

Supplementary Results for Length Distributions in Planar STIT Tessellations

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Formulas for the joint distribution of direction and length of the typical K - and J -segment in a planar, homogeneous, anisotropic, random tessellations that are stable under iteration (so-called STIT tessellations) are derived. They were announced in Thäle (2009), supplement the results from Mecke (2009) and Mecke *et al.* (2007) and complete the picture concerning length distributions of segments in the planar case.

Key words: Length Distribution, Iteration/Nesting, Random Tessellation, STIT Tessellation, Stochastic Geometry

1. Introduction

Since their introduction by Nagel and Weiß in the seminal paper (Nagel and Weiss, 2005), homogeneous (spatially stationary, that is stochastically translation invariant) iteration stable tessellations, STIT tessellations for short, have attracted considerable interest in stochastic geometry. In the planar case, these tessellations may be constructed in any bounded convex polygon $P \subset \mathbb{R}^2$ according to the following procedure: Let ϑ be a probability measure on the space \mathcal{H} of lines through the origin, such that $\vartheta(\{h\}) < 1$ for any $h \in \mathcal{H}$. Further, let μ be the induced translation invariant measure on the space \mathcal{G} of all lines in the plane and $L_A > 0$ be a fixed real number. To the polygon P , a random lifetime is assigned, which is exponentially distributed with parameter $\mu(\{g \in \mathcal{G} : g \cap P \neq \emptyset\})$. In the special case, where ϑ is the uniform distribution on \mathcal{H} , $\mu(\{g \in \mathcal{G} : g \cap P \neq \emptyset\})$ equals $1/\pi$ times the perimeter length of P . Upon expiry of the random lifetime of P , a random chord g is chosen according to the normalized measure μ restricted to $\{g \in \mathcal{G} : g \cap P \neq \emptyset\}$ and is introduced in P . Thus, the polygon P is split into two polygonal sub-cells P^+ and P^- . Now, the construction continues recursively and independently in both of the new cells P^+ and P^- until the time threshold L_A is reached, i.e. independent and exponentially random lifetimes are assigned to these two cells, random chords are chosen, which divide them further. It is important that each chord is chopped-off by the boundary of its mother cell. An illustration of the construction process is provided by Fig. 1. One of the main features of this class of tessellations is the property that the cells of a homogeneous STIT tessellation are not face-to-face, which means for example that a side of a cell can contain further nodes in its relative interior. This implies that there is more than one meaningful notion for segments of the tessellation. In the planar case, three possible types of line segments, so called I -, J - and K -segments (for definitions see below) were introduced in Mackisack and Miles (1996) and Miles and Mackisack (2002). Moreover, in Miles and Mackisack (2002) a model for homogeneous random pla-

nar tessellations was investigated, whose cells are also not face-to-face. It was shown there that the length distribution of the different types of segments are exponential distributions with different parameters. We will see below that this is not the case for STIT tessellations. Here, more involved distributions appear.

For homogeneous and isotropic random STIT tessellations, the length distributions of the three possible types of segments were determined by a technique related to balance equations. It was J. Mecke in Mecke (2009) who developed this method further and gave a formula for the *joint* distribution of direction and length of the typical I -segment of a homogeneous but anisotropic planar STIT tessellation. In the recent paper (Thäle, 2009), formulas for the joint distribution of direction and length of the typical K - and J -segment of such tessellations were announced and have already been used for a refined analysis of direction-dependent moments and moment relations. It is the aim of this note to give a proof of them in the spirit of Mecke (2009) and to make further advertisement for the this new tessellation model. Therefore, this note can be seen as a supplement to both of the papers Mecke (2009) and Thäle (2009).

For background material on random STIT tessellations we refer to the seminal paper (Nagel and Weiss, 2005) and also to the section on planar STIT tessellations in Thäle (2009). The notation used here are mostly adapted to that from Thäle (2009). As a general reference for stochastic geometry we cite the book (Stoyan *et al.*, 1995). We emphasize that by a homogeneous random tessellation in the plane we understand in this note a special planar random closed set, which is the union of cell boundaries that form a particle process of convex particles that have pairwise no common interior points and cover the whole plane.

