Rectilinear Distance in Rotated Regular Point Patterns

Masashi Miyagawa

Department of Ecosocial System Engineering, University of Yamanashi, 4-3-11 Takeda, Kofu, Yamanashi 400-8511, Japan
E-mail address: mmiyagawa@yamanashi.ac.jp

(Received November 2, 2009; Accepted February 3, 2010)

This paper examines the relationship between road directions and the rectilinear distance in regular point patterns. We derive the distributions of the rectilinear distances to the nearest and the second nearest points in rotated regular point patterns. These distributions demonstrate that road directions significantly affect the rectilinear distances. As an application of the nearest and the second nearest distances, we consider a facility location problem in which customers are serviced by either the nearest or the second nearest facility.
Key words: Road Direction, Distance Distribution, Average Distance, Maximum Distance

1. Introduction

Distance plays an important role in spatial analysis. In a point pattern analysis, the distance between neighbouring points is used for describing patterns for the distribution of various geographical objects (Cressie, 1993). In a spatial interaction model, the amount of flow or interaction between any two points decreases with distance (Taaffe *et al.*, 1996). In a facility location problem, the sum of distances from customers to their nearest facility is minimized (Drezner and Hamacher, 2002).

In these spatial models, the most frequently used distance is the Euclidean distance. Although the Euclidean distance is a good approximation for the actual travel distance, the rectilinear distance is more suitable for cities with a grid road network (Vaughan, 1987). In fact, the rectilinear distance has been used in various facility location models (Francis *et al.*, 1992; Macias and Perez, 1995; Aras *et al.*, 2008). The rectilinear distance R between two points (x_1, y_1), (x_2, y_2) is defined as

$$R = |x_1 - x_2| + |y_1 - y_2|. \quad (1)$$

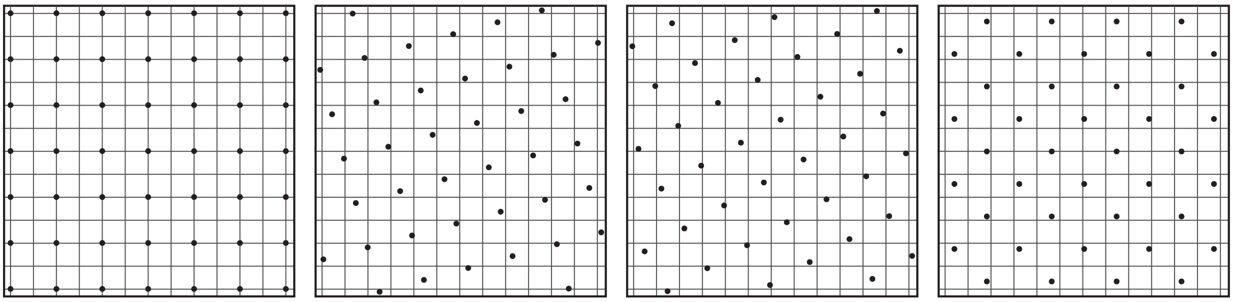
A significant characteristic of the rectilinear distance is that the distance depends on the direction of the coordinate axes. If grid roads exist everywhere and road directions are parallel to the coordinate axes, the rectilinear distance coincides with the road network distance. For the analysis using the rectilinear distance as an approximation of the network distance, the effect of road directions should be considered.

The purpose of this paper is to examine the relationship between road directions and the rectilinear distance. We focus on the rectilinear distance in rotated regular point patterns, as shown in Fig. 1. These regular patterns are obtained by rotating the square lattice at angle θ ($0 \le \theta \le \pi/4$). When $\theta = \pi/4$, the pattern is called the diamond lattice. These patterns are identical for the Euclidean distance, but not for the rectilinear distance. Examining the relationship between the rotation angle θ and the rectilinear distance leads to a deeper understanding of the rectilinear distance in regular point patterns. Regular patterns are important as a typical dispersed pattern. Larson and Odoni (1981) showed that the optimal facility location with rectilinear distances is the diamond lattice ($\theta = \pi/4$). The theoretical results of regular patterns will give a useful tool for the analysis of actual patterns. We assume that these regular patterns continue infinitely. This assumption allows us to examine the distance without taking into account the boundary effect. We also assume that grid roads run everywhere in north-south and east-west directions, as shown in Fig. 1.

