vertices participating in the 3-valent node. At first, tiles are classified into five types of convex pentagonal tiles called type 1, type 2,

Table 1. Classification focusing on the combination of vertices that constitute nodes.

Class symbol	**C22**	**C20**	**C12**	**C11**
Format of node restriction	$2X_1 + 2X_2 = 360^{\circ}$	$2X_1 + 2X_2 = 360^{\circ}$	$2X_1 + X_4 + X_5 = 360^{\circ}$	$2X_1 + X_3 + X_4 = 360^{\circ}$
	$2X_3 + X_5 = 360^{\circ}$	$X_3 + X_4 + X_5 = 360^{\circ}$	$2X_2 + X_4 = 360^{\circ}$	$2X_2 + X_5 = 360^{\circ}$
	$2X_4 + X_5 = 360^{\circ}$	$X_3 + X_4 + X_5 = 360^{\circ}$	$2X_3 + X_5 = 360^{\circ}$	$X_3 + X_4 + X_5 = 360^{\circ}$

Table 2. Conditions of tiles and properties of tilings.

Class symbol		Classification of Report I		4-valent node	3-valent nodes		Conditions of tile other than $a = b = c = d$	DOFs	Symmetry property of tile	Equilatral	Multipatterned	Smallest fundamental region		Figure number
Concentration group	Tile group	Connecting method	Case									Pieces	Symmetry group	
C22	**T1A**	*DE*-reversed	2	*DDEE*	*CCB*	*AAB*	$A + B + C = 360^{\circ}, A = C^{\#1}$ $(a = d, b = c^{\#2})$	1	Y	Y	Y[#9]	4	cmm	1 (a)[#10]
	T1E	*DE*-regular	7	*DDCC*	*BBA*	*EEA*	$E + A + B = 360^{\circ}, B = E$ $(a = b^{\#2})$	1	N	Y	Y	4	p2	1 (b)[#10]
	T2&4	*DE*-regular	10	*CCAA*	*DDB*	*EEB*	$B + D + E = 360^{\circ}, D = E^{\#3}$	1	Y	Y	N	4	p4g	1 (c)
C20	**T1A**	*DE*-regular	11	*DDEE*	*ABC*	*ABC*	$A + B + C = 360^{\circ}$ $(a = d^{\#2, \#4, \#5})$	2	N	Y	Y	2	cm	2 (a)[#10]
		DE-reversed	3	*DDEE*	*ABC*	*ABC*	$A + B + C = 360^{\circ}$ $(a = d^{\#2, \#4})$	2	N	Y	Y	2	p2	2 (b)[#10]
	T1C	*DE*-reversed	4	*AABB*	*CDE*	*CDE*	$C + D + E = 360^{\circ}$ $(a = c^{\#2})$	2	N	Y	Y	2	p2	3 (a)[#10]
	T2	*DE*-reversed	5	*CCAA*	*BDE*	*BDE*	$B + D + E = 360^{\circ}$ $(a = c, b = d^{\#2, \#6})$	2	N	Y	Y	4	p2	3 (b)[#10]
C12	**T1C**	*DE*-regular	14	*DDBC*	*EEB*	*AAC*	$C + D + E = 360^{\circ}, C = 2B^{\#7}$	1	N	Y	Y	4	p1	4 (a)
	T7	*DE*-regular	16	*DDCA*	*EEC*	*BBA*	$2E + C = 2B + A = 360^{\circ}$	1	N	Y	N	8	pgg	4 (b)
	T8	*DE*-regular	12	*DDAB*	*EEA*	*CCB*	$2C + B = 2E + A = 360^{\circ}$	1	N	Y[#8]	N	8	pgg	4 (c)
	T9	*DE*-regular	20	*CCAB*	*DDB*	*EEA*	$2D + B = 2E + A = 360^{\circ}$	1	N	N	N	8	pgg	4 (d)
C11	**T1A**	*DE*-regular	21	*DDAB*	*EEC*	*ABC*	$A + B + C = 360^{\circ}, C = 2D$	1	N	Y	N	4	p1	5 (a)
	T1C	*DE*-reversed	6	*AADE*	*BBC*	*CDE*	$C + D + E = 360^{\circ}, C = 2A$ $(a = c = d^{\#2})$	1	N	N	Y	4	pmg	5 (b)[#10]
	T2	*DE*-reversed	7	*AADE*	*CCB*	*BDE*	$B + D + E = 360^{\circ}, B = 2A$	1	N	N	N	4	p2	5 (c)

#1: $D = E = 90^{\circ}$ when $a = d, b = c$ is considered. #2: The condition for edges $a = b = c = d$ is not essential for tiling. #3: $A = C = 90^{\circ}$ when $a = b = c = d$ is considered. #4: Adding the relationship of $b = c$ to make tiles satisfy $a = d, b = c$ allows periodic tilings with contacting methods between the edges AB and BC that are different from a single condition of $a = d$ and multipatterned tilings. #5: The smallest fundamental region is formed of four pieces of tile if and only if the edge condition is only $a = d$. #6: Tiles satisfying $a = b = c = d$ allow multipatterned tilings. #7: The angular conditional equations are obtained from $a = b = c = d$ and the node restriction. On the contrary, use $a = b = c = d$ and the relationship of $C = 2E - 2D$ obtained from the angular conditional equation for deriving the node restriction from the angular conditional equations. #8: This will be the equilateral convex pentagonal tile belonging to type 8 and type 2 because of the angular relationship if and only if it is an equilateral pentagon. #9: It is considered to be the same pattern if there is no vertex sign because it is a pentagon that is symmetrical to the axis passing through the midpoint of vertex B and edge DE. #10: Convex pentagons satisfying the condition of edge $a = b = c = d$ are used in the figure of the report.

type 7, type 8, and type 9 in accordance with the idiomatic expressions. They are indicated by the signs T1, T2, T7, T8, and T9, respectively. Furthermore, T1 is classified into three categories by the positional relationship between the edges and angles, and T2 is classified into both the special and normal cases. T1 is a convex pentagonal tile (with parallel edges) where the three adjacent internal angles of a pentagon sum to 360°. In the 14 kinds of concentration methods, there are three kinds where the three adjacent internal angles sum to 360° (i.e., $A + B + C = 360^{\circ}$, $C + D + E = 360^{\circ}$, and $E + A + B = 360^{\circ}$). These three kinds were expressly indicated as T1A, T1C, and T1E because we perceive vertices D and E of a convex pentagon with four equal-length edges to be special. Tiles of DE-regular 10 classified as T2 require the condition of $a = b = c = d$ in accordance with the node restriction. Therefore, $A = C = 90^{\circ}$ is derived forming a tiling pattern with a higher symmetric property. Additionally, to be precise, it is a convex pentagonal tile that belongs to both type 2 and type 4 in the idiomatic expression (see Fig. 1 in Report I). Therefore, the tile of DE-regular 10 is expressed as T2&4. As a result, the tiles in the 14 kinds of concentration methods are classified into eight kinds of T1A, T1C, T1E, T2, T2&4, T7, T8, and T9. Table 2's column of tile group shows all eight varieties.

Tables 2 and 3 express the 14 kinds of concentration methods using new classification signs (nominal designations) and summarize the properties of each convex pentagonal tile and tiling (Sugimoto and Ogawa, 2004). C20-T1A falls under DE-regular 11 and DE-reversed 3 because these two tilings have the same node restriction and same tiles. Their only difference is whether they are DE-regular or DE-reversed. It should be noted that even the convex pentagons categorized into T1A, T1C, T1E, and T2 (because the sum of their internal angles is 360°) have different structures depending on each additional condition. As a result, there are 13 kinds of the convex pentagon tiles with four equal-length edges in this section.

Table 3. Relationship amog internal angles A, B, C, D, and E, and length of edge e as to pentagons in Table 2.

Class symbol	Internal angles of pentagon A	B	C	D	E	Additional informations about internal angles	Edge length ℓ and range (with $a=b=c=d=1$)
C22 -T1A	$90^\circ+\gamma$	$180^\circ-2\gamma$	$90^\circ+\gamma$	90°	90°	$0^\circ<\gamma<90^\circ$	$0<2\cos\gamma<2$
C22 -T1E	$180^\circ-2\alpha$	$90^\circ+\alpha$	$180^\circ-2\beta$	2β	$90^\circ+\alpha^{\#5}$	$\alpha^{\#6}=\cos^{-1}(\cos\beta\sin\beta)$, $0^\circ<\beta<90^\circ$	$0<2\cos^2\beta<2$
C22 -T2&4	90°	$90^\circ+\theta$	90°	$135^\circ-\theta/2$	$135^\circ-\theta/2$	$0^\circ<\theta<90^\circ$	$0<2\sqrt{2}\sin(\theta/2)<2$
C20 -T1A[#1]	$\gamma+D$	$180^\circ-2\gamma$	$180^\circ+\gamma-D$	D	$180^\circ-D$	$0^\circ<\gamma<90^\circ$, $\gamma<D<180^\circ-\gamma$	$0<2\cos\gamma<2$
C20 -T1C	$180^\circ-2\alpha$	$2\alpha^{\#2}$	$360^\circ-2\alpha-2D$	D	$D+2\alpha$	$90^\circ-D<\alpha<90^\circ-D/2$, $0^\circ<D<90^\circ$	$0<2\cos D<2$
C20 -T2	$180^\circ-2\alpha$	$90^\circ+\theta$	2α	$90^\circ-\alpha+\delta$	$180^\circ+\alpha-\theta-\delta$	$\delta=\tan^{-1}\left(\dfrac{\sin\theta}{\tan\alpha-\cos\theta}\right)$, $0^\circ<\alpha<90^\circ$, $0^\circ<\theta<90^{\circ\#7}$	$0<2\sqrt{1-\sin(2\alpha)\cos\theta}<2$
C12 -T1C	$180^\circ-2\alpha$	$2\alpha^{\#2}$	4α	$180^\circ-3\alpha$	$180^\circ-\alpha$	$30^\circ<\alpha<45^\circ$	$0<2\sin(3\alpha-90^\circ)<\sqrt{2}$
C12 -T7	$180^\circ-2\alpha$	$90^\circ+\alpha$	$180^\circ-2\beta$	$\alpha+\beta$	$90^\circ+\beta$	$\beta=\tan^{-1}\left(\dfrac{\tan\alpha-\cos\alpha}{\sin\alpha}\right)$, $38.1724^{\circ\#8}<\alpha<90^\circ$	$0<\dfrac{2\cos^2\alpha}{\sqrt{1-2\sin\alpha\cos^2\alpha}}<\dfrac{2\sqrt{2}}{\sqrt{-1+\sqrt{5}}}$
C12 -T8	$180^\circ-2\alpha$	4β	$180^\circ-2\beta$	$90^\circ+\alpha-2\beta$	$90^\circ+\alpha^{\#5}$	$\alpha=\tan^{-1}\left(\dfrac{\sin(2\beta)}{\cos(2\beta)+2}\right)+\beta$, $0^\circ<\beta<45^\circ$	$0<\dfrac{8\sin\beta\cos^2\beta}{\sqrt{1+8\cos^2\beta}}<2\sqrt{\dfrac{2}{5}}$
C12 -T9	$180^\circ-2\alpha$	$2\alpha+4\beta-180^\circ$	$180^\circ-2\beta$	$270^\circ-\alpha-2\beta$	$90^\circ+\alpha^{\#5}$	$\alpha=\tan^{-1}\left(\dfrac{1+\cos\beta\cos(3\beta)}{\cos\beta\sin(3\beta)}\right)$, $45^\circ<\beta<54.7356^{\circ\#9}$	$0<\dfrac{4(1-2\cos^2\beta)\cos^2\beta}{\sqrt{1-5\cos^2\beta+8\cos^4\beta}}<0.9428^{\#10}$
C11 -T1A	$180^\circ-2\alpha$	3α	$180^\circ-\alpha^{\#4}$	$90^\circ-\alpha/2$	$90^\circ+\alpha/2$	$0^\circ<\alpha<60^\circ$	$0<2\sin(3\alpha/2)<2$
C11 - T1C	$180^\circ-2\alpha$	$2\alpha^{\#2,\#3}$	$360^\circ-4\alpha$	α	3α	$45^\circ<\alpha<60^\circ$	$1<2\cos\alpha<\sqrt{2}$
C11 -T2	$180^\circ-2\alpha$	$360^\circ-4\alpha$	2α	$90^\circ-\alpha+\delta$	$5\alpha-\delta-90^\circ$	$\delta^{\#6}=\tan^{-1}\left(\dfrac{-\cos(4\alpha)}{\tan\alpha+\sin(4\alpha)}\right)$, $45^\circ<\alpha<60^\circ$	$1<2\sqrt{1+\sin(2\alpha)\sin(4\alpha)}<2$