We would like to remark that the development and the analysis of random tessellations whose cells are not face-to-face is of great importance. Applications of such structures in geology were motivated in Gray *et al.* (1976), where the geometry of so-called crack networks is discussed. One could imagine for example a terrain, in which cracks open

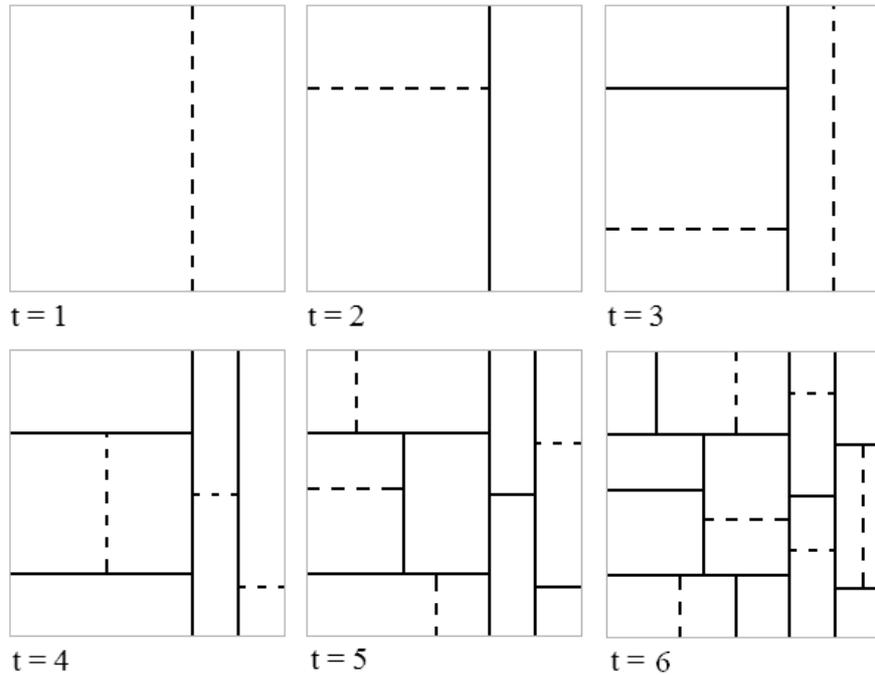


Fig. 1. States of the random cell division process for different time instants $t \in (0, 6]$ and for ϑ concentrated with equal weight only on two orthogonal directions; the respective new segments are dashed.

because of biochemical and mechanical processes. However, the appearing cracks have to stop at already existing ones, thus, the crack formation develops independently in the different regions of the terrain. Whereas modelling with line or Voronoi tessellations (which are the two standard models in stochastic geometry) is obviously inadequate, STIT tessellations have the potential to serve as new idealized mathematical model for such phenomena.

2. Background Material

The law of a homogeneous random STIT tessellation Φ in the plane is uniquely determined by the **edge length intensity** $L_A > 0$ of the tessellation, i.e. the mean edge length per unit area, and by its directional measure κ or directional distribution ϑ , see Nagel and Weiss (2005). Denote by $[\mathcal{H}, \mathfrak{H}]$ the measurable space of lines in \mathbb{R}^2 containing the origin and let for $B \in \mathfrak{B}$ (the Borel σ -field in \mathbb{R}^2), $\kappa(B)$ be the edge length intensity of those edges of Φ with direction in B , where by the **direction** $r(s)$ of a line segment s we understand the parallel line $r(g) \in \mathcal{H}$ through the origin of the line g containing the segment s . The mapping $B \mapsto \kappa(B)$ defines the **directional measure** κ of Φ on $[\mathcal{H}, \mathfrak{H}]$, which may be written as $\kappa = L_A \vartheta$ for a probability measure ϑ on the same space, the **directional distribution** of the tessellation. The probability measure ϑ can be interpreted as distribution of the direction of the segment through the *typical* point of Φ , when the tessellation is regarded under the Palm distribution of Φ with respect to the length measure induced by the tessellation, see Stoyan *et al.* (1995) for details.

