# Convex Pentagons for Edge-to-Edge Tiling, I

Teruhisa Sugimoto

*The Interdisciplinary Institute of Science, Technology and Art,*
*Suzukidaini-building 211, 2-5-28 Kitahara, Asaka-shi, Saitama 351-0036, Japan*
*E-mail address: ismsugi@gmail.com*

(Received January 30, 2012; Accepted September 14, 2012)

We introduce a plan toward a perfect list of convex pentagons that can tile the whole plane in edge-to-edge manner. Our strategy is based on Bagina's Proposition, and is direct and primitive: Generating all candidates of pentagonal tiles (several hundreds in number), classify them into the known 14 types, geometrically impossible cases, the cases that do not generate an edge-to-edge pentagonal tiling, and tentatively uncertain cases. At present, still 34 uncertain cases remain, but these case will be settled in near future. Some other results are also presented.
**Key words:** Convex Pentagon, Tiling, Tile, Monohedral Tiling, Edge-to-Edge Tiling

## 1. Introduction

Tiling by polygons is to cover the whole plane with polygons without gaps or overlaps. If all tiles in a tiling are of the same size and shape, then the tiling is called *monohedral*, and the polygon in the monohedral tiling is called the *prototile* of monohedral tiling (Grünbaum and Shephard, 1987), or simply, the *polygonal tile*. In the classification problem of convex polygonal tiles, only the pentagonal case remains unsettled. At present, known convex pentagonal tiles are classified into 14 types (see Fig. 1). However, it is not known whether the list of known types is perfect or not (Kershner, 1968; Gardner, 1975; Grünbaum and Shephard, 1987; Hallard, Kenneth and Richard, 1991; Wells, 1991; Sugimoto and Ogawa, 2005, 2006, 2009b).

Tiling by convex polygons is called *edge-to-edge* if any two convex polygons are either disjoint or share only one vertex or only one edge in common. In an edge-to-edge tiling, vertices are called *nodes*. Thus, at a node of an edge-to-edge tiling, vertices of several polygons meet, and the sum of the interior angles at the vertices gathered at a node is equal to 360°. More generally, let us call the multi-set of vertices of polygons a *spot*, if the sum of interior angles at the vertices in the multi-set is equal to 360° (see Fig. 2). Thus, a node of an edge-to-edge tiling presents a spot, but a spot is not necessarily corresponding to a node of edge-to-edge tiling, because, a spot is defined without considering the edge-lengths.

As shown in Fig. 1, let us label the vertices (angles) of the convex pentagon with labels $A$, $B$, $C$, $D$, $E$, and the edge of that with labels $a$, $b$, $c$, $d$, $e$ in the fixed manner. The interior angles at a vertex is denoted by the same letter (label) as the vertex. If $2A + B + C = 360°$ holds, then the multi-set $\{A, A, B, C\}$ constitute a spot, which is called the *label-set* of the spot. The number of labels (with allowing repetition) in the label-set is called the *valence* of the spot. For example, if the label-set of a spot is $\{A, A, B, C\}$ then the valence of the spot is four (or we say, the spot is 4-valent). The number of different labels of the label-set of a spot is called the *size* of the spot. Thus, a spot with the label-set $\{A, A, B, C\}$ has size 3.

Equilateral convex pentagonal tiles are completely characterized in the following way.

THEOREM 1 *(Hirschhorn and Hunt, 1985). An equilateral convex pentagon tiles the plane if and only if it has two angles adding to* 180°*, or it is the unique equilateral convex pentagon with angles A, B, C, D, E satisfying* $2B + A = 2E + C = 2D + A + C = 360°$ *(*$A \approx 70.88°$*,* $B \approx 144.56°$*,* $C \approx 89.26°$*,* $D \approx 99.93°$*,* $E \approx 135.37°$*).*

Bagina (2004) proved the following proposition, and she used it to present a different proof of Theorem 1.