#1: Two pentagons of DE-regular 11 and DE-reversed 3 correspond. #2: Quadrangle $ABCE$ is rhombic, and triangle CDE is an isosceles triangle with vertex angle $\angle ECD$. #3: Triangles ABE, CDB, and CDE are isosceles triangles with vertex angles of $\angle EAB$, $\angle BCE$, and $\angle ECD$, respectively. The triangles are congruent. #4: Divide a pentagon into parallelogram $ACDE$ and isosceles triangle ABC with vertex angle $\angle ABC$. $\alpha = 2\beta$ is derived from $A = \angle EAC + \angle CAB = 90^\circ - \beta + 90^\circ - 3\beta = 180^\circ - 4\beta$. #5: The triangle BDE is a right triangle because $\angle BED = 90^\circ$. #6: The value is not monotonous to the variable. #7: α and θ should satisfy the relationship of $\tan\alpha > \cos\theta$. #8: The accurate value of the lower limit is $\tan^{-1}\left(\sqrt{\left(-1+\sqrt{5}\right)/2}\right)$. #9: The accurate value of the upper limit is $\tan^{-1}\left(\sqrt{2}\right)$. #10: The accurate value of the upper limit is $\left(2/\sqrt{3}\right)\sin\left(\pi+\tan^{-1}\left(\left(\sqrt{3}+\cos\phi\right)/\sin\phi\right)+\phi\right)$, where $\phi = 3\tan^{-1}\left(\sqrt{2}\right)$.

Table 2's columns of 4-valent and 3-valent nodes show the node restriction of each tiling. For example, 4-valent node $DDEE$ and 3-valent nodes CCB and AAB of C22-T1A indicate that tiling is formed by only three nodes of $2D + 2E = 360^\circ$, $2C + B = 360^\circ$ and $2A + B = 360^\circ$. Table 2's columns on the conditions of tile other than $a = b = c = d$ show the necessary and sufficient (angular) conditions that allow tiling in accordance with the node restriction. However, alleviated conditions of edges are also shown if tiling is possible with the node restriction, instead of the convex pentagons with four equal-length edges (i.e., $a = b = c = d$) that are a focus of this study. That is, we started with the premise that DE edges border each other only in convex pentagons with four equal-length edges. However, in many cases, the equilateral convex pentagons are included, and if #2 is attached, three or four kinds of edge lengths are included. Table 2's column of DOFs shows the degree of freedom for the shape of each convex pentagon with four equal-length edges, assuming these pentagons satisfy the condition $a = b = c = d = 1$. The column of symmetry property of tile in Table 2 indicates "Y" when a pentagon is always symmetrical (e.g., a pentagon that is always symmetrical to the axis passing through the midpoint between vertex B and edge DE) and "N" for other cases. The column of equilateral indicates "Y" if the tiles can be equilateral convex pentagons without breaking the conditions and "N" for other cases. Note that, when C12-T8 tiles are equilateral convex pentagons, the tiles are symmetrical to the line connecting the midpoint of the vertex D and edge AB and become an equilateral convex pentagonal tile belonging to both type 8 and type 2 (Both type 8 and type 2 tiling are possible). Table 2's column of multipatterned indicates "Y" when multipatterned tiling is possible without breaking the node restriction and "N" for other cases. The column of pieces for the smallest fundamental region indicates the number of pieces of convex pentagons with four equal-length edges required for forming the smallest fundamental region. For reference purposes, the column of symmetry group indicates the kind of symmetry group for plane about the periodic tiling with the smallest fundamental region. The rightmost column of the figure number indicates corresponding figures.

In Figs. 1, 2, 3, 4, and 5, tiles of convex pentagons with four equal-length edges ($a = b = c = d$) were used (Sugimoto and Ogawa, 2004). The pale gray pentagons indicate the smallest fundamental region. The shape of the smallest fundamental region is selected based on the pair of DE-regular or DE-reversed. Note that some figures do not have periodic tiling patterns in the range shown. If the study's objective were restricted to the existence of periodic arrangement, that work would be straightforward. However, our interests include random arrangement, probability of quasi-periodic arrangements, and mutual relationships with other classifications that we mentioned earlier in the

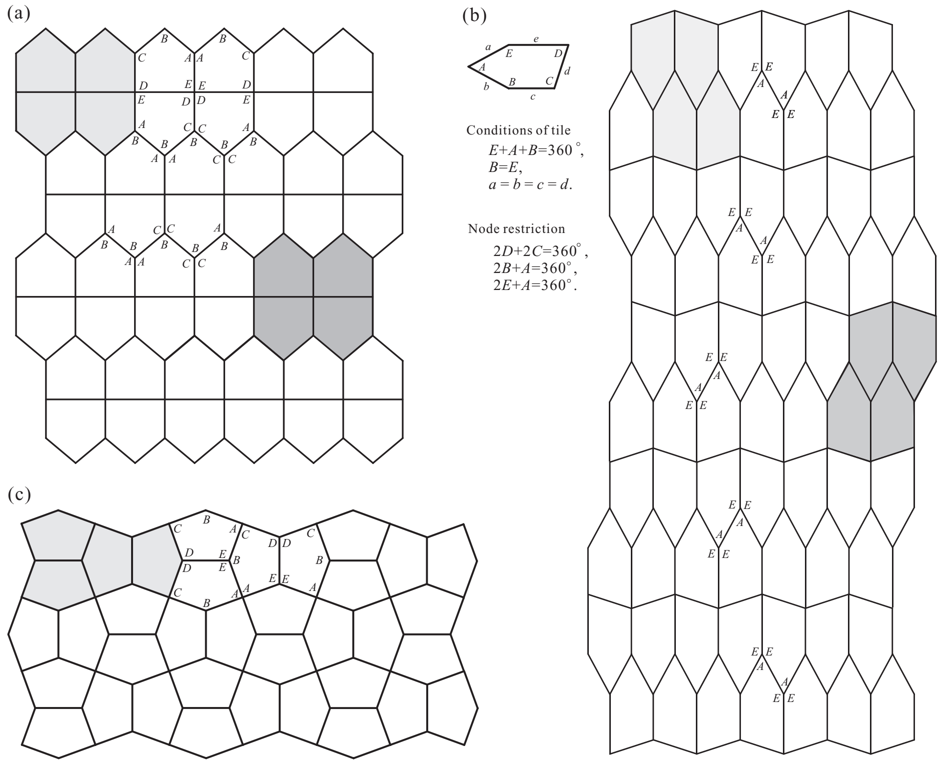


Fig. 1. Tilings of C22. (a) C22-T1A. (b) C22-T1E. (c) C22-T2&4.

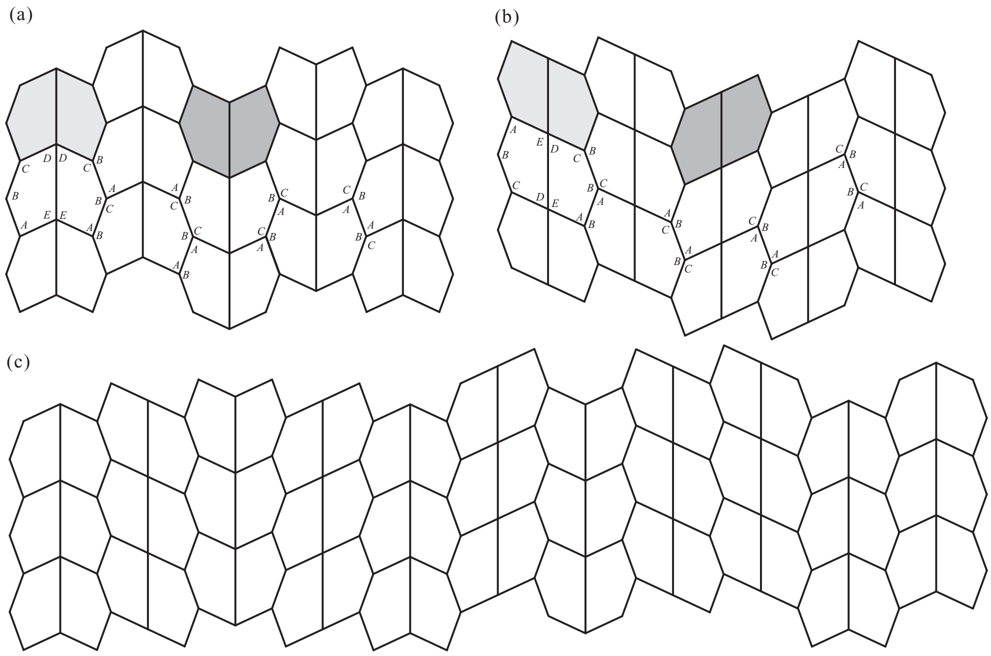


Fig. 2. Tilings of C20. Cases of C20-T1A. (a) DE-reglar pattern. (b) DE-reversed pattern. (c) Combination of DE-reglar pattern and DE-reversed pattern.

study in which the focus is restricted to convex pentagons with four equal-length edges. We would like to discuss these problems to some degree. In the cases that the multipatterned tiling is possible for a pentagonal tile, though it might be inconsiderate for crystallographers, the figure of periodic tiling will be omitted if we had considered a simple periodical arrangement to be apparent. In this report, we inserted these arrangements so that readers can understand that multipatterned tiling (random or quasi-periodic arrangements) would have been possible. Remember that the convex pentagon tiles used in this study always have a periodic arrangement. We expect that the periodic tiling using the smallest fundamental region can be easily imagined from the node restriction and illustrations.

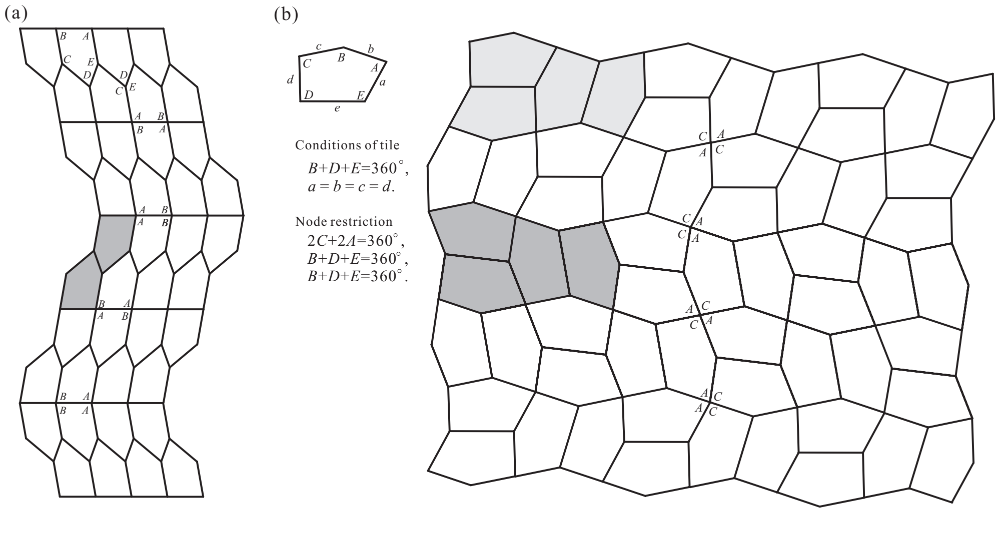


Fig. 3. Tilings of C20. (a) C20-T1C. (b) C20-T2.