It is assumed from now on that $L_A > 0$ and that ϑ is not concentrated on a single line in $[\mathcal{H}, \mathfrak{H}]$, a condition which ensures the existence of a homogeneous random STIT tes-

sellation in the plane with edge length intensity L_A and direction distribution (measure) ϑ (κ), see Nagel and Weiss (2005).

For fixed directional measure κ define the **rose of intersections** of κ by

$$s_\kappa(h) := \int_{\mathcal{H}} |\sin(h, h')| \kappa(dh'), \quad h \in \mathcal{H}$$

and furthermore the constant ζ_κ by

$$\zeta_\kappa := \int_{\mathcal{H}} \int_{\mathcal{H}} |\sin(h, h')| \kappa(dh) \kappa(dh'),$$

where by $\sin(h, h')$ we understand the sinus of the intersection angle between the two lines h and h' .

Recall from Nagel and Weiss (2005), that the intersection of Φ with an arbitrary line g in the plane induces a homogeneous random Poisson point process on that line with intensity $s_\kappa(r(g))$, where $r(g) \in \mathcal{H}$ is the direction of g , i.e. the line parallel to g containing the origin \mathcal{O} . Recall further that the cells of a random STIT tessellation are not face-to-face, which means that the intersection of two neighbouring cells that meet in a common line segment is not necessarily a side of both of the cells. This causes that there appear different types of line segments for the tessellation Φ , see Fig. 3.

By a **K -segment** we mean a linear segment of the tessellation which is bounded by nodes but with no further node in its relative interior, whereas by an **I -segment** we mean the maximal union of collinear and connected K -segments that cannot be enlarged by another K -segment. A **J -segment** is simply a side of a cell, see Fig. 3. The technique of Palm distribution, for which we refer to Stoyan *et al.* (1995), can now be used to define the **typical K -, I - and J -segment** of Φ and whenever the term “typical” appears, it refers to such a definition. We can also define the

