A Fluid-Erosion-Based Model of Landscape Evolution

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A new landscape evolution model is proposed which is composed of the shallow water equations for the fluid above the sediment and the mass conservation equation of the sediment. Numerical simulations of the formation of landscape and river network are carried out based on these equations. It is shown that steady patterns of river network are formed for the initial inclinations of slopes within 0.00005 and 0.005. The fractal dimensions of the river network and the exponent of Hack's law are obtained, which are consistent with values from observation.
Key words: Landscape Modeling, River Network, Pattern Formation, Fractal, Hack's Law

1. Introduction

Complex patterns are commonly found throughout nature, from snow crystals to leaf veins. Such pattern is the fractal, a complex geometric pattern that can be subdivided into parts, each of which is a smaller copy of the whole. Examples of fractals include landscapes, as well as snow crystals and leaf veins. The fractal dimension of a ridge-valley pattern in the nature is 1.7∼1.9. Such a complex pattern can be generated in a laboratory by pouring water on sand (Wittmann *et al.*, 1991). On the topic of fractal patterns, let us consider a river network, which is characterized by Hack's law as Eq. (1).

$$L \propto A^{0.6}. \tag{1}$$

This law gives the empirical relationship between the longest stream length L and the drainage area A for a given point on a river. The stream length is defined as the length which is measured headward from a given point on a stream to the divide. Because the river is composed of many streams there are many stream lengths going through the point, and L is defined as the longest among them. From a simple dimensional analysis the power index should be 0.5. However, the power index in Hack's law is 0.6. This implies that a river is neither smooth line nor plane but a complex pattern with a fractal nature. However, the mechanism underlying this law is not clear yet.

In the past, several models of landscape evolution have been proposed. Howard (1994) used a detachment-limited model, which combines mass movement and sediment transport, to simulate the steady state of landscape. Willgoose *et al.* (1991) used "the channelization equation" governing the development of channels. This is based on a biological model of leaf reticulation. Tucker *et al.* (2001) proposed simple numerical algorithms and transport equations to model the landscape network. The discrete models of river network, e.g. a random walk model, are also proposed (Smart and Morzzi, 1971; Stark, 1991; Coulthard *et al.*, 2002; Luo *et al.*, 2004). They can be easily simulated on a computer because the differential equations need not to be solved. However, none of the models mentioned above consider fluid dynamics, that is, it is impossible for these models to evaluate fluid motion such as flow velocity.

In this study, we consider the simple phenomenon of water flowing down a slope and develop a simple model that can produce a complex pattern.

2. A New Model of Landscape Evolution

Here, we propose a landscape evolution model that focuses on the erosion by fluid motion. Figure 1 shows the concept of the model. The velocity of the fluid flowing above the sediment is denoted by $\boldsymbol{u} = (u, v)$, the depth of the fluid by h, and the altitude of the bottom sediment by z. Here, x and y are the lateral and streamwise coordinates in the horizontal plane, and t denotes the time. By assuming that the depth of the fluid layer is small enough compared with the characteristic spatial variation in the flow velocity, the flow obeys the shallow water equations, which are given as follows (Izumi and Parker, 2000):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial (h+z)}{\partial x} - \frac{\tau_x}{\rho h} \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial (h+z)}{\partial y} - \frac{\tau_y}{\rho h} \tag{3}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0 \tag{4}$$

$$(\tau_x, \tau_y) = \rho C_f |\boldsymbol{u}|(u, v) \tag{5}$$

where g, ρ, and C_f denote the gravitational acceleration, the density of water, and the friction coefficient, respectively. (τ_x, τ_y) are the boundary shear stress. The friction coefficient C_f is a function of flow depth and the roughness, but in this model it is assumed to be a constant for simplicity. In general it is known C_f takes the order of 0.01 (Richards, 1982).