BAGINA'S PROPOSITION. *In each edge-to-edge tiling of the plane by uniformly bounded pentagons, there exists a tile with at least three nodes of valence three.*

"Uniformly bounded" means that there are fixed $R, r$ ($R > r > 0$) such that every tile contains a disk of radius $r$, and is enclosed by a disk of radius $R$. Since we consider only congruent convex pentagonal tiles, our tiling is always uniformly bounded.

The purpose of this research is to obtain a perfect list of types for convex pentagonal tiles that can generate an edge-to-edge tiling. Though our research is not completed yet, using our procedure or partial results obtained so far, we prove the following.

THEOREM 2 *If a convex pentagon in which the sum of three consecutive angles is never equal to* 360° *has exactly four different edge-lengths, and the two edges of equal length are not consecutive, then the convex pentagon cannot generate an edge-to-edge tiling.*

THEOREM 3 *If a convex pentagon can generate an edge-to-edge tiling with one 4-valent nodes and at most two 3-valent nodes, then the convex pentagon belongs to one (or more) of type* 1*, type* 2*, type* 6*, type* 7*, type* 8*, or type* 9*.*

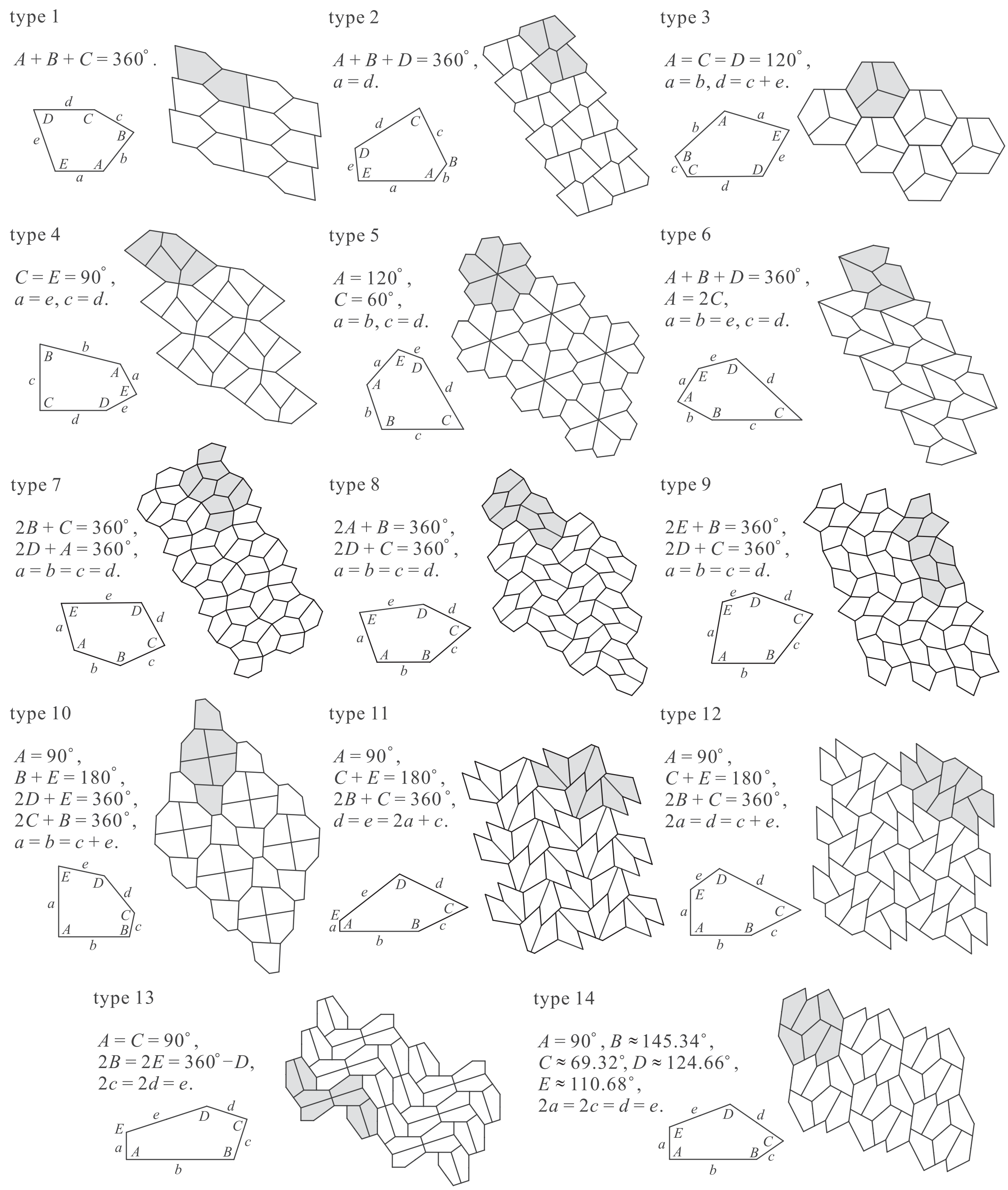


Fig. 1. Convex pentagonal tiles of 14 types. The pale gray pentagons in each tiling indicate the fundamental region (the unit that can generate a periodic tiling by translation only).

THEOREM 4 *Let $T$ be an edge-to-edge tiling by a convex pentagonal tile. If $T$ has only* 3*-valent nodes of size* 3 *and* 4*-valent nodes, then the convex pentagonal tile belongs to one (or more) of type* 1*, type* 2*, or type* 4.

The purpose of this paper is to introduce a plan to answer the following. Among the convex pentagons that can generate an edge-to-edge tilings, is there any one that does not belong to the known 14 types? Let us roughly explain our plan and procedure. Let $G = ABCDE$ be a candidate of convex pentagonal tile that can generate an edge-to-edge tiling. Then, by Bagina's Proposition, it has at least three vertices that will become 3-valent nodes in the tiling. We choose two of them, and consider conditions on angles, and edge lengths. By