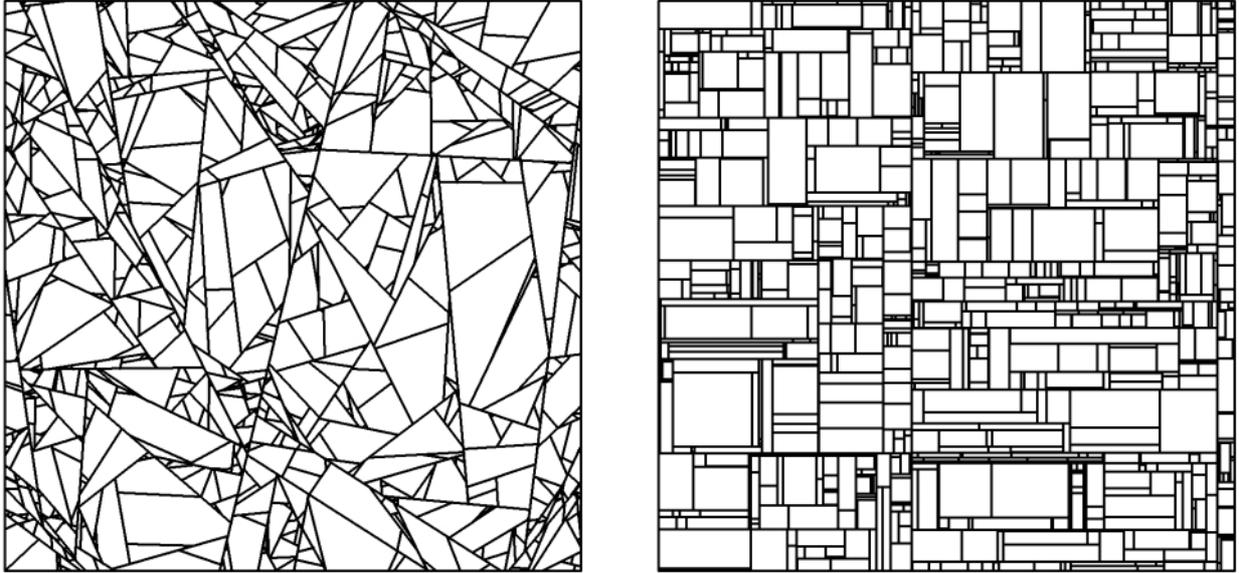


Fig. 2. Realizations of planar stationary STIT tessellations with different directional distributions (kindly provided by Joachim Ohser).

directional distribution or the distribution of the direction of the typical K -, I - or J -segment to be the distribution of the line in \mathcal{H} parallel to the line through the typical K -, I - or J -segment, respectively.

With these notions and notation we can now recall the main result from Mecke (2009):

The directional distribution of the typical I -segment has the density $\frac{1}{\zeta_\kappa} s_\kappa(\cdot)$ with respect to κ and the joint distribution of direction and length of the typical I -segment has the density

$$(h, x) \mapsto \frac{2}{\zeta_\kappa s_\kappa(h)} \int_0^{s_\kappa(h)} t^2 e^{-tx} dt, \quad h \in \mathcal{H}, \quad x > 0$$

with respect to the product measure $\kappa \otimes \ell_+$, where ℓ_+ stands for the Lebesgue measure on the positive real half-axis.

For completeness and for later reference, let us recall the concept of *iteration of tessellations*. Let Φ be a homogeneous random planar tessellation with cells C_k , $k \in \mathbb{N}$, and let $(\Phi_k)_{k \in \mathbb{N}}$ a family of independent and identically distributed random tessellation, which is also independent of Φ and for which each Φ_k has the same distribution as Φ . We define the iteration of Φ with the the sequence $(\Phi_k)_{k \in \mathbb{N}}$ to be the random tessellation

$$\Phi \cup \bigcup_{k \in \mathbb{N}} (\Phi_k \cap C_k).$$

The resulting tessellation will be denoted by $\Phi \odot \Phi$. The idea behind the definition is to associate to each cell C_k of the frame tessellation Φ a component tessellation Φ_k and to subdivide each cell C_k by its component tessellation Φ_k , i.e. to make local superpositions of the tessellations Φ and Φ_k inside the cells C_k . In this sense, a homogeneous random tessellation Φ is called *iteration stable*, if $2(\Phi \odot \Phi)$ has the same distribution as Φ , where $2(\Phi \odot \Phi)$ stands for the random tessellation $\Phi \odot \Phi$ dilated by a factor 2. In

other words, Φ is iteration stable if its distribution does not change under *rescaled* iteration. It was shown in Nagel and Weiss (2005) that for finite areas, the class of STIT tessellations as constructed in the Introduction is exactly the same as the class of iteration stable random tessellations. The stochastic stability of STIT tessellations under rescaled iteration will be the crucial tool to derive a balance equation for a certain distribution function related to the length and directional distribution of the typical K -segment below.

3. The Result for the Typical K -Segment

We establish in this section a similar formula for the joint density of direction and length of the typical K -segment as that one for the typical I -segment and we will use to this end the technique developed in Mecke (2009). We fix from now on a homogeneous random STIT tessellation in the plane with edge length intensity $L_A > 0$ and directional measure κ . From κ we can derive a translation invariant measure on the space $[\mathcal{G}, \mathfrak{G}]$ of all lines in the plane by putting $\mu := \kappa \otimes \ell_+$, where we have used the parametrization of a line $g \in \mathcal{G}$ by its direction $r(g) \in \mathcal{H}$ —the factor κ —and its distance from the origin—the factor ℓ_+ . We define now a new length measure l_κ in the plane by