Not only the distance to the nearest point but also the distance to the kth nearest point has been used in spatial analysis. The distance from an arbitrary location to the kth nearest point is called the kth nearest distance. An application of the kth nearest distance is found in a facility location problem with closing of facilities. Classical facility location models usually assume that customers always use their nearest facility. Facilities might, however, be closed or disrupted due to accidents or disasters. Customers then have to use more distant facilities. Thus, when locating facilities, the distance to the kth nearest facility should also be taken into account.

The probability density functions of the kth nearest distance have been obtained for several patterns. The nearest distance was derived in Clark and Evans (1954) for the random pattern, Persson (1964) for the square lattice, and Holgate (1965) for the triangular lattice. The kth nearest distance was derived in Dacey (1968) for the random pattern, Koshizuka (1985) for $k = 1, 2, 3$ for the square lattice, and Miyagawa (2009) for $k = 1, 2, \ldots, 7$ for the square, triangular, and hexagonal lattices. The nearest rectilinear distance was derived in Larson and Odoni (1981) for the random pattern. The kth nearest rectilinear distance was derived in Miyagawa (2008) for $k = 1, 2, \ldots, 8$ for the square and diamond lattices.

In this paper, we derive the distributions of the nearest and the second nearest rectilinear distances for the rotated regular patterns. The present paper extends Miyagawa (2008) by introducing the rotation angle θ. We also consider



(a) $\theta=0$ (b) $\theta=\frac{\pi}{12}$ (c) $\theta=\frac{\pi}{6}$ (d) $\theta=\frac{\pi}{4}$

Fig. 1. Rotated regular point patterns.

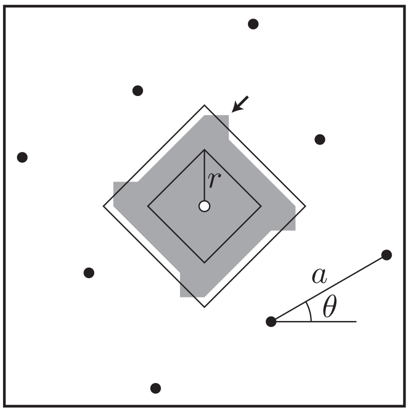


Fig. 2. Region where the white point is the nearest.

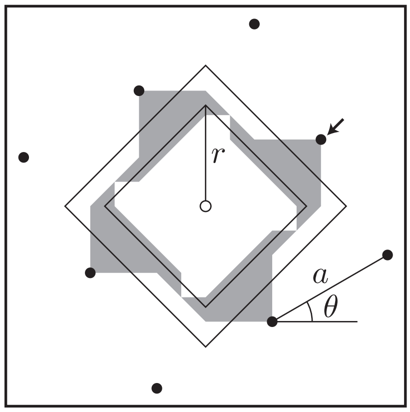


Fig. 3. Region where the white point is the second nearest.

a facility location problem in which customers are serviced by the nearest or the second nearest facility.

A number of facility location models that explicitly include the possibility of closing have been proposed. Weaver and Church (1985) addressed the vector assignment p-median problem, where a certain percentage of customers could be serviced by the kth nearest facility. Pirkul (1989) studied a similar problem in which customers are served by two facilities designated as primary and secondary facilities. Drezner (1987) generalized the p-median and p-center problems by considering the possibility that some of the facilities become inactive. Berman *et al.* (2007) extended Drezner's work and demonstrated that the probability of facility failure has a strong effect on the optimal facility location. Snyder and Daskin (2005) presented two reliability models based on the p-median problem and the uncapacitated fixed-charge location problem. Church *et al.* (2004) developed a interdiction model to identify the set of facilities that, if interdicted, causes the greatest loss. Church and Scaparra (2007) extended the model to generate the range of possible failures and impacts. A survey of facility location problems under uncertainty is provided in Snyder (2006).