these conditions, we can produce 465 patterns of pentagons. Examine these pentagons one by one, and classify them into (i) geometrically impossible cases, (ii) the cases that cannot generate an edge-to-edge tiling, (iii) known types, and (iv) remainders. If there is no remainder, then the list of known types will be a perfect list,

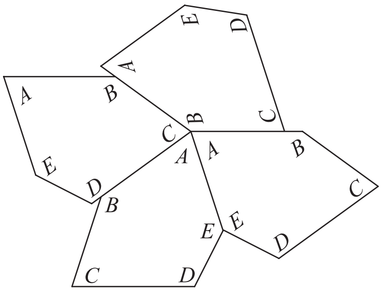


Fig. 2. 4-valent spot of size 3.

otherwise, new type will be obtained in the remainder.

At present, our plan is not completed yet, and there remain 34 uncertain cases (unknown whether a convex pentagon can generate an edge-to-edge tiling). I think I can settle these 34 cases soon.*1

## 2. Present Classification and EE Convex Pentagonal Tiles

As mentioned in Section 1 and shown in Fig. 1, the known convex pentagonal tiles can be classified into 14 types. (The pale gray convex pentagons in each tiling of Fig. 1 indicate the fundamental region, which is the unit that can generate a periodic tiling by translation only.) For example, a convex pentagonal tile of type 1 satisfies that the sum of three consecutive angles is equal to 360° (or the sum of the remaining two consecutive angles is equal to 180°). This condition is expressed as $A + B + C = 360^\circ$ in Fig. 1. According to the present classification rule of pentagonal tiles, one convex pentagonal tile may belong to more than one type. For example, the convex pentagons of Fig. 3 belong to both type 1 and type 7. Note also that the classification problem of types of convex pentagonal tiles and the classification problem of pentagonal tilings (tiling patterns) are quite different.

In the 14 types shown in Fig. 1, the convex pentagons of types 4, 5, 6, 7, 8, and 9 can generate an edge-to-edge tiling. On the other hand, the tilings of types 3, 10, 11, 12, 13, and 14 of Fig. 1 are non-edge-to-edge with allowing vertices lying in the interior of edges. Tilings of type 1 or type 2 are generally non-edge-to-edge as shown in Fig. 1, however, in special cases, the convex pentagonal tiles of type 1 or type 2 can generate edge-to-edge tilings with one 3-valent node with size 3 and one 4-valent node with size 2, as shown in Fig. 4.