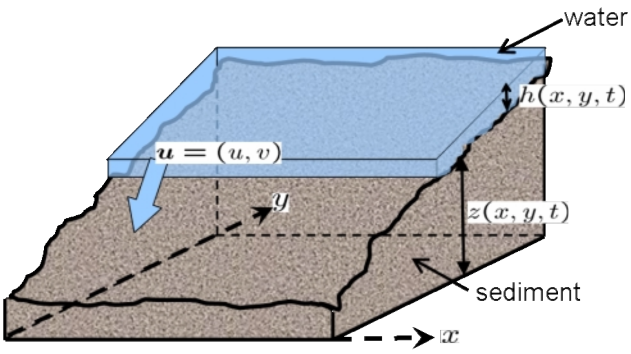

Fig. 1. Definitions of variables in the fluid erosion.

Table 1. Parameters used in the simulation.

Parameter	Value
c_f	0.01
D_f	0.005
k	1.2
α	0.8
β	0.7
h_0	0.0001
Δx	0.1
Δy	0.1
Δt	0.1
N_x	256
N_y	512

For the sake of simplicity, we do not consider the mixed flow dynamics of water and sediment. Evolution of the elevation z is expressed by

$$\frac{\partial z}{\partial t} = -q_s + D_f \nabla^2 z \tag{6}$$

$$q_s = k Q^\alpha S^\beta \tag{7}$$

where q_s is the rate of eroded sediment; D_f is the diffusion coefficient; and k, α, and β are constants. Q and S denote the discharge, i.e. the local flow rate of the fluid phase, and the local slope gradient, respectively, such that

$$Q = h\ |\boldsymbol{u}|/h_0 \tag{8}$$

$$S = \sqrt{(\partial z/\partial x)^2 + (\partial z/\partial y)^2}. \tag{9}$$

The first term on the right-hand side of Eq. (6) is associated with channel erosion, and the second term is associated with diffusion. Eq. (7) is based on a detachment-limited model (Howard, 1994). In a detachment-limited model q_s is propotional to $A^\alpha S^\beta$, where A denotes the drainage area. However, because it is hard to calculate the drainage area A in every numerical step, in our model the discharge Q is used instead of A. Note that the relation between the drainage area and the discharge is $Q \propto A$ (Leopold *et al.*, 1964). The exponents α, β, and the coefficient k are 0.8, 0.7, and 1.2, respectively, based on Howard (1994).