Most of the previous studies concerning facility location problems with closing of facilities used discrete network models, in which demand occurs only at nodes of a network. Since discrete models can easily describe realistic situations, the focus is on developing algorithms and obtaining numerical solutions. This paper, in contrast, uses a continuous model, in which demand occurs anywhere on a plane. Continuous models often yield simple closed form solutions, which provide fundamental relationships between variables.

The rest of this paper is organized as follows. The next section derives the distributions of the nearest and the second nearest rectilinear distances for the rotated regular patterns. Section 3 presents an application to a facility location problem. The final section summarizes our main results.

2. Nearest and Second Nearest Distance Distributions

Let R_1, R_2 be the rectilinear distances from an arbitrary location in a study region to the nearest and the second nearest points. Let $f_1(r)$, $f_2(r)$ be the probability density functions of R_1, R_2. We call $f_1(r)$, $f_2(r)$ the nearest distance distribution and the second nearest distance distribution, respectively. In this section, we derive $f_1(r)$, $f_2(r)$ for the rotated regular patterns.

Let $F_1(r)$ be the cumulative distribution function of the nearest distance R_1. $F_1(r)$ is the probability that $R_1 \leq r$, which is given by

$$F_1(r) = \frac{S_1(r)}{S} \tag{2}$$

where S and $S_1(r)$ are the area of the study region and the area of the region such that $R_1 \leq r$ in the study region, respectively. Differentiating Eq. (2) with respect to r yields

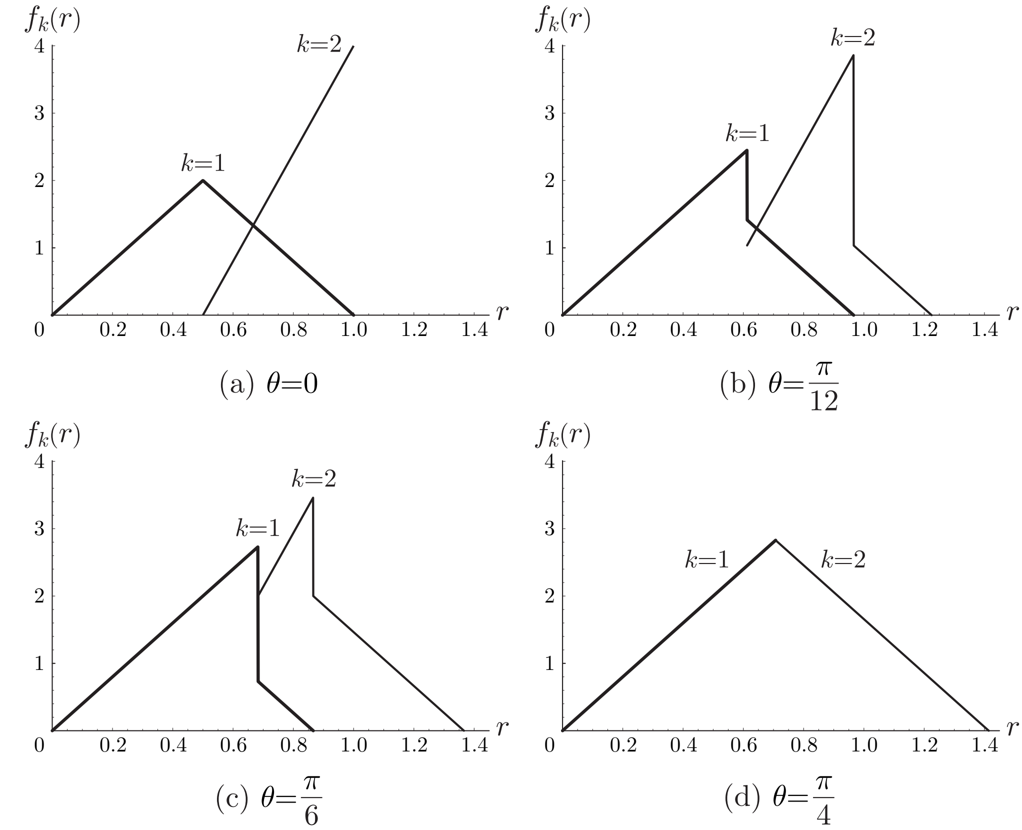


Fig. 4. Nearest and second nearest distance distributions.