The term "convex pentagonal tile" is used to stand for a convex pentagon that generate an edge-to-edge tiling, or a non-edge-to-edge tiling or both. So, let us call a convex pentagonal tile that can generate an edge-to-edge tiling an *EE convex pentagonal tile* in this paper.

*1 In 2012, we found the perfect list of types of convex pentagonal tiles that can generate an edge-to-edge tiling. We have known that a same result was obtained by Bagina (Bagina, 2011) after the manuscript in which the list (Sugimoto, 2012) is shown was submitted to Forma.

## 3. Angle Conditions

### 3.1 Definition of $N_1, N_2, N_3, N_4$ and $N_g(G), N_t(G)$ for a pentagon $G$

Hereafter, the term "pentagon" means only a five-vertex polygon, not necessarily a pentagonal tile. The symbol $G$ is used for a convex pentagon.

The total number of possible label-sets of 3-valent spots of a convex pentagon $ABCDE$ is 35. We divides these 35 3-valent spots into four sets $N_1$, $N_2$, $N_3$, and $N_4$ as follows. (The reason why the label-sets with size 3 are divided into two sets $N_1$ and $N_2$ will become clear in Subsection 3.2).

$$\begin{aligned}
N_1 &= \{\{A, B, C\}, \{B, C, D\}, \{C, D, E\}, \{D, E, A\}, \{E, A, B\}\}.\\
N_2 &= \{\{A, B, D\}, \{B, C, E\}, \{C, D, A\}, \{D, E, B\}, \{E, A, C\}\}.\\
N_3 &= \{\{A, A, B\}, \{A, A, C\}, \{A, A, D\}, \{A, A, E\}, \{B, B, A\},\\
&\quad \{B, B, C\}, \{B, B, D\}, \{B, B, E\}, \{C, C, A\}, \{C, C, B\},\\
&\quad \{C, C, D\}, \{C, C, E\}, \{D, D, A\}, \{D, D, B\}, \{D, D, C\},\\
&\quad \{D, D, E\}, \{E, E, A\}, \{E, E, B\}, \{E, E, C\}, \{E, E, D\}\}.\\
N_4 &= \{\{A, A, A\}, \{B, B, B\}, \{C, C, C\}, \{D, D, D\}, \{E, E, E\}\}.
\end{aligned}$$

$N_1$ is the set of 3-valent spots of size 3 consisting of three consecutive vertices. $N_2$ is the set of 3-valent spots of size 3 consisting of three non-consecutive vertices. $N_3$ is the set of 3-valent spots of size 2, and $N_4$ is the set of 3-valent spots of size 1.

**Example 3.1.** Let $G_0 = ABCDE$ be a convex pentagon such that $A = 148^\circ$, $B = 72^\circ$, $C = 140^\circ$, $D = 108^\circ$, $E = 72^\circ$, $a = d$ $(\neq b \neq c \neq a)$. This pentagon $G_0$ has two 3-valent spots $\{A, B, C\} \in N_1$ and $\{E, A, C\} \in N_2$, and since $A+B+C = 360^\circ$, $a = d$, it can generate an edge-to-edge tiling with 3-valent node $\{A, B, C\}$ as shown in Fig. 4(a). Thus, $G_0$ is an EE convex pentagonal tile.

For a convex pentagon $G$, define $N_g$, $N_t$ as follows:

$N_g := N_g(G) =$ the set of 3-valent spots of $G$.

$N_t := N_t(G) =$ the set of 3-valent spots of $G$ given by nodes (provided that $G$ is an EE convex pentagonal tile).

For example, $N_g(G_0) = \{\{A, B, C\}, \{E, A, C\}\}$ and $N_t(G_0) = \{\{A, B, C\}\}$. So, $G_0$ is a convex pentagon satisfying $N_g \cap N_1 \neq \emptyset$ and a convex pentagonal tile satisfying $N_t \cap N_1 \neq \emptyset$.