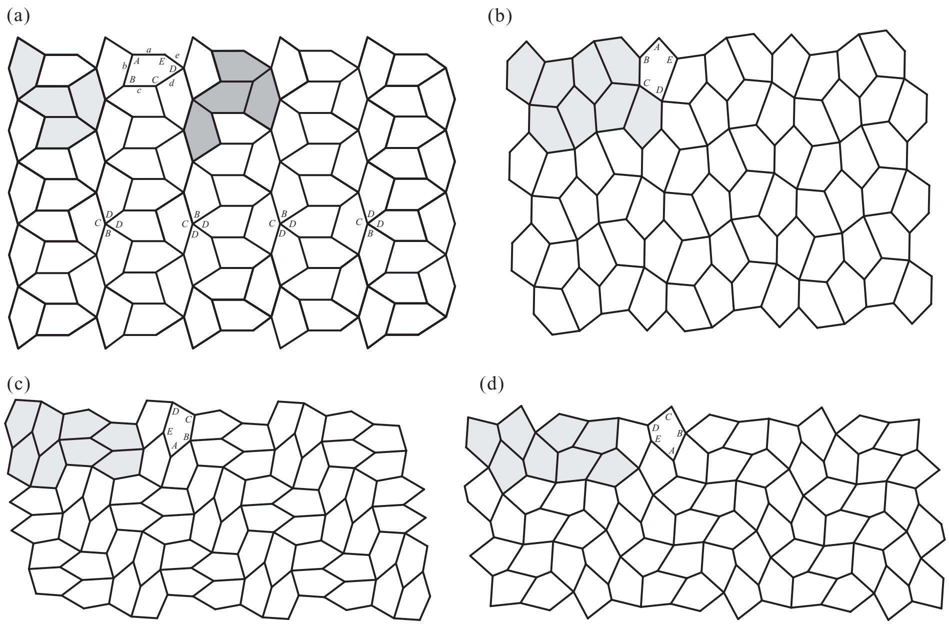


Fig. 4. Tilings of C12. (a) C12-T1C. (b) C12-T7. (c) C12-T8. (d) C12-T9.

Table 3's column concerning the internal angles of tiles and additional information about internal angles indicate the relationship among internal angles (A, B, C, D, E) using α, β, γ, δ, and θ (see Fig. 6 for positions) can be determined by dividing the inside of a pentagon, as in the geometrical discussion in Section 5 of Report I. The column of edge length ℓ and range in Table 3 shows the equation and the range expressing the length ℓ of e (edge DE), which alone has a different length assuming a convex pentagon with four equal-length edges with $a = b = c = d = 1$.

3.2 Discussion

Figure 2(a) is a tiling with C20-T1A tiles in DE-regular. As shown in this figure, in DE-regular, the C20-T1A tiles satisfying $a = d$, $b = c$ form the periodic tiling with the smallest fundamental region (the pale gray region). In addition they allow multipatterned tiling by freely combining a module of the smallest fundamental region and that of the smallest fundamental region rotated by 180° (the dark gray region). If the C20-T1A tiles satisfy only the condition of edge $a = d$, four pentagons (one smallest fundamental region and one smallest fundamental region rotated by 180° in Fig. 2(a)) can form the smallest fundamental region and allow only periodical tiling. Note that the smallest fundamental region differs with the tiles which satisfy only $a = d$, and the tiles which satisfy $a = d$, $b = c$.

Figure 2(b) is tiling with C20-T1A tiles in the DE-reversed pattern. The following allow periodic tiling using the smallest fundamental region (the pale gray region) and multipatterned tiling by using reflective symmetry of the smallest fundamental region (the dark gray region) concurrently: the cases of C20-T1A satisfying $a = d$, $b = c$ in DE-reversed (Fig. 2(b)), C20-T1C (Fig. 3(a)), C20-T2 satisfying $a = b = c = d$ (Fig. 3(b)), and C12-T1C (Fig. 4(a)). C20-T1A tiles satisfying only the edge con-

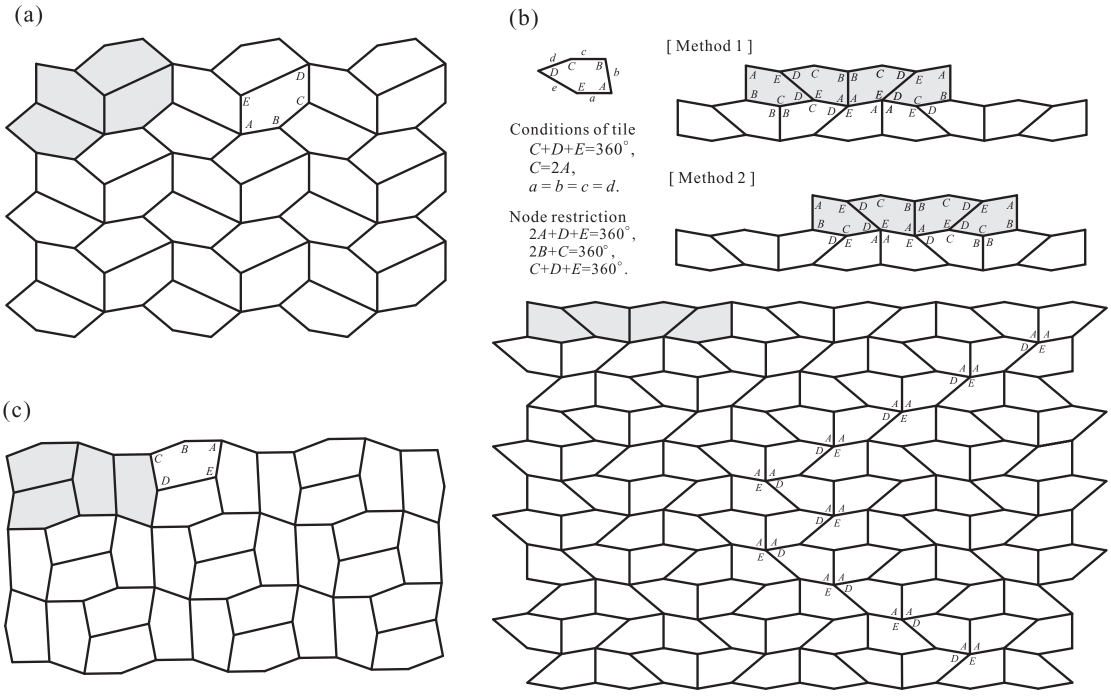


Fig. 5. Tilings of C11. (a) C11-T1A. (b) C11-T1C. (c) C11-T2.

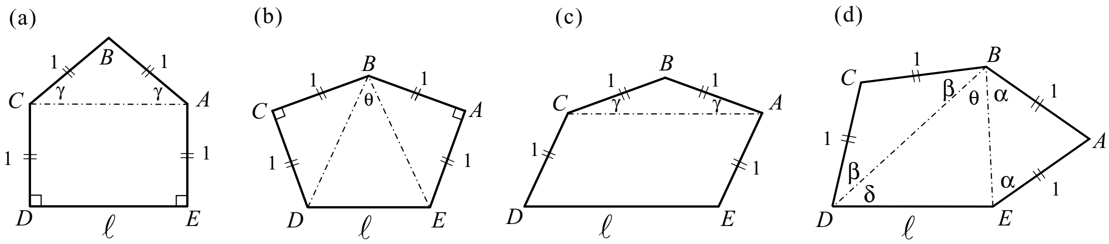


Fig. 6. Relationship between internal angles of tiles. (a) C22-T1A. (b) C22-T2&4. (c) C20-T1A. (d) Other.

dition of $a = d$ and C20-T2 tiles satisfying only $a = c$, $b = d$ both allow only periodic tiling using the smallest fundamental region.

As shown in Fig. 2(c), C20-T1A tiles satisfying $a = d$, $b = c$ allow multipatterned tiling with DE-regular and DE-reversed mixed without breaking the node restriction.

C22-T1A tiles in Fig. 1(a) and C22-T1E tiles in Fig. 1(b) allow periodic tiling using the smallest fundamental region (the pale gray region) as well as multipatterned tiling by using the smallest fundamental region and the smallest fundamental region rotated by 180° or mirrored (the dark gray region). As each figure shows, the smallest fundamental region rotated by 180° has the same shape as the smallest fundamental region mirrored. In the C22-T1A tiling pattern in Fig. 1(a), the pentagon is symmetrical to the axis passing through vertex B and edge DE. Additionally, the smallest fundamental region is symmetrical. Therefore, all patterns can be considered to be the same if the vertex sign is not entered. On the other hand, two C22-T1E tiles bonded on the edge d (CD-reversed) are a hexagon. If we consider that the element of tiling is the hexagonal shape, we can see that tiling is the always same pattern.

C22-T2&4 tiles forming the tiling pattern in Fig. 1(c) cannot allow multipatterned tiling under the node restriction. The pentagon is symmetrical to the axis passing through the midpoint between vertex B and edge DE. Therefore, C22-T2&4 tiles can form the tiling pattern as in Fig. 1(c) without observing the node restriction. (As mentioned above, we can form the tiling with the nodes used in type 4 because C22-T2&4 tiles also belong to type 4.) They all have the same pattern unless tile vertex signs are entered.

As shown in Fig. 5(b), the C11-T1C tile has two methods for combining the smallest fundamental regions through translation. Therefore, C11-T1C tiles allow multipatterned tiling only with the translation operation. However, the C11-T1C tile consists of three congruent isosceles triangles: ABE, CEB, and CDE. If we consider that tiling is formed of the smallest fundamental region composed of four bonded congruent isosceles triangles with different directions, a tiling is periodic with the same pattern at all times.

Figure 4(b) C12-T7, (c) C12-T8, and (d) C12-T9 correspond to the idiomatic expressions of type 7, type 8, and type 9, respectively. The tilings with C12-T1C in Fig. 4(a), C11-T1A in Fig. 5(a), and C11-T2 in Fig 5(c) are also shown by Marjorie Rice (Schattschneider, 1978, 1981;

Grünbaum and Shephard, 1987).

The tilings by C22-T1E in Fig. 1(b), C20-T2 in Fig. 3(b), and C11-T1C in Fig. 5(c) are known for several periodic tilings. However, we have not found descriptions that they allow multipatterned tiling using the convex pentagons of this study. Therefore, they are new pentagon tiling patterns. On the other hand, we found the descriptions in Schattschneider (1978) that C20-T1A, C20-T1C, and C12-T1C tiles allow multipatterned tiling in the same manner as our study. However, we concluded on our own that C20-T1A, C20-T1C, and C12-T1C tiles allow multipatterned tiling. The quasi-periodic or random tiling patterns made by multipatterned tiling in this report are one-dimensional.

4. Tilings Using 13 Kinds of Convex Pentagons with Four Equal-length Edges without Applying Node Restriction

4.1 Conditions of tile

Before getting into the theme of this section, we should have an exact understanding of the conditions of tile. We should first confirm the conditions of tile mentioned in the idiomatic expressions. For example, the convex pentagon tile belonging to type 1 in the idiomatic expression is expressed by a condition where the adjacent internal angles in a convex pentagon total 360° (e.g., $A + B + C = 360^\circ$). Therefore, all convex pentagons satisfying that condition are categorized into the convex pentagon tile of type 1. That is, that the adjacent internal angles in a convex pentagon sum to 360° is a necessary and sufficient condition for convex pentagonal tiles belonging to type 1 in order to allow tiling. Note that, in the case of type 1, the tile requires a condition of edge in addition to the condition of angle mentioned above to be an EE tiling. For example, if a convex pentagon's adjacent internal angles total 360°, and expressed as $A+B+C = 360^\circ$, the tile requires the condition of edge to be $a = d$ in order to be EE tiling. On the other hand, the conditions of each convex pentagon with four equal-length edges (that are mentioned in Table 2) are necessary and sufficient conditions for allowing tiling under the node restriction. These conditions are shown in Sec. 3. The conditions of convex pentagon with four equal-length edges (that are mentioned in Table 2) sometimes completely coincide with the conditions of tile described in the idiomatic expressions. However, in most cases, these conditions differ from one another. In this section, we focus on the difference in the meanings of these conditions. We also discuss what tilings can be formed of only 3- and 4-valent nodes when the 13 kinds of convex pentagon with four equal-length edges do not apply the node restrictions. We then summarize the results.