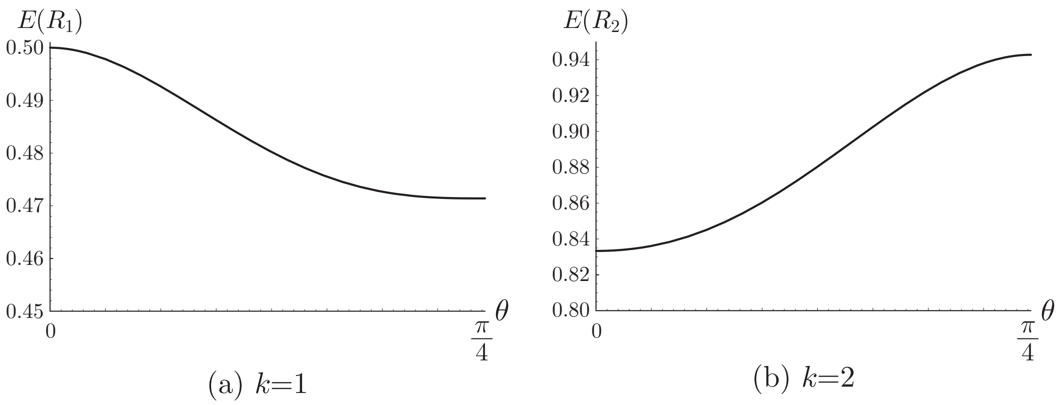


Fig. 5. Average nearest and second nearest distances.

the nearest distance distribution $f_1(r)$ as

$$f_1(r) = \frac{1}{S}\frac{\mathrm{d}S_1(r)}{\mathrm{d}r}. \tag{3}$$

The study region can be confined to the region where a point is the nearest, as shown in Fig. 2. This is because we assume that regular patterns continue infinitely. This region corresponds to a Voronoi polygon with rectilinear distances (see Okabe *et al.*, 2000).

Let a be the distance between two adjacent points. The area of the region in Fig. 2 is $S = a^2$. $S_1(r)$ is the area of the rectilinear circle, which is a square rotated at angle $\pi/4$, centred at the white point with radius r in the region, as shown in Fig. 2. Then we have

$$S_1(r) = \begin{cases} 2r^2 \\ \qquad \left(0 < r \leq \frac{a}{2}(\sin\theta + \cos\theta)\right) \\ a^2 - 2(a\cos\theta - r)^2 \\ \qquad \left(\frac{a}{2}(\sin\theta + \cos\theta) < r \leq a\cos\theta\right). \end{cases} \tag{4}$$