### 3.2 EE convex pentagonal tiles with spots in $N_1$ or $N_4$

If a convex pentagon has a spot in $N_1$, then it belongs to type 1. Thus, a convex pentagon $G$ that satisfies $N_g \cap N_1 \neq \emptyset$ belongs to type 1 and can be excluded from further investigation in this paper. This is why we divide the set of 3-valent spots of size 3 into $N_1$ and $N_2$.

For a convex pentagonal tile satisfying $N_t \cap N_4 \neq \emptyset$, we have the following Lemma.

LEMMA 1 *If an EE convex pentagonal tile satisfies $N_t \cap N_4 \neq \emptyset$ then it also satisfies that $N_t \cap (N_1 \cup N_2 \cup N_3) \neq \emptyset$.*

**Proof**. An internal angle at a vertex in a spot presented by a 3-valent node of size 1 is 120°, and two edges adjacent to the vertex have the same length. Then, an EE convex pentagonal tile has at most three 3-valent spots, for otherwise, it has four vertex of angle 120° and all edge-lengths must be equal, but such pentagon cannot exist.

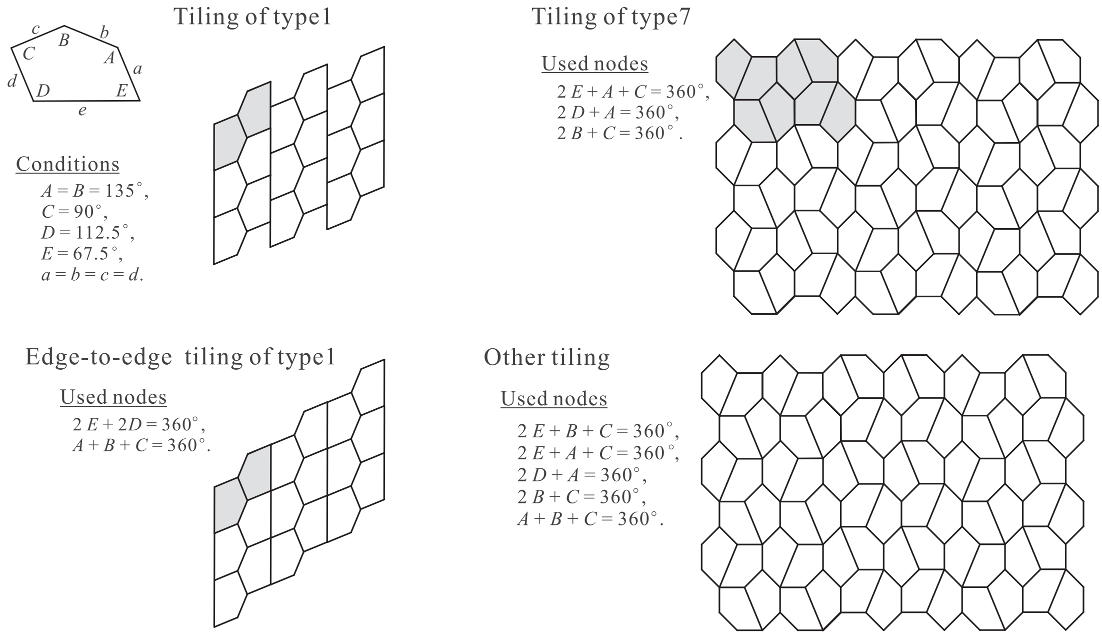


Fig. 3. Convex pentagonal tile that belongs to both type 1 and type 7, and the example of tilings that are generated by the tile. The pale gray pentagons in each tiling indicate the fundamental region.