$$l_\kappa(s) := \mu\{g \in \mathcal{G} : g \cap s \neq \emptyset\} = s_\kappa(h)l(s)$$

for any line segment $s \subset \mathbb{R}^2$, where $h \in \mathcal{H}$ is parallel to $g \in \mathcal{G}$, $s \subset g$, and l denotes the usual Euclidean length. We will call $l_\kappa(s)$ the κ -**length** of the segment s . Observe, that the intersection property of STIT tessellations mentioned above can now be formulated as follows: *The intersection of a homogeneous random STIT tessellation having directional measure κ with a line $g \in \mathcal{G}$ is a homogeneous random point process on g with κ -intensity 1, i.e. the mean number of points per unit κ -length equals 1.*

Let Φ be the previously fixed random STIT tessellation

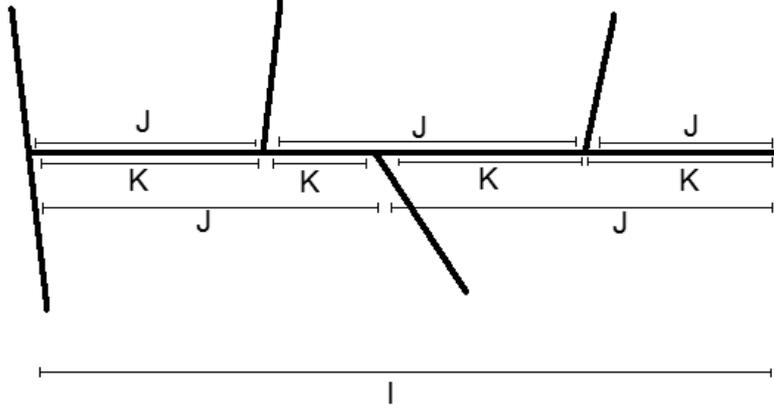


Fig. 3. Different types of segments in planar STIT tessellations.

and consider the Palm distribution

$$\mathbb{Q}(C) := \frac{1}{L_A} \iint \mathbf{1}_{[0,1]^2}(z) \mathbf{1}_C(\phi - z) dz P_\Phi(d\phi)$$

of Φ , where C is a closed subset of \mathbb{R}^2 and P_Φ denotes the distribution of the tessellation Φ (cf. Stoyan *et al.*, 1995). Note that \mathbb{Q} is the Palm distribution of Φ with respect to the length measure induced by Φ . Under the distribution \mathbb{Q} we have $\mathcal{O} \in \Phi$ (\mathcal{O} denotes again the origin), which is to say \mathcal{O} belongs with probability one to the tessellation under \mathbb{Q} . In this case, \mathcal{O} can be interpreted as the typical point of the tessellation. In this sense, the distribution \mathbb{Q} may be interpreted as the conditional distribution of the tessellation under the condition that the origin is a point of its edge-network. Let now K be the almost surely uniquely determined K -Segment of Φ under \mathbb{Q} with $\mathcal{O} \in K$ (or alternatively the K -segment through the typical point of Φ). The **typical remaining K -Segment** K_r is defined as the intersection of K with the closed upper half-plane (or alternatively as the intersection with the half-space bounded by the line parallel through the x -axis that contains the typical point). The κ -length of K_r will be denoted by $l_\kappa(K_r)$. For $B \in \mathfrak{F}$ and $x \in (0, \infty)$ we denote by

$$G_K(B, x) = \mathbb{Q}(r(K_r) \in B; l_\kappa(K_r) \geq x)$$

the probability that the direction of the typical remaining K -segment K_r is in B and its κ -length exceeds x (when Φ is regarded under the Palm distribution \mathbb{Q}). Let for $s, t > 0$ us think of the rescaled STIT tessellation $\frac{1}{s+t}\Phi$ as a frame tessellation, whose cells are subdivided during an iteration process by tessellations having the same distribution as $\frac{1}{t}\Phi$. The result is—because of the stochastic iteration stability property of the tessellation Φ —a homogeneous random tessellation with the same law as $\frac{1}{s+t}\Phi$.