Substituting Eq. (4) and $S = a^2$ into Eq. (3) gives the

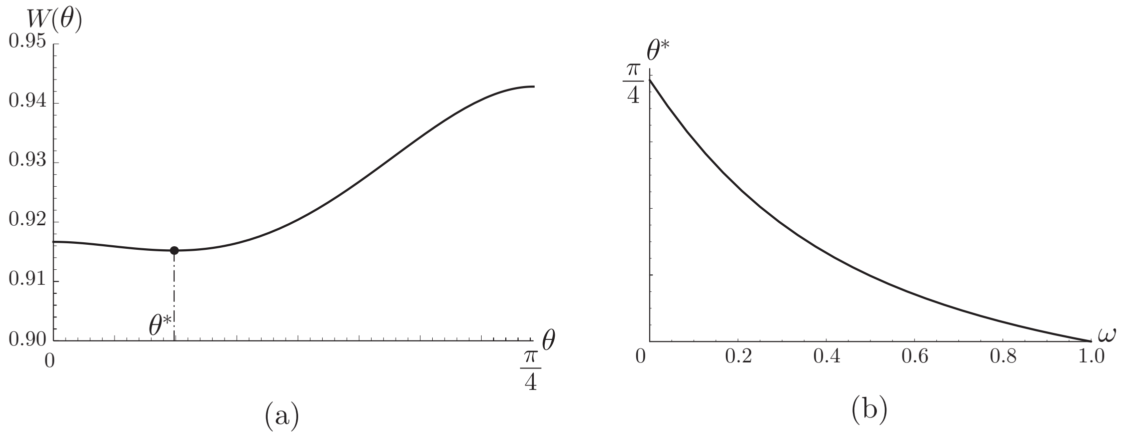


Fig. 6. Weighted sum of the average distances ($\omega = 0.5$) (a) and optimal rotation angle (b).

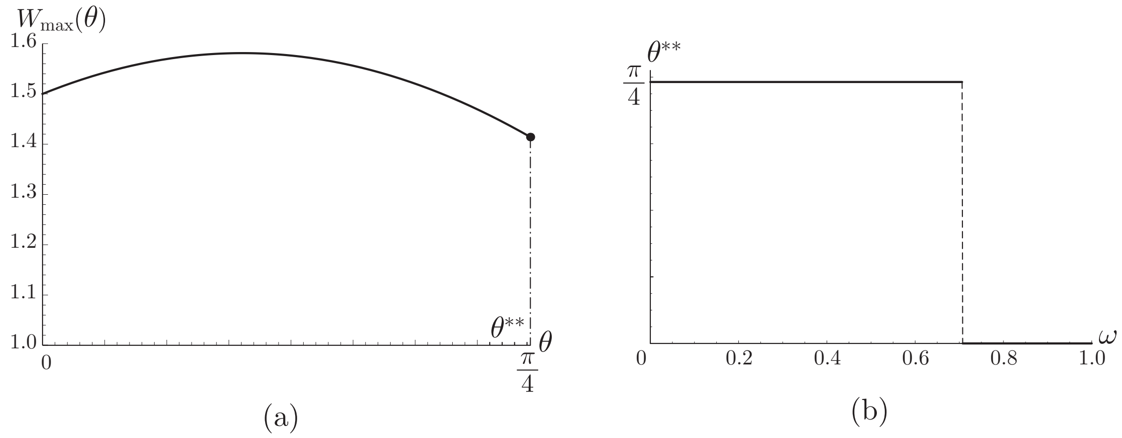


Fig. 7. Weighted sum of the maximum distances ($\omega = 0.5$) (a) and optimal rotation angle (b).

nearest distance distribution $f_1(r)$ as

$$f_1(r) = \begin{cases} 4\rho r \\ \quad \left(0 < r \le \frac{1}{2\sqrt{\rho}}(\sin\theta + \cos\theta)\right) \\ -4\rho r + 4\sqrt{\rho}\cos\theta \\ \quad \left(\frac{1}{2\sqrt{\rho}}(\sin\theta + \cos\theta) < r \le \frac{\cos\theta}{\sqrt{\rho}}\right) \end{cases} \tag{5}$$

where $\rho\ (= 1/a^2)$ is the density of points. If $\theta = 0$ ($\theta = \pi/4$), Eq. (5) reduces to the nearest distance distribution of the square (diamond) lattice obtained in Miyagawa (2008). The average nearest distance $E(R_1)$ is given by

$$\begin{aligned} E(R_1) &= \int_0^\infty r f_1(r)\,\mathrm{d}r \\ &= \frac{1}{6\sqrt{\rho}}(2\sin^3\theta + 3\cos\theta). \end{aligned} \tag{6}$$