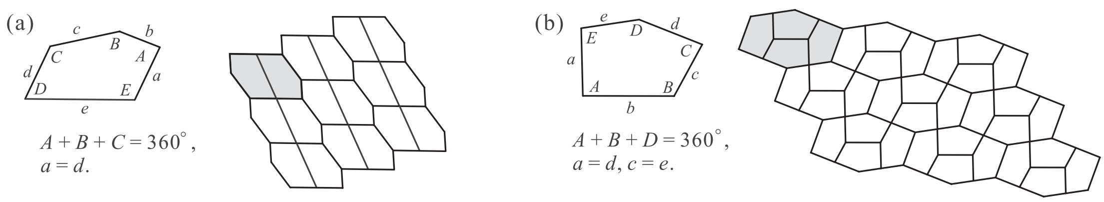


Fig. 4. Examples of edge-to-edge tilings by convex pentagonal tiles that belong to type 1 or type 2. The pale gray pentagons in each tiling indicate the fundamental region. (a) Convex pentagonal tiles that belong to type 1. (b) Convex pentagonal tiles that belong to type 2.

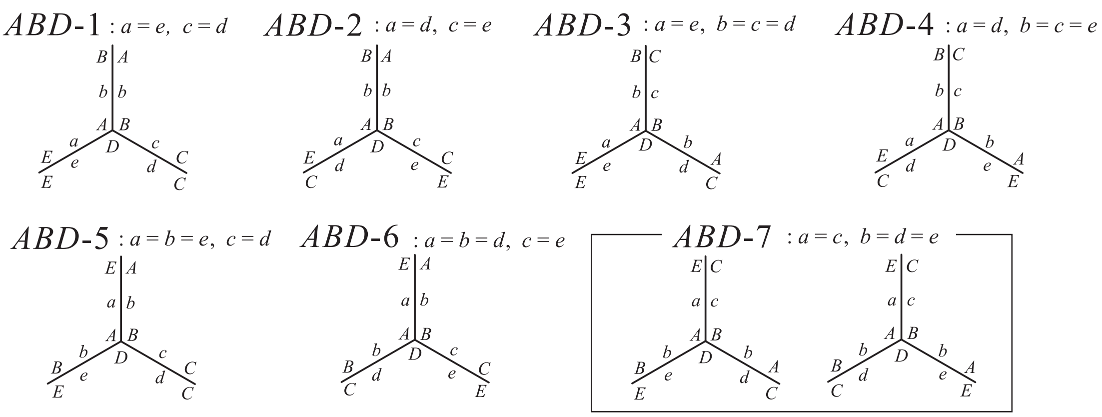


Fig. 5. Seven sub-cases of tentative 3-valent node of $\{A, B, D\}$.

If an EE convex pentagonal tile has three 3-valent nodes of size 1, it belongs to type 1. This can be seen as follows: If the three different vertices (vertex-labels) in the three label-sets of size 1 are consecutive (for example, $A = B = C = 120^\circ$), then the pentagonal tile clearly belongs to type 1. (There is no EE convex pentagonal tile with three label-sets of 3-valent nodes of size 1 such that three different vertices in the three label-sets with size 1 are not consecutive, e.g., $A = B = D = 120^\circ$, see Appendix A).

If an EE convex pentagonal tile has one or two label-sets

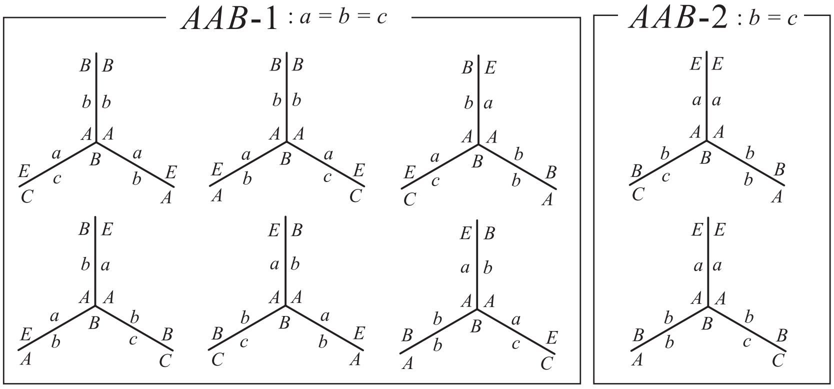


Fig. 6. Two sub-cases of tentative 3-valent node of $\{A, A, B\}$.