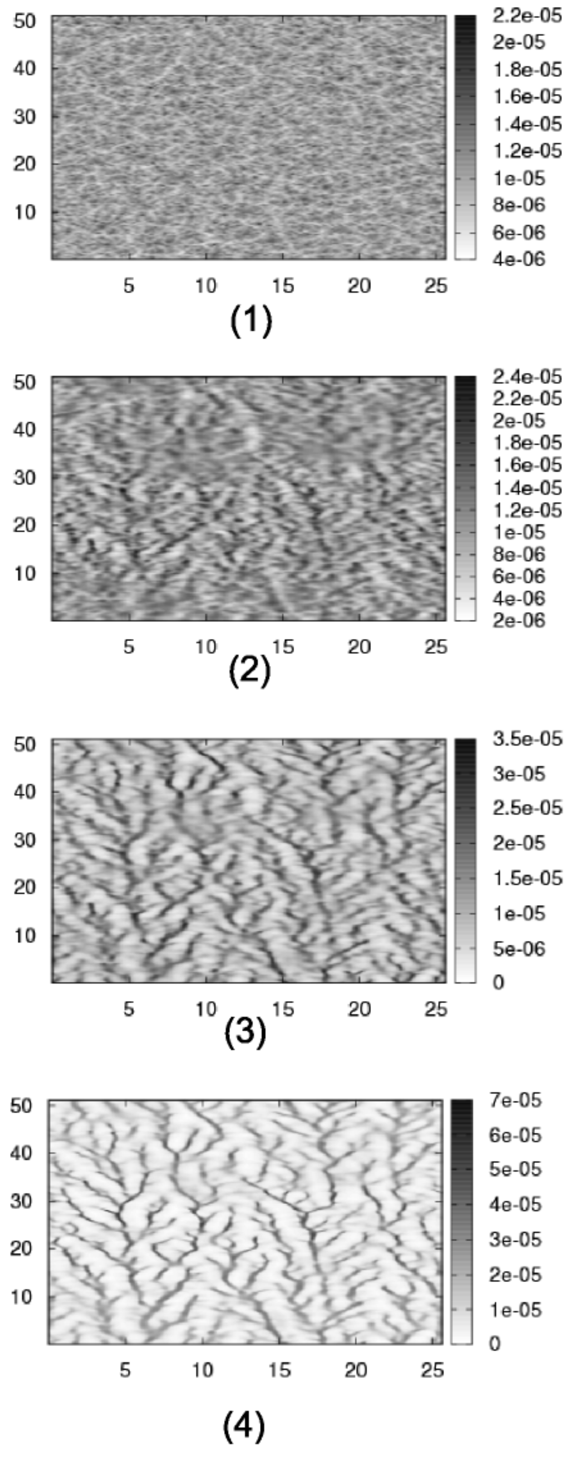

Fig. 2. Visualization of distributions of the water depth for the initial gradient $S_0 = 0.0001$. The darker areas indicate deeper region. (1) $t = 100$, (2) $t = 1100$, (3) $t = 1800$, and (4) $t = 2300$.

3. Numerical Calculation

We apply the CIP (cubic interpolated pseudoparticle) method Ogata and Yabe, 2004) to the advection term in the shallow water equations (2)–(4). We apply the ADI (alternating direction implicit) method and the cyclic reduction algorithm to the diffusion term in Eq. (6), the central difference in space, and the forward difference in time. The grid sizes in the x and y directions are Δx and Δy, and the number of grid points is given by $N_x \times N_y$.

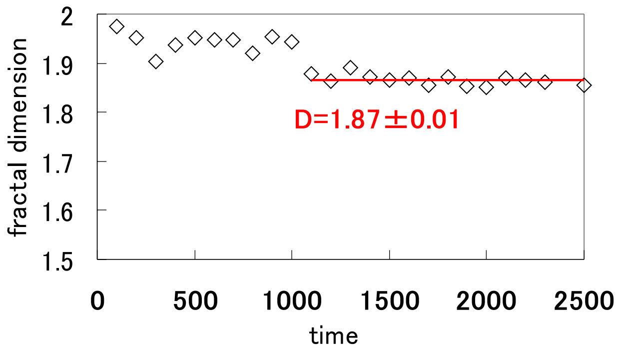

Fig. 3. Time variation of the fractal dimension for $S_0 = 0.0001$. The average value of the fractal dimension for $t \geq 1100$ is 1.87.

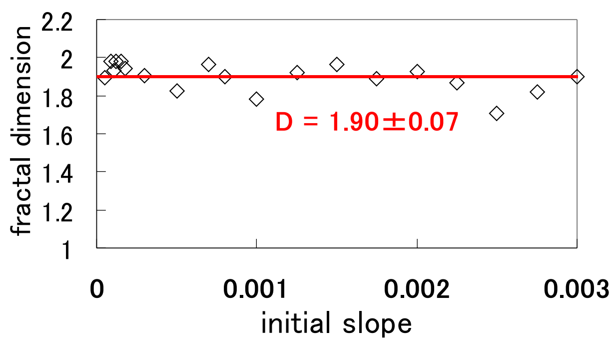

Fig. 4. Fractal dimension as a function of the initial slope.

As an initial condition, we add a minute disturbance

$$\tilde{z}(x, y) = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} a_{ij} \sin \frac{2\pi i x}{L_x} \cos \frac{2\pi j y}{L_y} \tag{10}$$

to the inclined plane $z(x, y) = z(0, 0) + S_0 y$, where S_0 is the initial slope and a_{ij} is a uniform random factor with a maximum value of 0.0001. Note that $\tilde{z}$ has some smoothness because of omitting high wave number modes. A sheet flow with the depth h_0 and zero velocity are set on the surface initially.