The second nearest distance distribution $f_2(r)$ is similarly obtained by calculating the area of the rectilinear circle in the region where a point is the second nearest, as shown in Fig. 3. Then we have

$$f_2(r) = \begin{cases} 8\rho r - 4\sqrt{\rho}\cos\theta \\ \quad \left(\frac{1}{2\sqrt{\rho}}(\sin\theta + \cos\theta) < r \le \frac{\cos\theta}{\sqrt{\rho}}\right) \\ -4\rho r + 4\sqrt{\rho}(\sin\theta + \cos\theta) \\ \quad \left(\frac{\cos\theta}{\sqrt{\rho}} < r \le \frac{1}{\sqrt{\rho}}(\sin\theta + \cos\theta)\right) \end{cases} \tag{7}$$

which under $\theta = 0$ ($\theta = \pi/4$) reduces to the second nearest distance distribution of the square (diamond) lattice obtained in Miyagawa (2008). The average second nearest distance $E(R_2)$ is given by

$$E(R_2) = \frac{1}{6\sqrt{\rho}}\{(2\sin\theta - \cos\theta)(\sin 2\theta + 1) + 6\cos\theta\}. \tag{8}$$

The nearest and the second nearest distance distributions $f_1(r)$, $f_2(r)$ and the average distances $E(R_1)$, $E(R_2)$ are shown in Figs. 4 and 5, where the density of points is $\rho = 1$. Note that $E(R_1)$ has a minimum at $\theta = \pi/4$ (diamond lattice), whereas $E(R_2)$ has a minimum at $\theta = 0$ (square lattice).

3. Application to a Facility Location Problem

In this section, we provide an application of the nearest and the second nearest distances to a facility location problem. Suppose that customers are uniformly distributed on a plane. If all customers are serviced by their nearest facility, the optimal facility location is the diamond lattice ($\theta = \pi/4$), as shown in Larson and Odoni (1981). This is because the average distance to the nearest facility $E(R_1)$ of the diamond lattice is the smallest (see Fig. 5(a)). If some facilities are closed and customers are serviced by their second nearest facility, however, it is uncertain whether or not the diamond lattice is still optimal. Recall that the average distance to the second nearest facility $E(R_2)$ of the diamond lattice is the greatest among the rotated regular patterns (see Fig. 5(b)).

Let us find the rotation angle that minimizes weighted sum of the average distances. The problem is formulated as follows:

$$\begin{aligned}\text{min.}\ W(\theta) &= E(R_1) + \omega E(R_2)\\ &= \frac{1}{6\sqrt{\rho}}\{2\sin^3\theta + 3(1+2\omega)\cos\theta\\ &\quad +\omega(2\sin\theta - \cos\theta)(\sin 2\theta + 1)\}\end{aligned} \tag{9}$$

where ω ($0 \le \omega \le 1$) is a weight. If $\omega = 0$, the distance to the second nearest facility is of no importance. This corresponds to the case where all customers use their nearest facility. As ω increases, the second nearest facility becomes more important. If $\omega = 1$, the second nearest facility is as important as the nearest facility.

The first and second derivatives of $W(\theta)$ are

$$\begin{aligned}\frac{\mathrm{d}W(\theta)}{\mathrm{d}\theta} &= \frac{1}{4\sqrt{\rho}}[(\sin 2\theta + \cos 2\theta - 1)\\ &\quad \cdot\{(1+3\omega)\sin\theta - (1-\omega)\cos\theta\}],\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\mathrm{d}^2W(\theta)}{\mathrm{d}\theta^2} &= \frac{1}{4\sqrt{\rho}}\{(1+\omega)(3\sin 3\theta - \sin\theta)\\ &\quad +6\omega\cos 3\theta - 2(1+2\omega)\cos\theta\},\end{aligned} \tag{11}$$

respectively. From

$$\frac{\mathrm{d}W(\theta)}{\mathrm{d}\theta} = 0 \Leftrightarrow \theta = 0,\ \frac{\pi}{4},\ \arccos\frac{3\omega+1}{\sqrt{10\omega^2+4\omega+2}} \tag{12}$$