We will regard now the typical remaining K -segment during the iteration $\frac{1}{s+t}\Phi \odot \frac{1}{t}\Phi$, where by \odot we denote the operation of iteration of tessellations. Note that in contrast to the case of the typical remaining I -segment considered in Mecke (2009), the typical remaining K -segment can become shorter during the iteration process.

With probability $\frac{s}{s+t}$ the typical point of the tessellation with the same distribution as $\frac{1}{s+t}\Phi$ (again with respect to the

Palm distribution of $\frac{1}{s+t}\Phi$ induced by the length measure of this tessellation) lies on the frame $\frac{1}{s}\Phi$ and with probability $\frac{t}{s+t}$ on a tessellation nested inside the cells of the frame tessellation. Thus, the distribution we are seeking for is a mixture with weights $\frac{s}{s+t}$ and $\frac{t}{s+t}$ of the distribution of the typical remaining K -segment of the frame tessellation and the distribution of the truncated typical remaining K -segment of the nested tessellations, respectively.

Consider now the line $g_r \in \mathcal{G}$ through K_r and the two cells adjacent to K_r . Inside these two cells we iterate tessellations and each of them induces a homogeneous Poisson point process on g_r with κ -intensity 1. The superposition of these two processes is again a homogeneous Poisson point process on g_r . It has κ -intensity 2, i.e. the κ -length between two points on g_r is exponentially distributed with parameter 2. Now, the typical remaining K -segment of $\frac{1}{s+t}\Phi$ is either the segment from \mathcal{O} to the next point of this Poisson point process or the typical truncated remaining K -segment of a tessellation with the same distribution as $\frac{1}{t}\Phi$, which was nested inside the cells of the frame tessellation, see Fig. 4. Taking into account the STIT property and the scaling factors s and t we arrive at the following crucial functional equation for G_K , namely

$$G_K(B, (s+t)x) = \frac{s}{s+t} G_K(B, sx) e^{-2tx} + \frac{t}{s+t} G_K(B, tx) e^{-sx}. \quad (1)$$

For its solution we introduce the two functions

$$\begin{aligned} u(y) &:= y G_K(B, y), \\ v(y) &:= u(y) e^y \end{aligned}$$

and the abbreviations $a := sx$ and $b := tx$. A simple calculation shows that (1) is equivalent to

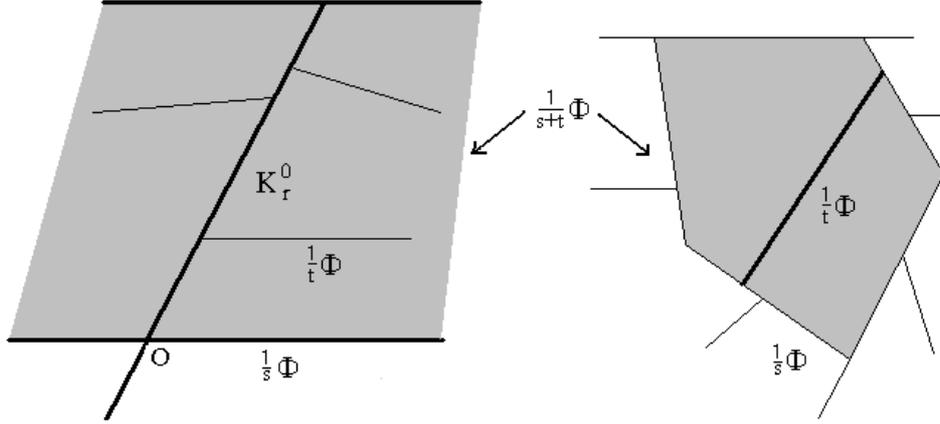
$$u(a+b) = u(a) e^{-2b} + u(b) e^{-a}$$

or—by multiplying with the factor e^{a+b} —to

$$v(a+b) = v(a) e^{-b} + v(b).$$

Interchanging the rôle of a and b we obtain $v(b) e^{-a} + v(a) = v(a+b) = v(a) e^{-b} + v(b)$, whence

$$\frac{v(a)}{e^{-a} - 1} = \frac{v(b)}{e^{-b} - 1}$$

Fig. 4. The segment K_r under the operation of iteration.