4.2 Tilings allowed by 3- and 4-valent nodes

More specifically, we discuss what kinds of tilings are possible with C22-T1A convex pentagons with four equal-length edges that are shown in Table 2. As mentioned in Sec. 3, C22-T1A tiles allow the tilings using one 4-valent node, $DDEE$ ($2D + 2E = 360^\circ$), and two 3-valent nodes, CCB ($2C + B = 360^\circ$) and AAB ($2A + B = 360^\circ$). However, as mentioned above, the angular condition for C22-T1A tiles belonging to type 1 is $A + B + C = 360^\circ$ for allowing tiling. ($A = C$, which was included in the original condition, is not required.) Meanwhile, a convex pentagon with four equal-length edges with an angular condition of $A + B + C = 360^\circ$ falls under the C20-T1A tile mentioned in Sec. 3. This C20-T1A tiles allow tilings using 4-valent node $DDEE$ and 3-valent node ABC ($A+B+C = 360^\circ$). Therefore, C22-T1A tiles allow tiling using the nodes "$DDEE$, CCB, AAB" as well as the nodes "$DDEE$, ABC." In contrast, C20-T1A tile allows tiling using the nodes "$DDEE$, CCB, and AAB" if and only if it satisfies the relationship of $A = C$.

Table 4 summarizes the results of the discussion mentioned above concerning the 13 kinds of convex pentagons with four equal-length edges from Table 2 (Sugimoto and Ogawa, 2006a). Note that C12-T8 and C12-T9 tiles in Table 2 have no bearing on this section because they do not allow tiling in a node restriction other than that which is applied.

The column of combinations of nodes usable in the tiling in Table 4 shows all combinations of 3- and 4-valent nodes usable in the tiling of each number. Tilings with the numbers of 1, 3, 5, 6, 7, 9, 10, and 12 satisfy the simplest node condition. Other tilings with the numbers of 2, 4, 8, 11, and 13 do not satisfy the simplest node condition if all nodes in the table are used. However, even in the case that the simplest node condition is not satisfied, the relative ratio of the total number of 3- and 4-valent nodes in the pattern $V_3 : V_4 \approx 2 : 1$ (V_k: total number of nodes of valence k) is true if a tiling pattern is maximum and finite. Table 4's column of connecting method shows how tiling is performed, namely either by using DE-regular or by using DE-reversed. Both methods are entered into the numbers 1, 2, and 11 because both DE-regular and DE-reversed can be used for tiling. In the case of number 1, either DE-regular, DE-reversed, or both can be used for tiling using the nodes "$DDEE$, ABC" in the table. On the other hand, in the case of numbers 2 and 11, both DE-regular and DE-reversed should be used for all five kinds of 4-valent nodes and three kinds of 3-vaent nodes in one tiling. However, if used nodes are selected and restricted, tilings of either connection method are allowed. Note that the number of combinations of usable nodes is larger for numbers 2 and 11 than other numbers because those pentagons are symmetrical to the axis passing through the midpoint of the vertex B and edge DE (see Figs. 7(a) and 8(a)). Therefore, in the case of numbers 2 and 11, all tiling patterns can be considered to be the same unless there are vertex signs. Table 4's multipatterned column indicates "Y" for the case in which multipatterned tiling is possible when using the combination of nodes entered in the corresponding column and "N" for other cases. In the column of figure of corresponding tiling pattern, related figure numbers were showed. The column of conditions of tile (case to realize the combination of nodes in tiling) other than $a = b = c = d$ shows the necessary and sufficient conditions that convex pentagon tiles should satisfy using the combinations of usable nodes. Similarly in Table 2, even if the convex pentagon does not have edges $a = b = c = d$, which is the focus of this study, the conditions of edges that are alleviated as possible were surrounded by parentheses if tiling is possible with the node restriction. Table 4's rightmost column, "Pertinent tiles in

Table 4. Tiling (where node restrictions have not been applied) using the 13 kinds of convex pentagons with four equal-length edges that are discussed in Sec. 2.

List number	Combinations of nodes usable in tiling: 4-valent nodes	Combinations of nodes usable in tiling: 3-valent nodes	Connecting method	Maltipterned	Figure of corresponding tiling pattern	Conditions of tile other than $a = b = c = d$ (case to realize the combination of nodes in tiling)	Pertinent tiles in Table 2
1	*DDEE*	*ABC*	*DE*-reular / #1 *DE*-reversed	Y	Fig. 2	$A + B + C = 360^\circ$ ($a = d$ ^{#4, #5})	C20-T1A, C22-T1A, C11-T1A, C12-T7 (iff $A = 90^\circ$)#10
2	*DDEE*, *DEEE*, *DDDE*, *DDDD*, *EEEE*	*ABC*, *CCB*, *AAB*	*DE*-reular & #2 *DE*-reversed	Y#3	Fig. 7 (a)	$A + B + C = 360^\circ$, $A = C$ #6 ($a = d$, $b = c$ #4)	C22-T1A, C20-T1A (iff $A = C$)#10
3	*DDAB*	*ABC*, *EEC*	*DE*-regular	N	Fig. 5 (a)	$A + B + C = 360^\circ$, $C = 2D$	C11-T1A, C20-T1A (iff $C = 2D$)#10, C12-T7 (iff $A = 90^\circ$)#10
4	*DDAB, DDCA*	*ABC, EEC*, *BBA*	*DE*-regular	Y	Fig. 7 (b)	$A = 90^\circ$, $B = C = 135^\circ$, $D = 67.5^\circ$, $E = 112.5^\circ$	C11-T1A (iff $A = 90^\circ$)#10, C12-T7 (iff $A = 90^\circ$)#10
5	*DDCA*	*EEC*, *BBA*	*DE*-regular	Y	Fig. 4 (b)	$2E + C = 2B + A = 360^\circ$	C12-T7, C11-T1A (iff $A = 90^\circ$)#10
6	*AABB*	*CDE*	*DE*-reversed	Y	Fig. 3 (a)	$C + D + E = 360^\circ$ ($a = c$ #4)	C20-T1C, C12-T1C, C11-T1C
7	*AADE*	*CDE*, *BBC*	*DE*-reversed	Y	Fig. 5 (b)	$C + D + E = 360^\circ$, $C = 2A$ ($a = c = d$#4)	C11-T1C, C20-T1C (iff $C = 2A$)#10
8	*AADE*, *AABB*	*CDE*, *BBC*	*DE*-reversed	Y	Fig. 7 (c)		
9	*DDBC*	*EEB*, *AAC*	*DE*-regular	Y	Fig. 4 (a)	$C + D + E = 360^\circ$, $C = 2B$ #7	C12-T1C, C20-T1C (iff $C = 2B$)#10
10	*CCAA*	*BDE*	*DE*-reversed	Y	Fig. 3 (b)	$B + D + E = 360^\circ$ ($a = c$, $b = d$ #4, #8)	C20-T2, C11-T2, C22-T2&4
11	*CCAA*, *CAAA*, *CCCA*, *AAAA*, *CCCC*	*BDE*, *DDB*, *EEB*	*DE*-reular & #2 *DE*-reversed	Y #3	Fig. 8 (a)	$B + D + E = 360^\circ$, $D = E$ #9	C22-T2&4, C20-T2 (iff $D = E$)#10
12	*AADE*	*BDE*, *CCB*	*DE*-reversed	N	Fig. 5 (c)	$B + D + E = 360^\circ$, $B = 2A$	C11-T2, C20-T2 (iff $B = 2A$) #10
13	*AADE*, *CCAA*	*BDE*, *CCB*	*DE*-reversed	Y	Fig. 8 (b)		

#1: Using either the regular pattern, the reversed pattern, or both enables tilings using the 4- and 3-valent nodes in the corresponding columns. #2: If there are all five kinds of 4-valent nodes and three kinds of 3-valent nodes in a single tiling, the tiling needs to use both the regular pattern and reversed pattern. #3: It is considered to be the same pattern if there is no vertex sign because it is a pentagon that is symmetrical to the axis passing through the midpoint of vertex B and edge DE. #4: The condition for edges $a = b = c = d$ is not essential for tiling. #5: Adding the relationship of $b = c$ to make tiles satisfy $a = d$, $b = c$ allows periodic tilings with contacting methods between the edges AB and BC that are different from a single condition of $a = d$ and multipatterned tilings. #6: $D = E = 90^\circ$ when $a = d$, $b = c$ is considered. #7: The angular conditional equations are obtained from $a = b = c = d$ and the node restriction. On the contrary, use $a = b = c = d$ and the relationship of $C = 2E - 2D$ obtained from the angular conditional equation for deriving the node restriction from the angular conditional equations. #8: Tiles satisfying $a = b = c = d$ allow multipatterned tilings. #9: $A = C = 90^\circ$ when $a = b = c = d$ is considered. #10: Tilings are enabled using the 4- and 3-valent nodes entered in the corresponding column if and only if the tile satisfies the relationship described after "iff" in the parenthesis.

Table 2," shows the pertinent convex pentagons with four equal-length edges of Table 2. In this column, C12-T7 (iff $A = 90^\circ$) or C20-T1A (iff $A = C$) indicate that the C12-T7 tile allows the tilings of numbers 1, 3, and 4 if and only if it satisfies $A = 90^\circ$, or C20-T1A tile allows the tilings of number 2 if and only if it satisfies $A = C$.

For the same reasoning as that of the previous section, it should be noted that some tiling patterns are not periodical in the range of Figs. 7 and 8. In addition, the shape of the smallest fundamental region is selected based on the pair of DE-regular or DE-reversed. This shape is colored in the figures. However, Figs. 7(a) and 8(a) have the same tiling pattern unless there are the vertex signs. Therefore, there is not a mark of the smallest fundamental regions or fundamental regions. The eight pentagons colored in Figs. 7(b) and 8(b) are required for forming the smallest fundamental region for tiling that allows two kinds of 4-valent nodes with numbers 4 and 13 in Table 4. However, we should consider that the pentagons, including the four pale gray pentagons shown in Figs. 7(b) and 8(b) and the four pentagons in dark gray (i.e., the region rotated by 180° or mirrored), are multipatterned tilings by two kinds of basic elements. This situation arises because we can use two kinds of 4-valent nodes allowing multipatterned tiling due to the fact that we can combine freely two kinds of one-dimensional modules consisting of the four basic elements of pentagons. Meanwhile, two kinds of 4-valent nodes are used for tiling of number 8 in Table 4. As shown in Fig. 7(c), three different layouts of the vertical directions of the figure are enabled by the parallel shift of the smallest fundamental region that consists of four pentagons, signifying that multiple tiling is possible.

4.3 Discussion

We are not aware of any reports pointing out the properties of multipatterned tiling with convex pentagons using the combinations of nodes of numbers 4, 8, and 13 of Table 4. Therefore, we consider that this combination is a new tiling pattern of pentagons.