We use the periodic boundary conditions of u, v, h, and $z(x, y) - S_0 y$ in the x and y directions with periods $L_x = N_x \Delta x$ and $L_y = N_y \Delta y$. The values of the parameters are listed in Table 1.

4. Simulation Results

Figure 2 shows the simulation result for the initial value of the slope 0.0001. A complex pattern appears gradually as time passes; finally, a steady pattern is obtained. We convert the data of elevation z, shown in Fig. 2(4), into a binary image in order to calculate the fractal dimesion D using the box couting method. The fractal dimension D of this pattern becomes constant for large time, and the time-averaged value of D is about 1.87 for $t \geq 1100$ (Fig. 3). Moreover, the dimension remains almost constant at 1.90 when the initial slope is changed (Fig. 4). This corresponds to the phenomenon in which similar branching patterns appear in various geographical features. However, when the initial slope is smaller than 0.00005, the pattern is not clearly formed because erosion does not occur and the fractal dimension is not computable. In contrast, when the initial slope is steeper than 0.005 the pattern consists of straight streams; a complex pattern is not formed because of the large gravitational force and thus the fractal dimension is one.

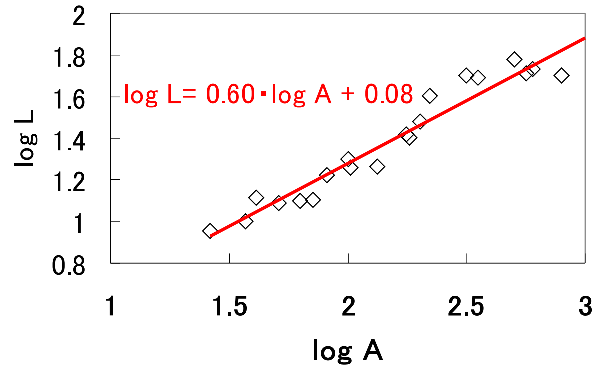

Fig. 5. Relationship between drainage area A and stream length L. This result shows the Hack's law.

Next, we consider the relationship between the longest stream length L and the drainage area A. The longest stream length is easily determined from the binary image. The drainage area for the longest stream is calculated as follows. First, a marker is placed on every grid point and the heights of eight neighbors are compared. Next, the markers are moved to the lowest point among the eight neighbors, and this procedure is repeated. In most cases, the marker reaches the outlet, while in a few cases, it stops before it reaches the outlet. In any case, one counts the marker in the drainage area if it is in the drainage basin of the stream considered. Finally, the drainage area A is obtained as the number of markers in the area related to the stream considered. As shown in Fig. 5, this result is consistent with Hack's law (1). The discrepancy between Hack's law and the simulation results for $\log L \geq 1.6$ arises due to the periodic boundary condition in the y direction. Since the size of region in the y drection L_y is 51.2, i.e. $\log L_y = 1.7$, this discrepancy is reasonable when L is large. Note that Fig. 5 is plotted for various initial slopes from 0.00005 to 0.003.

5. Conclusion

In this study, we propose a simple model for determining landscape evolution; this model employs shallow water equations and the mass conservation of sediment. In the simulation, this model generates steady pattern of river network. Numerical simulation shows that the fractal dimension of the river pattern is 1.90, which is independent of the initial slope and is close to the value of 1.7∼1.9 in the geographical features. This result is in good agreement with Hack's law, and hence, the proposed model is appropriate for pattern generation. Therefore, it is clarified that the mechansm responsible for the complex pattern is mass movement by fluid motion. The present model is simple, but rather impractical. However, more realistic effects such as non-periodic conditions and precipitation can be easily incorporated into the model. In future, we intend to explore the precise mechanism of fractal patterns and Hack's law.

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