and we see that the expression $v(a)/(e^{-a} - 1)$ does not depend on a (the same of course for b) and must therefore be a constant, which can depend on B . From the definition of G_K it follows immediately (let x tend to 0) that this constant must be equal to $\vartheta(B)$, the probability that the direction of K_r is in B . From this fact we deduce that the unique solution of the functional equation (1) is given by

$$G_K(B, x) = \vartheta(B) \frac{1 - e^{-x}}{x} e^{-x},$$

where ϑ is the directional distribution of Φ . Here, uniqueness is rather easy to show by contradiction and can also be deduced from Mecke *et al.* (2007, lemma 6). Thus, we obtain as a partial result: The direction and the κ -length of the typical remaining K -segment of the STIT tessellation Φ are independent. Moreover, the tail function of the κ -length of K_r is given by

$$x \mapsto \frac{1 - e^{-x}}{x} e^{-x}.$$

The distribution of the direction of K_r coincides with ϑ .

Replacing the κ -length by the usual Euclidean length, we obtain

$$\hat{G}_K(B, x) = \int_B \frac{1 - e^{-s_\kappa(h)x}}{s_\kappa(h)x} e^{-s_\kappa(h)x} \vartheta(dh)$$

with $\hat{G}_K(B, x) = \mathbb{Q}(r(K_r) \in B; l(K_r) > x)$, $B \in \mathfrak{H}$ and $x > 0$. From the last formula it is not hard to see that the joint distribution of direction and Euclidean length of the typical remaining K -segment has the density

$$(h, x) \mapsto s_\kappa(h) \int_1^2 t e^{-s_\kappa(h)tx} dt \quad (2)$$

with respect to the product measure $\vartheta \otimes \ell_+$. In the calculation we have used that fact that

$$\int_1^2 t e^{-ut} dt = \frac{1}{u^2} (1 + u - (1 + 2u)e^{-u}) e^{-u}$$

for some real parameter u .

Denote now by D_K be the joint distribution of direction and length of the typical K -segment and by D'_K the same

distribution for the typical remaining K -segment K_r . It follows from the general theory of Palm distributions that for any measurable function $f: \mathcal{H} \times (0, \infty) \rightarrow [0, \infty)$ we have

$$\begin{aligned} & \int f(h, x) dD'_K(h, x) \\ &= \frac{3\zeta_\kappa}{2L_A} \iint_0^x f(h, y) dy dD_K(h, x), \end{aligned} \quad (3)$$

see Stoyan *et al.* (1995) for the technical framework and Mecke (2009) for the related I -segment result, as well as the references given therein. Note in particular that the prefactor of the right-hand side of (3) equals the mean length of the typical K -segment of Φ , which is known from Nagel and Weiss (2006) (for comparison note that in Nagel and Weiss (2006) the constant $\zeta = L_A^{-2} \zeta_\kappa$ was used instead of our ζ_κ). Plugging the density (2) into (3) and changing the order of integration, we get

$$\begin{aligned} & \frac{3\zeta_\kappa}{2L_A} \iint_0^x f(h, y) dy dD_K(h, x) \\ &= \iint_0^\infty f(h, x) s_\kappa(h) \int_1^2 t e^{-s_\kappa(h)tx} dt dx \vartheta(dh) \end{aligned}$$

and since

$$-\frac{d}{dx} \left[s_\kappa(h) \int_1^2 t e^{-s_\kappa(h)tx} dt \right] = s_\kappa^2(h) \int_1^2 t^2 e^{-s_\kappa(h)tx} dt$$

we infer that D_K has density

$$(x, h) \mapsto \frac{2L_A}{3\zeta_\kappa} s_\kappa^2(h) \int_1^2 t^2 e^{-s_\kappa(h)tx} dt$$

with respect to $\vartheta \otimes \ell_+$. Taking now into account $\kappa = L_A \vartheta$, we arrive at the analogue to the main result from Mecke (2009):