In the multipatterned tiling using five kinds of nodes "$DDAB$, $DDCA$, EEC, BBA, ABC" of number 4 in Table 4, the convex pentagons with four equal-length edges in the case where the C11-T1A tile is the same shape as the C12-T7 tile are used. Their tiles satisfy the conditions $A = 90^\circ$, $B = C = 135^\circ$, $D = 67.5^\circ$, $E = 112.5^\circ$, and $a = b = c = d$ (i.e., they belong to both type 1 and type 7 of the idiomatic expression). This multipatterned tiling of number 4 in Table 4 is generated since a tiling can contain all node (the node restriction "$DDAB$, EEC, ABC" of C11-T1A and the node restriction "$DDCA$, EEC, BBA" of C12-T7 in Table 2). Then the multipatterned tiling using four kinds of nodes "$AABB$, $AADE$, BBC, CDE" of number 8 in Table 4 uses the convex pentagons with four equal-length edges, satisfying the angular conditions: $C + D + E = 360^\circ$ and $C = 2A$. (Actually, the convex pentagon may not have four equal-length edges. See Table 4, #4.) These conditions, $C + D + E = 360^\circ$ and $C = 2A$, fall under the C11-T1C tile in Table 2. Therefore,

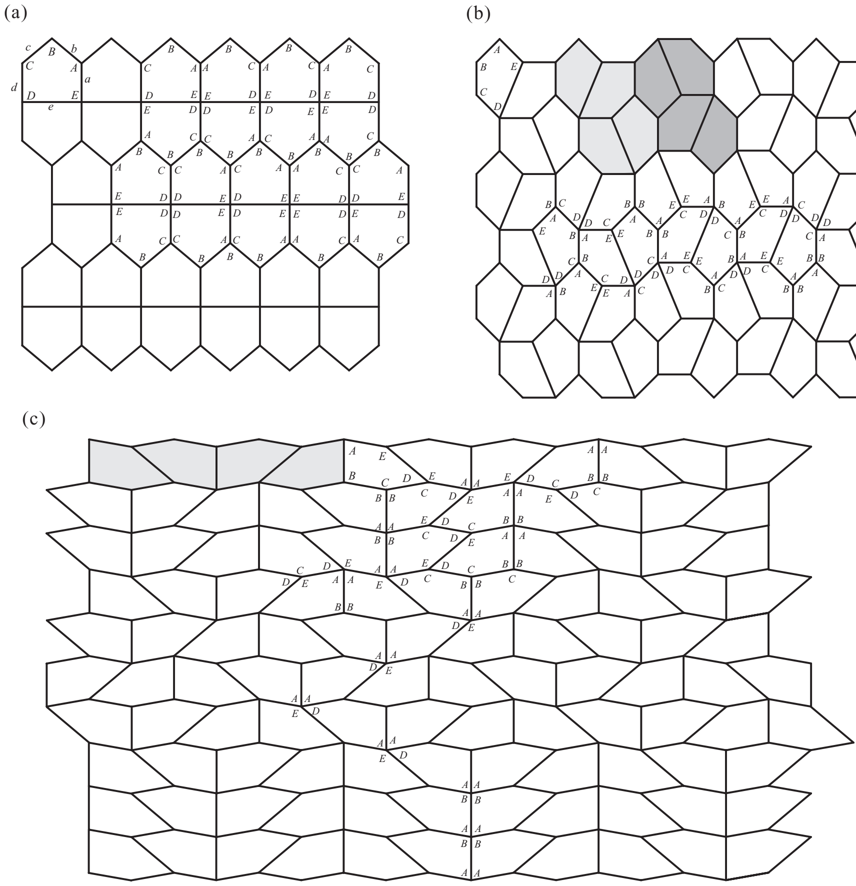


Fig. 7. Tiling where node restrictions have not been applied, part 1. (a) Tiling with C22-T1A tiles using nodes "$DDEE$, $DEEE$, $DDDE$, $DDDD$, $EEEE$, ABC, CCB, AAB." (b) Tiling with C11-T1A tiles using nodes "$DDAB$, $DDCA$, ABC, EEC, BBA." (c) Tiling with C11-T1C tiles using nodes "$AADE$, $AABB$, CDE, BBC."

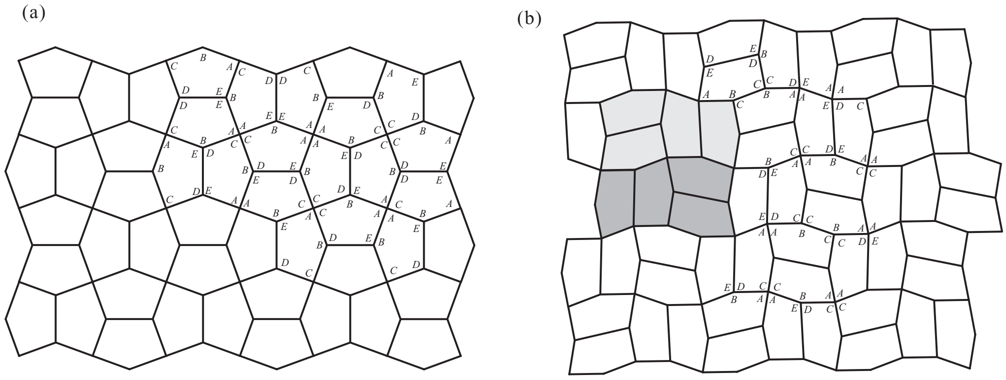


Fig. 8. Tiling where node restrictions have not been applied, part 2. (a) Tiling with C22-T2&4 tiles using nodes "$CCAA$, $CAAA$, $CCCA$, $AAAA$, $CCCC$, BDE, DDB, EEB." (b) Tiling with C11-T2 tiles using nodes "$AADE$, $CCAA$, BDE, CCB."

they are the necessary and sufficient angular conditions for tiling that satisfy the node restriction of "$AADE$, BBC, CDE." On the other hand, since C11-T1C tiles belong to type 1 (i.e., the condition of type 1 is $C + D + E = 360°$), these tiles can allow the tiling with nodes "$AABB$, CDE." That is, the convex pentagons with four equal-length edges that satisfy $C + D + E = 360°$, $C = 2A$ enables tiling using nodes "$AADE$, BBC, CDE" and tiling using nodes "$AABB$, CDE." Further, the multipatterned tiling of number 8 in Table 4 is generated because both nodes can be simultaneously used in tiling. However, as explained in Subsec. 3.2, the convex pentagon tiles used in the tiling consist of three congruent triangles. If we consider that the tiling is formed of these triangles, we see that the pattern will always be the same. The multipatterned tiling use four node kinds, "$CCAA$, $AADE$, BDE, CCB" in Table 4, number 13 uses convex pentagons with four equal-length edges that satisfy the angular condition, $B + D + E = 360°$, $B = 2A$. These convex pentagons with four equal-length edges fall under C11-T2 tiles of Table 2 and belong to type 2. Therefore, the convex pentagons with four equal-length edges satisfying $B + D + E = 360°$, $B = 2A$ enable tiling using nodes

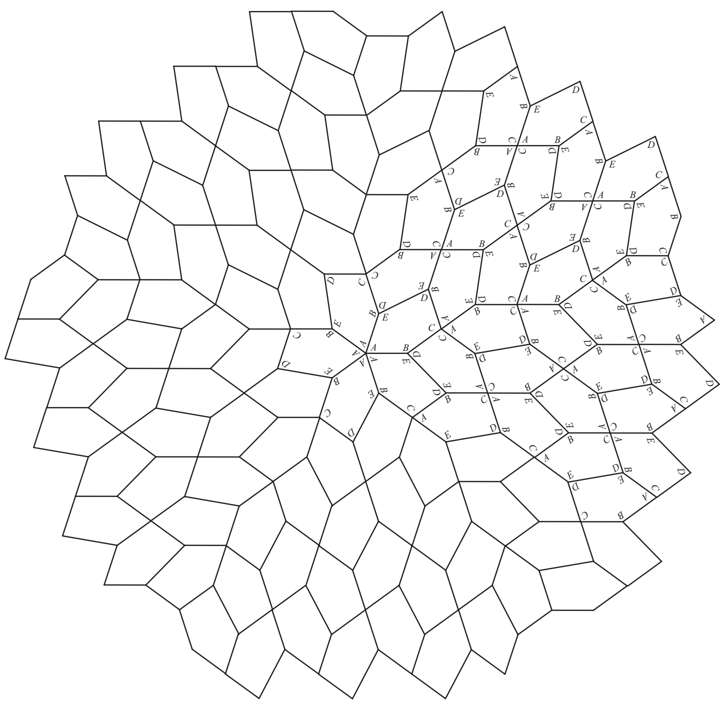


Fig. 9. Special tiling by convex pentagons of C20-T2 with $A = 72°$

"$AADE$, BDE, CCB," just as C11-T2 and also tiling using nodes "$CCAA$, BDE" as a type 2 (pentagonal tile that satisfies $B + D + E = 360°$). Furthermore, the multipatterned tiling of number 13 in Table 4 is generated because both nodes were used simultaneously in the tiling.

At the conclusion of this section, we introduce a special tiling using convex pentagons with four equal-length edges of C20-T2 and C11-T2 with $A = 72°$. As shown in Fig. 9, the tiling has only one 5-valent node in the tiling pattern. This tiling always uses the DE-reversed. One kind of 5-valent $AAAAA$ node, one kind of 4-valent $AACC$ node, and one kind of 3-valent BDE node are used. What has no periodicity, however, is a tiling of 5-rotation symmetry. The tiling in Fig. 9 is a special case in which a 5-valent node appears only in one center even if the tiling is continued beyond the illustration. In terms of maximum and finite tiling, the 5-valent node, which can be ignored on its own, and tiling in Fig. 9 can be considered to satisfy the relationship of $V_3 : V_4 \approx 2 : 1$ (Sugimoto and Ogawa, 2006b). Note that, based on Table 3, C20-T2 tiles reserve freedom θ only by assuming $A = 72°$ ($\alpha = 54°$). C11-T2 tiles with $A = 72°$ fall under C20-T2 tiles satisfying $\alpha = 54°$, $\theta = 54°$. Additionally, a C20-T2 tile becomes an equilateral pentagon when $\alpha = 54°$, $\theta = \cos^{-1}(3/(4\sin(2\pi/5))) \approx 37.945°$. The tiling in Fig. 9 is also enabled by using this equilateral pentagon (Hirschhorn and Hunt, 1985).

5. Discussion of Convex Pentagons with Four Equal-length Edges Eliminated by Topological Judgment but Confirmed by the Geometric Judgment

In Report I, we discussed 33 cases in Tables 3 and 4 of Report I by (i) topological judgment (graph theory) to investigate the possibility of tiling using symbolized notation without breaking down the order of pentagonal meshes, and (ii) geometric judgment to investigate the possibility of the existence of the convex pentagon in Euclidean space. As a result, according to Report I, the convex pentagons of DE-regular 1–5, 8 in Table 3 and DE-reversed 1, 8 in Table 4 are eliminated by the topological judgment. But they are confirmed by the geometric judgment (i.e., the convex pentagons exist though their tilings are impossible under the node restriction). Here, we correct the results of Report I. In Table 7 of Report I, we classified DE-regular 13, 17, 18 into "N" of the geometric judgment. But the classification was a mistake. The cases of DE-regular 13, 17, 18 are confirmed by the geometric judgment and are eliminated by the topological judgment. Therefore, in this section, we consider the convex pentagons of DE-regular 1–5, 8, 13, 17, 18 in table 3 and DE-reversed 1, 8 in Table 4 of Report I.

Tables 5 and 6 summarize the properties of convex pentagons with four equal-length edges of DE-regular 1–5, 8, 13, 17, 18 and DE-reversed 1, 8. In addition, they show the probability of tiling without the node restriction (Sugimoto and Ogawa, 2006b).

Table 5's column of node restriction that was applied in Report I shows 3- and 4-valent nodes of DE-regular 1–5, 8, 13, 17, 18 and DE-reversed 1, 8. They are mentioned in Tables 3 and 4 of Report I. The next column shows the necessary and sufficient conditions that convex pentagons with four equal-length edges should satisfy so that the relationship of the internal angles of each node restriction is realized. Based on the conditions, DE-regular 2 and DE-reversed 8 convex pentagons with four equal-length edges are the same as Table 2's C22-T1A and C12-T1C tiles, respectively. Therefore, DE-regular 2 and DE-reversed 8 convex pentagons are pentagonal tiles even though they do

Table 5. Convex pentagons with four equal-length edges that eliminated by topological judgement, but confirmed by geometric judgment.