and

$$\frac{\mathrm{d}^2W(0)}{\mathrm{d}\theta^2} < 0,\ \frac{\mathrm{d}^2W\left(\frac{\pi}{4}\right)}{\mathrm{d}\theta^2} < 0,\ \frac{\mathrm{d}^2W\left(\arccos\frac{3\omega+1}{\sqrt{10\omega^2+4\omega+2}}\right)}{\mathrm{d}\theta^2} > 0, \tag{13}$$

we have the optimal rotation angle θ^* as

$$\theta^* = \arccos\frac{3\omega+1}{\sqrt{10\omega^2+4\omega+2}}. \tag{14}$$

Figure 6(a) shows the weighted sum of the average distances $W(\theta)$ for $\omega = 0.5$. It can be seen that $W(\theta)$ has a minimum at $\theta^* = \arccos(2.5/\sqrt{6.5}) \approx 0.197$. Figure 6(b) depicts the optimal rotation angle θ^* as a function of weight ω. The optimal angle decreases continuously from $\theta^* = \pi/4$ to $\theta^* = 0$. That is, any rotated pattern as well as the square and diamond lattices has the possibility of being the best depending on the weight.

Next, let us find the rotation angle that minimizes weighted sum of the maximum distances. This is an optimization from an equity point of view. The farthest points from the nearest and the second nearest facilities are shown as arrows in Figs. 2 and 3. Let $U(R_1)$, $U(R_2)$ be the maximum distances to the nearest and the second nearest facilities, respectively. From Eqs. (5) and (7), we have

$$U(R_1) = \frac{\cos\theta}{\sqrt{\rho}} \tag{15}$$

$$U(R_2) = \frac{\sin\theta + \cos\theta}{\sqrt{\rho}}. \tag{16}$$

The problem is formulated as follows:

$$\begin{aligned}\text{min.}\quad W_{\max}(\theta) &= U(R_1) + \omega U(R_2)\\ &= \frac{1}{\sqrt{\rho}}\{\omega\sin\theta + (1+\omega)\cos\theta\}.\end{aligned} \tag{17}$$

$W_{\max}(\theta)$ is concave for $0 \le \theta \le \pi/4$, because

$$\frac{\mathrm{d}^2W_{\max}(\theta)}{\mathrm{d}\theta^2} = -\frac{1}{\sqrt{\rho}}\{\omega\sin\theta + (1+\omega)\cos\theta\} < 0. \tag{18}$$

Then the optimal rotation angle θ^{**} is obtained as

$$\theta^{**} = \begin{cases}\frac{\pi}{4} & \left(0 \le \omega \le \frac{1}{\sqrt{2}}\right)\\ 0 & \left(\frac{1}{\sqrt{2}} < \omega \le 1\right).\end{cases} \tag{19}$$

Figure 7(a) shows the weighted sum of the maximum distances $W_{\max}(\theta)$ for $\omega = 0.5$. It can be seen that $W_{\max}(\theta)$ has a minimum at $\theta^{**} = \pi/4$. Figure 7(b) depicts the optimal rotation angle θ^{**} as a function of weight ω. The best facility pattern is either the diamond lattice or the square lattice. This result makes a clear contrast with the average distance case.

4. Conclusion

This paper has examined the relationship between road directions and the rectilinear distance in regular point patterns. We have derived the distributions of the rectilinear distances to the nearest and the second nearest points in rotated regular point patterns. These distributions demonstrate that road directions significantly affect the rectilinear distances. Road directions should therefore be incorporated into spatial analysis based on rectilinear distances.

As an application of the nearest and the second nearest distances, we have considered a facility location problem in which customers are serviced by the nearest or the second nearest facility. The objectives of the problem are to minimize the weighted sum of the average distances and the weighted sum of the maximum distances. In the former case, any rotated pattern as well as the square and diamond lattices has the possibility of being the best. In the latter case, in contrast, either the diamond lattice or the square lattice is the best. These findings will give an insight into the further studies on facility location problems with closing of facilities.