The joint distribution of direction and Euclidean length of the typical K -segment is a probability measure on $\mathcal{H} \times (0, \infty)$ with density

$$(x, h) \mapsto \frac{2}{3\zeta_\kappa} s_\kappa^2(h) \int_1^2 t^2 e^{-s_\kappa(h)tx} dt$$

with respect to the product measure $\kappa \otimes \ell_+$. Moreover, the directional distribution of the typical K -segment coincides with that of the typical I -segment.

In the particular isotropic case, where κ is L_A times the uniform distribution on $[\mathcal{H}, \mathfrak{H}]$, direction and length of the typical K -segment are independent and we get back the formula from Mecke *et al.* (2007), namely

$$\begin{aligned} x &\mapsto \frac{4L_A}{3\pi} \int_1^2 u^2 e^{-\frac{2}{\pi}L_A u x} du \\ &= \frac{\pi^2}{3L_A^2 x^3} \left(1 + \frac{2}{\pi}L_A x + \frac{2}{\pi^2}L_A^2 x^2 \right. \\ &\quad \left. - \left(1 + \frac{4}{\pi}L_A x + \frac{8}{\pi^2}L_A^2 x^2 \right) e^{-\frac{2}{\pi}L_A x} \right) e^{-\frac{2}{\pi}L_A x}. \end{aligned}$$

In the case, where κ is concentrated with equal weight to the two coordinate directions, we obtain for the density

$$\begin{aligned} x &\mapsto \frac{2}{3L_A^2 x^3} \left(8 + 4L_A x + L_A^2 x^2 \right. \\ &\quad \left. - (8 + 8L_A x + 4L_A^2 x^2) e^{-\frac{1}{2}L_A x} \right) e^{-\frac{1}{2}L_A x} \end{aligned}$$

for the typical K -segment in one of the two possible directions.

4. The Result for the Typical J -Segment

In the case of the typical J -segment of the tessellation, the situation is much simpler. First, from lemma 4 in Nagel and Weiss (2006) we infer that the directional distribution of the typical J -segment coincides with that of the typical K -segment and hence with that of the typical I -segment. Moreover, from the fact that STIT tessellations have Poisson typical cells, see Nagel and Weiss (2005), we conclude in connection with example 1.5 in Baumstark and Last (2009) that

The joint distribution of direction and Euclidean length of the typical J -segment is a probability measure on $\mathcal{H} \times (0, \infty)$ with density

$$(h, x) \mapsto \frac{s_\kappa^2(h)}{\zeta_\kappa} e^{-s_\kappa(h)x}$$

with respect to the product measure $\kappa \otimes \ell_+$, which is to say that the length distribution of the typical J -segment

having a given direction h is an exponential distribution with parameter $s_\kappa(h)$.

Again, in the isotropic case we get back the known formula from Mecke *et al.* (2007) and for the case, where κ is concentrated with equal weight on the two coordinate directions, we have the density

$$x \mapsto \frac{1}{2}L_A e^{-\frac{1}{2}L_A x}$$

for both possible directions.

5. Discussion

For the planar case, the picture concerning the length distribution of the different types of line segments appearing in STIT tessellations is now complete. We have in the isotropic case the special formulas from Mecke *et al.* (2007) and in the general anisotropic case that from Mecke (2009) and ours obtained above. These distributions yield a very precise description of the geometry of segments in planar STIT tessellations in the general anisotropic regime.

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