Classification of Report I		Node restriction that was applied in Report I			Condition that pentagons should satisfy so that the applied node restriction is realized	Is EE tiling possible in other nodes?	DOFs	Equilatral	Comments
Connecting method	Case	4-valent node	3-valent nodes						
DE-regular *DE*-reversed	1 1	*DDEE*	*AAC*	*BBC*	$A+B+C=360^\circ$, $A=B$	Y[#1]	1	N	See Fig. 10. This convex pentanal tile is expressed as C20-T1A$_{A=B}$.
DE-regular	2	*DDEE*	*CCB*	*AAB*	$A+B+C=360^\circ$, $A=C$	Y[#2]	1	Y	It has the same shape as C22-T1A (*DE*-reversed 2) tile.
DE-regular	3	*DDAA*	*BBC*	*EEC*	$B+C+E=360^\circ$, $B=E$	N[#3]	1	Y	It will belong to type 1 when $A=B=E=108^\circ$. It will be an equilateral pentagon when $C=D\approx 98.71^\circ$.
DE-regular	4	*DDAA*	*CCB*	*EEB*	$B+C+E=360^\circ$, $C=E$	N[#3]	1	Y	It will belong to type 1 when $B=D=72^\circ$. It will be an equilateral pentagon when $A=B\approx 98.71^\circ$.
DE-regular	5	*DDBB*	*AAC*	*EEC*	$E+A+C=360^\circ$, $A=E$	N[#3]	1	Y	It will belong to type 1 when $A=B=E=108^\circ$. It will be an equilateral pentagon when $B=D=90^\circ$.
DE-regular	8	*DDBB*	*CCA*	*EEA*	$E+A+C=360^\circ$, $C=E$	N[#3]	1	Y	It will be an equilateral pentagon when $B=D\approx 98.71^\circ$.
DE-regular	13	*DDAB*	*EEB*	*CCA*	$2E+B=2C+A=360^\circ$	N[#4]	1	Y	It will belong to both type 1 and type 7 when $B=C=135^\circ$. It will belong to both type 2 and type 8 when $a=b=c=d=e$.
DE-regular	17	*DDCA*	*EEA*	*BBC*	$2E+A=2B+C=360^\circ$	N[#5]	1	Y	It will belong to type 8 when $B=C=120^\circ$. It will belong to type 2 when $C=E\approx 137.05^\circ$.
DE-regular	18	*AABC*	*DDB*	*EEC*	$2D+B=2E+C=360^\circ$	N[#6]	1	Y	It will belong to type 1 when $a=b=c=d=e$. It will belong to type 2 when $B=E\approx 139.40^\circ$.
DE-reversed	8	*BBDE*	*AAC*	*CDE*	$C+D+E=360^\circ$, $C=2B$	Y	1	Y	It has the same shape as C12-T1C (*DE*-regular 14) tile.

#1: Tilings with the nodes "$DDEE$, ABC" are enabled if the condition of the edge is "$a=d$" or "$a=d$, $b=c$." #2: Tilings with the nodes "$DDEE$, CCB, AAB" are enabled if the condition of the edge is "$a=d$, $b=c$." #3: Convex pentagon belongs to type 2 and tiling by the pentagons is NEE. However, in the special cases, the pentagon can form EE tiling. #4: Pentagon can form EE tiling if and only if it satisfies "$B=E$" or "$a=b=c=d=e$ ($A=B\approx 98.71^\circ$)." #5: Pentagon can form EE tiling if and only if it satisfies "$B=C$." Pentagon can form NEE tiling if and only if it satisfies "$C=E$." #6: Pentagon can form EE tiling if and only if it is equilateral ($C=80^\circ$, $D=100^\circ$). Pentagon can form NEE tiling if and only if it satisfies "$B=E$."

Table 6. Relationship among internal angles A, B, C, D, and E, and length of edge e as to pentagons in Table 5.

Classification of Report I		Internal angles of pentagon					Additional informations about internal angles	Edge length ℓ and range (with $a=b=c=d=1$)
Connecting method	Case	A	B	C	D	E		
DE-regular	1	$180^\circ-2\alpha$	$180^\circ-2\alpha$	4α	$180^\circ-3\alpha$	3α	$0^\circ<\alpha<45^\circ$	$\sqrt{2}<2\cos\alpha<2$
DE-regular	3	$180^\circ-2\alpha$	$90^\circ+\beta$	$180^\circ-2\beta$	2α	$90^\circ+\beta$	$\alpha=-\tan^{-1}\left[\dfrac{2\cos^2\beta-\sin\beta-\sqrt{1+3\cos^2\beta}}{2\cos\beta\sin\beta}\right]$, $0^\circ<\beta<90^\circ$	$0<\dfrac{2\cos\beta\cos\alpha}{\cos(\beta-\alpha)}<2$
DE-regular	4	$180^\circ-2\alpha$	4β	$180^\circ-2\beta$	2α	$180^\circ-2\beta$	$\alpha=\tan^{-1}\left[\dfrac{-1+\sqrt{1+12\cos^2\beta\sin^2\beta}}{2\cos\beta\sin\beta}\right]$, $0^\circ<\beta<45^\circ$	$0<\dfrac{2\cos\beta\sin(3\beta-\alpha)}{\sin(2\beta+\alpha)}<2$
DE-regular	5	$180^\circ-2\alpha$	$90^\circ-\alpha+\theta$	4α	$90^\circ+\alpha-\theta$	$180^\circ-2\alpha$	$\theta=3\alpha-\sin^{-1}\left[\dfrac{\cos\alpha\sin(3\alpha)}{\sin(2\alpha)}\right]$, 30° [#1] $<\alpha<45^\circ$	$0<\dfrac{2\sin(2\alpha)\sin\theta}{\sin(3\alpha)}<2.732$ [#2]
DE-regular	8	4β	$90^\circ-\beta+\theta$	$180^\circ-2\beta$	$90^\circ+\beta-\theta$	$180^\circ-2\beta$	$\theta=\cos^{-1}(2\sin\beta)$, $0^\circ<\beta<30^\circ$ [#3]	$0<2\cos\beta\sqrt{1-4\sin^2\beta}<2$
DE-regular	13	4β	$90^\circ-\beta+\theta$	$180^\circ-2\beta$	$135^\circ-\dfrac{3\beta}{2}-\dfrac{\theta}{2}$	$135^\circ+\dfrac{\beta}{2}-\dfrac{\theta}{2}$	$\theta=2\tan^{-1}\left[\dfrac{(1-2\sin\beta)\left[\tan\left[\frac{5\beta}{2}\right]+1\right]}{(1+2\sin\beta)\left[\tan\left[\frac{5\beta}{2}\right]-1\right]}\right]$, 20.70° [#4] $<\beta<30^\circ$	$0<\dfrac{2\cos\beta\sin\theta}{\sin\left[45^\circ+\frac{5\beta}{2}-\frac{\theta}{2}\right]}<2.645$ [#5]
DE-regular	17	$180^\circ-2\alpha$	$90^\circ+\beta$	$180^\circ-2\beta$	$\alpha+\beta$	$90^\circ+\alpha$	$\beta=\cos^{-1}\left[\dfrac{\cos\alpha}{\sin\alpha}\right]$, 45° [#6] $<\alpha<90^\circ$	$0<\dfrac{2\cos^2\alpha}{\sin\alpha}<\sqrt{2}$
DE-regular	18	$180^\circ-2\alpha$	$4\alpha+2\beta-180^\circ$	$180^\circ-2\beta$	$270^\circ-2\alpha-\beta$	$90^\circ+\beta$	$\alpha=-\tan^{-1}\left[\dfrac{\sin\beta(4\cos^2\beta-1)-\sqrt{16\cos^4\beta-9\cos^2\beta+1}}{2\cos\beta(2\cos^2\beta-1)}\right]$, $45^\circ<\beta<50.18^\circ$ [#7]	$0<\dfrac{-2\cos\beta\sin(3\alpha+\beta)}{\cos(\beta-\alpha)}<1.199$ [#8]

#1: $\alpha>30^\circ$ because $\theta>0^\circ$. #2: Edge length is the upper limit when $\alpha=45^\circ$. The accurate value of the upper limit is $2\sqrt{2}\sin(5\pi/12)$. #3: $\beta<30^\circ$ because $2\sin\beta<1$. #4: From $B<180^\circ$, the lower limit is $2\tan^{-1}\left(\sqrt{4-\sqrt{14}}/\sqrt{4+\sqrt{14}}\right)\approx 20.70^\circ$. #5: Edge length if the upper limit when $\beta\approx 20.70^\circ$. The equation of upper limit is complicated. Therefore, it is omitted. #6: $\alpha>45^\circ$ because $\cos\alpha/\sin\alpha<1$. #7: From $B<180^\circ$, the lower limit is $2\tan^{-1}\left(\sqrt{2\left(7-\sqrt{17}\right)}\Big/\left(1+\sqrt{17}\right)\right)\approx 50.18^\circ$. #8: Edge length is the upper limit when $\beta\approx 50.18^\circ$. The equation of upper limit is complicated. Therefore it is omitted.

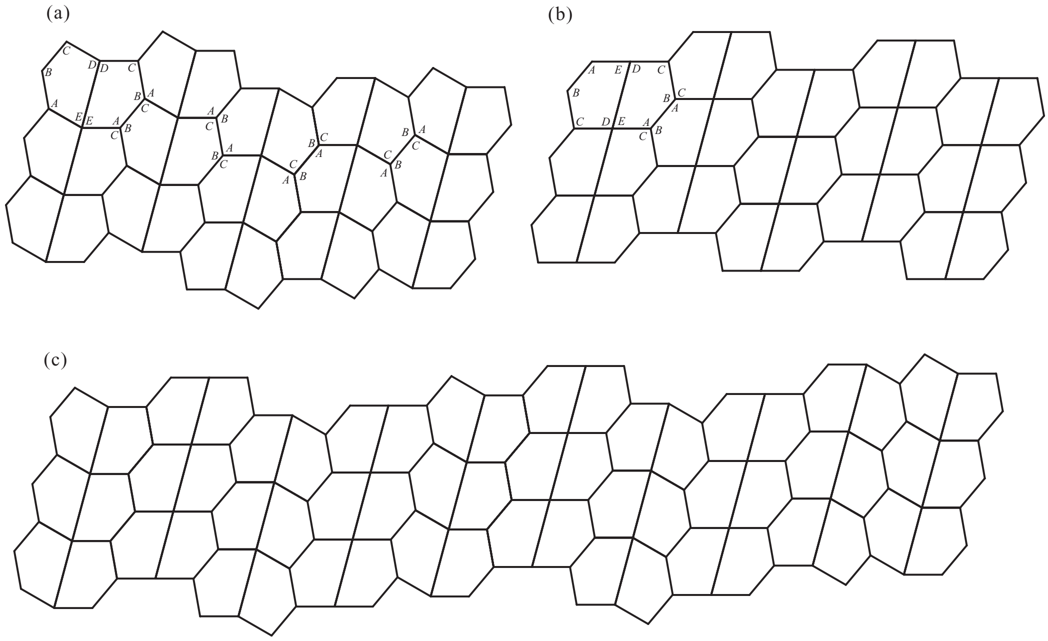
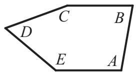
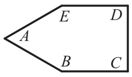
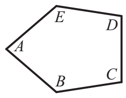
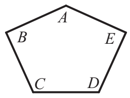
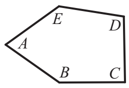


Fig. 10. Tilings with C20-T1A$_{A=B}$ with using nodes "$DDEE$ and ABC." (a) Tiling by DE-regular pattern. (b) Tiling by DE-reversed pattern. (c) Tiling by the combination of DE-reglar pattern and DE-reversed pattern.