The assumption that customers are serviced by the nearest or the second nearest facility might be invalid in disastrous situations where many facilities are disrupted simultaneously. The higher order distances will also be required for future research.

Acknowledgments. I wish to thank Takeshi Koshizuka, Yoshiaki Ohsawa, and an anonymous referee for their valuable comments.

References

Aras, N., Orbay, M. and Altinel, I. (2008) Efficient heuristics for the rectilinear distance capacitated multi-facility Weber problem, *Journal of the Operational Research Society*, **59**, 64–79.

Berman, O., Krass, D. and Menezes, M. (2007) Facility reliability issues in network p-median problems: Strategic centralization and co-location effects, *Operations Research*, **55**, 332–350.

Church, R. and Scaparra, M. (2007) Analysis of facility systems' reliability when subject to attack or a natural disaster, in *Critical Infrastructure: Reliability and Vulnerability* (eds. A. Murray and T. Grubesic), pp. 221–241, Springer-Verlag, Berlin.

Church, R., Scaparra, M. and Middleton, R. (2004) Identifying critical infrastructure: The median and covering facility interdiction problems, *Annals of the Association of American Geographers*, **94**, 491–502.

Clark, P. and Evans, F. (1954) Distance to nearest neighbor as a measure of spatial relationships in populations, *Ecology*, **35**, 85–90.

Cressie, N. (1993) *Statistics for Spatial Data*, John Wiley & Sons, New York.

Dacey, M. (1968) Two-dimensional random point patterns: A review and an interpretation, *Papers of the Regional Science Association*, **13**, 41–55.

Drezner, Z. (1987) Heuristic solution methods for two location problems with unreliable facilities, *Journal of the Operational Research Society*, **38**, 509–514.

Drezner, Z. and Hamacher, H. (eds.) (2002) *Facility Location: Applications and Theory*, Springer-Verlag, Berlin.

Francis, R., McGinnes, F. and White, J. (1992) *Facility Layout and Location: An Analytical Approach*, Prentice Hall, Upper Saddle River.

Holgate, P. (1965) The distance from a random point to the nearest point of a closely packed lattice, *Biometrika*, **52**, 261–263.

Koshizuka, T. (1985) On the relation between the density of urban facilities and the distance to the nearest facility from a point in a given area, *Journal of the City Planning Institute of Japan*, **20**, 85–90 (in Japanese).

Larson, R. and Odoni, A. (1981) *Urban Operations Research*, Prentice-Hall, Englewood Cliffs.

Macias, R. and Perez, J. (1995) Competitive location with rectilinear distances, *European Journal of Operational Research*, **80**, 77–85.

Miyagawa, M. (2008) Analysis of facility location using ordered rectilinear distance in regular point patterns, *FORMA*, **23**, 89–95.

Miyagawa, M. (2009) Order distance in regular point patterns, *Geographical Analysis*, **41**, 252–262.

Okabe, A., Boots, B., Sugihara, K. and Chiu, S. (2000) *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, John Wiley & Sons, Chichester.

Persson, O. (1964) Distance methods: the use of distance measurements in the estimation of seedling density and open space frequency, *Studia Forestalia Suecica*, **15**, 1–68.

Pirkul, H. (1989) The uncapacitated facility location problem with primary and secondary facility requirements, *IIE Transactions*, **21**, 337–348.

Snyder, L. (2006) Facility location under uncertainty: a review, *IIE Transactions*, **38**, 537–554.

Snyder, L. and Daskin, M. (2005) Reliability models for facility location: The expected failure cost case, *Transportation Science*, **39**, 400–416.

Taaffe, E., Gauthier, H. and O'Kelly, M. (1996) *Geography of Transportation*, Prentice Hall, Upper Saddle River.

Vaughan, R. (1987) *Urban Spatial Traffic Patterns*, Pion, London.

Weaver, J. and Church, R. (1985) A median location model with nonclosest facility service, *Transportation Science*, **19**, 58–74.