not allow tiling with each node restriction of Repot I. On the other hand, DE-regular 1 and DE-reversed 1 convex pentagons with four equal-length edges have the same conditions (the same shape) despite their different connecting methods used in tiling (i.e., DE-regular or DE-reversed). The DE-regular 1 convex pentagon with four equal-length edges belongs to type 1 in the idiomatic expression because of the condition $A + B + C = 360°$, $A = B$. Therefore, as described in Sec. 4, its tiling is possible by using 4-valent node $DDEE$ and 3-valent node ABC (see Fig. 10). DE-regular 1 tile is called C20-T1A$_{A=B}$ because tiling as C20-T1A tile in Table 2 is possible. Convex pentagons with four equal-length edges of DE-regular 3, 4, 5, 8 belong to type 2 in the idiomatic expression. Tilings by these pentagonal tiles are non-edge-to-edge (NEE) because of the edge's condition $a = b = c = d$ (i.e., DE-regular 3, 4: $b \neq e$, DE-regular 5, 8: $c \neq e$). However, in the special cases, the pentagonal tiles can form edge-to-edge (EE) tiling. When the pentagons of DE-regular 3, 5 satisfy $A = B = E = 108°$, they have the relations $A + B + C = 360°$ and $a = b = c = d$. Therefore, the pentagons belong also to type 1 and can form EE tiling. If the pentagon of DE-regular 4 has $B = D = 72°$, it belongs also to type 1 and can form EE tiling because of the relations $C + D + E = 360°$ and $a = b = c = d$. In addition, each DE-regular 3, 4, 5, 8 convex pentagons can be an equilateral pentagon. Therefore, those pentagons that belong to type 2 allow EE tilings if and only if they are equilateral pentagons. On the other hand, DE-regular 13, 17, 18 convex pentagons with four equal-length edges cannot allow to tile the plane (these pentagons are not pentagonal tiles in almost all cases) even if NEE tiling is allowed. However, in the special cases, these pentagonal will be tileable. DE-regular 13 convex pentagon with $A = 90°$, $B = C = 135°$, $D = 67.5°$, $E = 112.5°$, and $a = b = c = d$ belongs to both type 1 and type 7 (i.e., this pentagon is C12-T7 (iff $A = 90°$). See Table 4), and can form EE tiling. If DE-regular 13 convex pentagon is an equilateral pentagon, it has the relations $A = B \approx 98.71°$ and $A + C + E = B + C + E = 360°$. Thus, the equilateral pentagon of DE-regular 13 belongs to both type 2 and type 8, and tilings by the equilateral pentagons are edge-to-edge. When the convex pentagon with four equal-length edges of DE-regular 17 satisfies $B = C = 120°$, it belongs to type 8 (tiling by the pentagonal tile of type 8 is edge-to-edge). If DE-regular 17 pentagon with $a = b = c = d$ has $C = E \approx 137.05°$ (i.e., this pentagon has $E + A + C = 360°$), it belongs to type 2 and allows NEE tiling. Note that an equilateral pentagon of DE-regular 17 is not tileable. When DE-regular 18 pentagon is an equilateral pentagon, it has the relation $E + A + B = 360°$ ($A = 60°$, $B = 160°$, $C = 80°$, $D = 100°$, $E = 140°$). Thus the equilateral pentagon of DE-regular 18 belongs to type 1 and can form EE tiling. Then, DE-regular 18 convex pentagon with $B = E$ belongs to type 2 and allows NEE tiling because of the relations $B + C + E = 360°$ and $a = b = c = d$. Based on the above discussions, Table 5 column titled "Is EE tiling possible in other nodes?" indicates "Y" if EE tiling is possible in all cases and "N" if EE tiling is impossible without the special cases. Table 5's column of DOFs shows the degree of freedom concerning the shape of convex pentagons with four equal-length edges, assuming that these convex pentagons satisfy the condition $a = b = c = d = 1$. The column of the equilateral indicates "Y" if a convex pentagon with four equal-length edges can be an equilateral convex pentagon without breaking the conditions that a convex pentagon with four equal-length edges should satisfy. The column indicates "N" for other cases. Each DE-regular 3, 4, 5, 8, 13, 18 convex pentagons can be an equilateral pentagon and is tileable. Details of equilateral pentagonal tiles are

Table 7. Classification of the equilateral convex pentagonal tiles obtained from our convex pentagons with four equal-length edges.

Class symbol	Figure of equilateral convex pentagonal tile	Internal angles	Pertinent our convex pentagons	Angular relationship when tiles are equilateral pentagons (each of α, β, γ, θ, and D corresponds to the contents of Tables 3 and 6).
ET1		$A = 100°$, $B = 80°$, $C = 160°$ $D = 60°$, $E = 140°$	C12-T1C (*DE*-regular 14) C11-T1A (*DE*-regular 21) *DE*-regular 18	$\alpha = 40°$ $\alpha = 20°$ $\beta = 50°$
ET1-*R*		$A = 60°$, $B = E = 150°$, $C = D = 90°$	C22-T1A (*DE*-reversed 2) C22-T1E (*DE*-regular 7) C20-T1A (*DE*-regular 11)[#1] C20-T1C (*DE*-reversed 4)[#2]	$\gamma = 60°$ $\alpha = 60°$ $\gamma = 60°$, $D = 90°$ [#3] $D = 60°$ [#3], $\alpha = 45°$
ET2		$A \approx 81.29°$, $B = E \approx 130.645°$, $C = D \approx 98.71°$	C12-T8 (*DE*-regular 12) *DE*-regular 3 *DE*-regular 4 *DE*-regular 8 *DE*-regular 13	$\beta \approx 24.68°$ [#4] $\alpha \approx 49.35°$, $\beta \approx 40.65°$ $\alpha \approx 40.65°$, $\beta \approx 24.68°$ $\beta \approx 24.68°$ $\beta \approx 24.68°$
ET2-*R*		$A \approx 131.41°$, $B = E = 90°$, $C = D \approx 114.295°$	C22-T2&4 (*DE*-regular 10) C20-T2 (*DE*-reversed 5) *DE*-regular 5	$\theta = \cos^{-1}(3/4) \approx 41.41°$ $\alpha \approx 37.76°$, $\theta \approx 39.23°$ [#5] $\alpha \approx 32.85°$
ET7		$A \approx 70.88°$, $B \approx 144.56°$, $C \approx 89.26°$, $D \approx 99.93°$, $E \approx 135.37°$	C12-T7 (*DE*-regular 16)	$\alpha \approx 54.56°$ [#6]

#1: According to Table 3, this pentagon is equilateral if $\gamma = 60°$ and has the remaining degree of freedom of 1 ($A = 60° + D$, $B = 60°$, $C = 240° - D$, $E = 180° - D$). $D = 90°$ is also required for this pentagon to be ET1-*R*. #2: According to Table 3, this pentagon is equilateral if $D = 60°$ and has the remaining degree of freedom of 1 ($A = 180° - 2\alpha$, $B = 2\alpha$, $C = 240° - 2\alpha$, $E = 60° + 2\alpha$). $\alpha = 45°$ is also required for this pentagon to be ET1-*R*. #3: This D corresponds to the layout of pentagons with four equal-length edges. #4: The accurate value is $\beta = \cos^{-1}\left(\sqrt{\left(3+\sqrt{13}\right)/8}\right)$. #5: The accurate value is $\alpha = \sin^{-1}\left(\sqrt{6}/4\right)$, $\theta = \cos^{-1}\left(3\Big/\left(4\sin\left(2\sin^{-1}\left(\sqrt{6}/4\right)\right)\right)\right)$. #6: The accurate value is omitted because it cannot be an equation.

explained in Sec. 6.

Table 6 shows the relationship among the internal angles A, B, C, D, E obtained by separating DE-regular 1, 3, 4, 5, 8, 13, 17, 18 convex pentagons with four equal-length edges into three triangles (see Fig. 6(d) for positions of α, β, and θ). This table also shows the equation of the length ℓ of edge e with $a = b = c = d = 1$ and the range of values.

6. Equilateral Convex Pentagonal Tile Obtained from Convex Pentagons with Four Equal-length Edges

In this section, we introduce the possible types of convex pentagons when they are equilateral pentagons and are tileable. These pentagons are among 16 kinds of convex pentagons with four equal-length edges that can be the equilateral pentagonal tiles discussed in Secs. 3 and 5. However, we introduce only the cases of equilateral convex pentagonal tile obtained from the convex pentagons with four equal-length edges that we have previously discussed. We do not discuss details of the equilateral convex pentagonal tile itself.

Currently, the theorem shown below is known to concern the equilateral convex pentagon.

THEOREM 1 *(Hirschhorn and Hunt, 1985). An equilateral convex pentagon tiles the plane if and only if it has two angles adding to* $180°$*, or it is the unique equilateral convex pentagon with angles* A, B, C, D, E *satisfying* $2B + A = 2E + C = 2D + A + C = 360°$ *(*$A \approx 70.88°$*,* $B \approx 144.56°$*,* $C \approx 89.26°$*,* $D \approx 99.93°$*,* $E \approx 135.37°$*).*

Therefore, the tiles of tilings by congruent equilateral convex pentagons belong to type 1, type 2, or type 7. The equilateral convex pentagonal tile obtained from our convex pentagons with four equal-length edges also follows the above theorem.

We researched the shapes of 16 kinds of convex pentagons with four equal-length edges (that are mentioned in Tables 2 and 5) that can be equilateral pentagonal tiles. Then, these that can be equilateral pentagons are classified into five kinds in Table 7. We named these five kinds of equilateral convex pentagonal tiles ET1, ET1-*R*, ET2, ET2-*R*, and ET7, as shown in Table 7 (Sugimoto and Ogawa, 2006a). "E" stands for "equilateral." "T1," "T2," and "T7" refer to type 1, type 2, and type 7, respectively, in the idiomatic expression. "*R*" refers to the pentagon that has a right angle. An equilateral convex pentagon has no problem in the positional relationship between the internal angles and the edges. Therefore, in some cases, the positions of the angles A, B, C, D, E of the equilateral convex pentagon in Table 7 differ from those of angles A, B, C, D, E of the convex pentagon with four equal-length edges from which the equilateral convex pentagon was derived. ET1, ET1-*R*, ET2, ET2-*R*, and ET7 in Table 7 have no freedom other

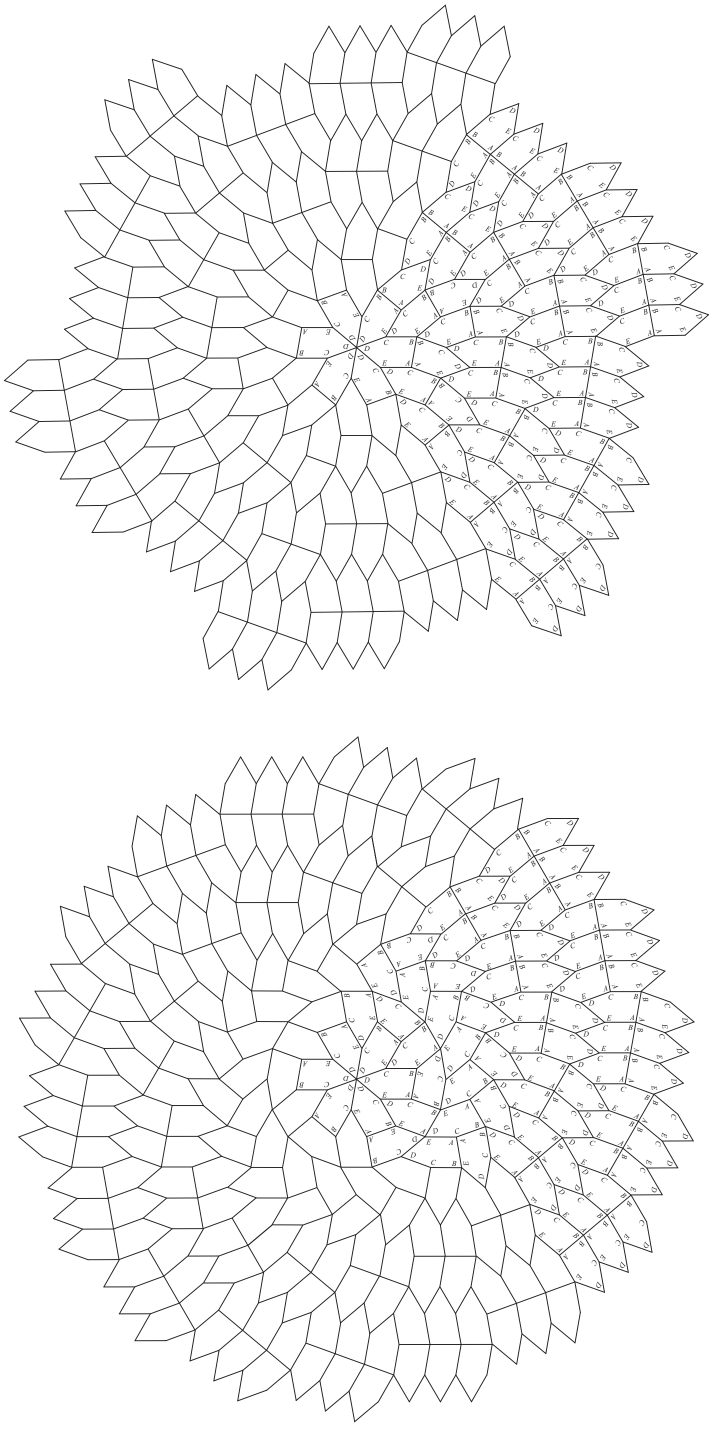


Fig. 11. Special tilings by equilateral convex pentagons of ET1.

than size. The convex pentagon with four equal-length edge tiles of C20-T1A and C20-T1C has a freedom of 2 and the equation of the length ℓ of edge includes only one variable (see Tables 2 and 3). If their conditions are $\gamma = 60^\circ$ and $D = 60^\circ$, they become equilateral convex pentagonal tiles with a freedom of 1 remaining. In this report, the remaining variables of C20-T1A and C20-T1C convex pentagons were selected to $D = 90^\circ$ and $\alpha = 45^\circ$ and classified into ET1-R and ET2-R as shown in Table 7, respectively.

It is apparent that the equilateral convex pentagonal tile (and the C20-T1A and C20-T1C equilateral convex pentagonal tiles having the degree of freedom of 1) in Table 7 can enable most of the tiling that can be performed by the convex pentagon with four equal-length edges from which these tiles were derived. For example, ET1 can perform most tilings that are allowed by C12-T1C and C11-T1A tiles. However, ET1 derived from C12-T1C and C11-T1A tiles cannot perform the tiling that is enabled by C11-T1A tile satisfying $A = 90^\circ$ of number 5 in Table 4, because of the internal angle problem. Detailed explanations about the possible tilings already mentioned in this study when the equilateral convex pentagonal tile in Table 7 is used will be

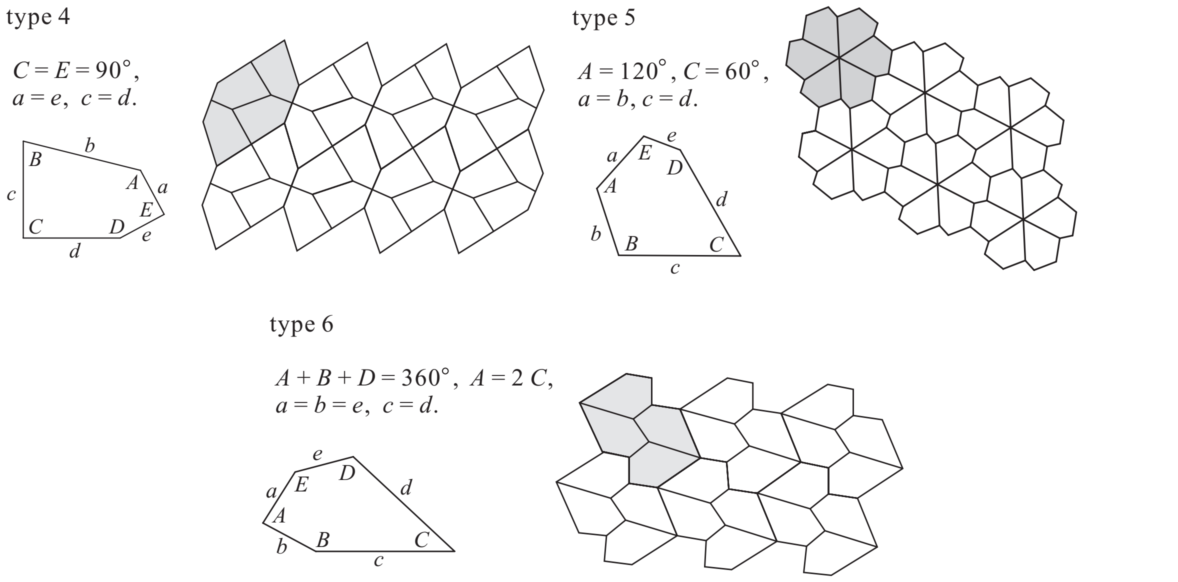


Fig. 12. Convex pentagonal tiles of type 4, type 5, and type 6.

omitted.

Now, we introduce special tilings using the equilateral convex pentagonal tiles. Those are tilings in which only one 6-valent node using ET1 exists in each tiling pattern as shown in Fig. 11 (Schattschneider, 1978, 1981; Hirschhorn and Hunt, 1985). If the vertex sign of ET1 in Table 1 is used, these tilings use one kind of 6-valent node $DDDDDD$, two kinds of 4-valent nodes $BBDE$, $AABB$, and two kinds of 3-valent nodes CDE, AAC. They have no periodicity, however, shows tilings with six-rotational symmetry. The tilings in Fig. 11 are special cases in which a 6-valent node appears only in a center even if each tiling is continued beyond the illustration. That is, the 6-valent node can be ignored if a tiling is maximum and finite. Therefore, the tilings in Fig. 11 can be considered to satisfy the relationship of $V_3 : V_4 \approx 2 : 1$ (Sugimoto and Ogawa, 2006b).

7. Conclusions

This report explains 14 kinds (i.e., 13 kinds of tile in Table 2 and C20-T1A$_{A=B}$ tile in Table 5) of convex pentagonal tiles with four equal-length edges that are obtained from setting the (edge-to-edge) tiling in accordance with the simplest node condition by convex pentagons with four equal-length edges. This report also lists the tiling that uses these pentagons and is enabled by 3- and 4-valent nodes (Sugimoto and Ogawa, 2004, 2006a). We have confirmed that these tiles belong to at least one of the convex pentagonal tiles called type 1, type 2, type 4, type 7, type 8, and type 9 in the idiomatic expression. That is, the new convex pentagonal tile which is not fitted to the idiomatic expression did not exist.

In this report, we have pursued the subject of four equal-length edges. The only assumption used in this premise is that the edges e (edge DE) necessarily have contact with one another. In some cases of the node restriction, the condition of $a = b = c = d$ is not required in tiling, practically speaking, when the conditions of edges are alleviated and tiles other than those with four equal-length edges are already included (see Tables 2 and 4). In our results of this report, tilings of type 4, type 5, and type 6 (see Fig. 12) did not come out. Here, we should discuss the relationship between the convex pentagonal tiles of Table 2 and those of type 4, type 5, and type 6. At first, the tiling of type 4 is deviated from the simplest node condition because it requires two kinds of 4-valent nodes. Note that, as to the tiling of type 4, the node ratio of $V_3 : V_4 \approx 2 : 1$ is established because four same kind of 3-valent nodes exist as compared to two kinds of 4-valent nodes in tiling (Sugimoto and Ogawa, 2006b). However, as mentioned above, we have derived a convex pentagonal tile called C22-T2&4 that belongs to both type 2 and type 4. This C22-T2&4 tile is a special case in which type 4 convex pentagonal tiles satisfy the conditions of $D = E$ and $a = b = c = d$. Then, type 5, which uses the 6-valent node in tiling, is never included because of the simplest node condition. However, the convex pentagonal tile of type 5 is already included in the C20-T2 tile if the convex pentagonal tile of type 5 is assumed as the convex pentagon with four equal-length edges tile (in the case of $A = 60°$, $C = 120°$). The convex pentagonal tile of type 6 with the tile conditions $A + B + D = 360°$, $A = 2C$, and $a = b = e$, $c = d$ cannot be a pentagon with four equal-length edges or an equilateral pentagon. Therefore, it cannot be directly connected to our convex pentagon with four equal-length edges as are type 4 and type 5. However, the tiling of type 6 satisfies the simplest node condition because one kind of 4-valent node ($2C + B + D = 360°$) and two kinds of 3-valent nodes ($2E + A = A + B + D = 360°$) are used. (It falls under C11 in the classification signs in Table 1.) Further, if the internal angles of type 6 convex pentagonal tile are $A = B = D = E = 120°$ and $C = 60°$, the tile belongs to both type 6 and type 5 and enable both tilings (i.e., there is a relationship between type 5 and type 6 convex pentagonal tiles) (Sugimoto and Ogawa, 2000). On the other hand, the convex pentagon with four equal-length edges tile that satisfies the conditions of $A = 90°$, $B = C = 135°$, $D = 67.5°$, $E = 112.5°$, $a = b = c = d$

and was mentioned in Sec. 4 belongs to both type 1 and type 7 (i.e., there is a relationship between type 1 and type 7 convex pentagonal tiles). As mentioned in Secs. 3 and 5, the convex pentagonal tile of type 8 (i.e., C12-T8 tile) with five equal-length edges belongs also to type 2 (i.e., there is a relationship between type 8 and type 2 convex pentagonal tiles). On the contrary, the convex pentagonal tile of type 9 has no relationship with other types. It has seemed that the convex pentagonal tiles of type 7, type 8, and type 9 are different from the convex pentagonal tile belonging to type 1 or type 2. However, we consider that type 9 is really the only convex pentagonal tile that is incompatible with all other types.

We have found the connection between all convex pentagonal tiles allowing the known EE tiling and the convex pentagonal tiles in this study just through the above discussions even if the freedom of the structure has not been fully explained. We assert that previous researchers' results of convex pentagonal tiles can be summarized by keeping on the focus on the specific edge e. However, we do not think that this list is perfect because a case of negation is not exhausted logically.

We advanced our own research by classifying the tiling patterns systematically and discussing them with an aim at an exhaustive study of the convex pentagonal tile. To that end, it is important to have a grasping of the properties of tiling and classifications of tiling patterns. We consider that the conditions of a convex pentagonal tile cannot be expressed without the classification of tiling patterns. Therefore, in this study, we discussed and introduced all the tilings using our convex pentagons with four equal-length edges tiles that can allow multipatterned tiling.

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Appendix A. Correction as the Result of Report I

Although the combinations for DE-regular patterns were enumerated in Table 3 of Report I, we find the mistake and show the correction below. We considered the symbolic symmetry in the simultaneous $D \leftrightarrow E$ and $A \leftrightarrow C$ reflections within the range allowed by the symbols. As a result, 23 regular patterns were shown in Table 3 of Report I. However, the statement was a mistake. In Report I, we judged that the case "$BBCA$, DDA, EEC" is equivalent to DE-regular 19 ($BBCA$, DDC, EEA). But, the case "$BBCA$, DDA, EEC" is independent of "$BBCA$, DDC, EEA." Therefore, it is necessary to add the case "$BBCA$, DDA, EEC" to Table 3 of Report I.

Here, the case "$BBCA$, DDA, EEC" is investigated. The conditions of convex pentagon that satisfies the node restriction "$BBCA$, DDA, EEC" are expressed as follows.

$$2D + A = 2E + C = 360^\circ, a = b = c = d. \qquad (A.1)$$

Pentagons with four equal-length edges can be divided into two isosceles triangles EAB and BCD, and other triangle BDE (see Fig. 6(d)). The base angles of the two isosceles triangles EAB and BCD are denoted α and β. The interior angles of the pentagon that satisfies (A.1) can be expressed as follows.

$$\begin{cases} A = 180^\circ - 2\alpha, \\ B = \alpha + \beta, \\ C = 180^\circ - 2\beta, \\ D = 90^\circ + \alpha, \\ E = 90^\circ + \beta. \end{cases}$$

Therefore, because of $\angle EBD = 0^\circ$, the convex pentagon that satisfies (A.1) cannot exist in geometry.

Thus, our result (the number of concentration methods that allow the EE tiling in accordance with node restriction) in Report I is unchanging. Moreover, the case "$BBCA$, DDA, EEC" (the convex pentagon that satisfies (A.1)) is the outside for consideration also in this paper.

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