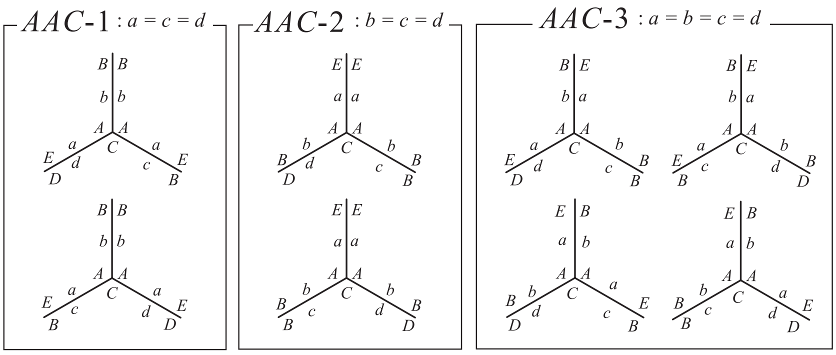


Fig. 7. Three sub-cases of tentative 3-valent node of $\{A, A, C\}$.

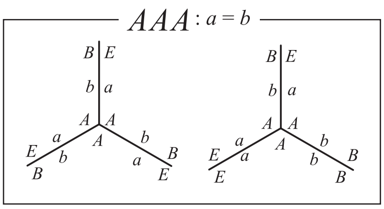


Fig. 8. One sub-case of tentative 3-valent node of $\{A, A, A\}$.

of 3-valent node with size 1, then by Bagina's Proposition, we must have $N_t \cap (N_1 \cup N_2 \cup N_3) \neq \emptyset$. □

### 3.3 Tentative node, label-sets $v_1$, $v_2$, and the sets $\mathbf{G}_i$, $i = 1, \ldots, 5$

Let $G = ABCDE$ be a candidate of an EE convex pentagonal tile. A spot of $G$ that is supposed to become a 3-valent node of the supposed edge-to-edge tiling is called a *tentative 3-valent node* of $G$. Then, by Bagina's Proposition, $G$ should have at least three tentative 3-valent nodes (in other words, $G$ should be able to form at least three tentative 3-valent nodes at the same time around itself). If $G$ generates an edge-to-edge tiling, and has at least one 3-valent node of size 1, then it satisfies that $N_t \cap (N_1 \cup N_2 \cup N_3) \neq \emptyset$ by Lemma 1. Therefore, if $G$ generates an edge-to-edge tiling, $G$ has at least one 3-valent node of size 2 or 3. Let $v_1$ be the label-set of a tentative 3-valent node of $G$ that has *size 2 or 3*, and let $v_2$ be the label-set of any tentative 3-valent node of $G$. Note that $v_1$ may not be uniquely defined, and $v_1 = v_2$ is possible. If $G$ has a spot consisting of three consecutive vertices, and generates a tiling, then $G$ also belongs to type 1. Hence, we may suppose that $N_g(G) \cap N_1 = \emptyset$. Denote by $(G, v_1, v_2)$ a convex pentagon in which $v_1, v_2$ are specified as the spots of tentative 3-valent nodes, and define the sets of such convex pentagons $(G, v_1, v_2)$ in the following way.

$$\mathbf{G}_1 := \{(G, v_1, v_2) \mid v_1, v_2 \in N_g \cap N_2,\ N_g \cap N_1 = \emptyset\}.$$
$$\mathbf{G}_2 := \{(G, v_1, v_2) \mid v_1, v_2 \in N_g \cap N_3,\ N_g \cap N_1 = \emptyset\}.$$
$$\mathbf{G}_3 := \{(G, v_1, v_2) \mid v_1 \in N_g \cap N_3,\ v_2 \in N_g \cap N_2,\ N_g \cap N_1 = \emptyset\}.$$
$$\mathbf{G}_4 := \{(G, v_1, v_2) \mid v_1 \in N_g \cap N_2,\ v_2 \in N_g \cap N_4,\ N_g \cap N_1 = \emptyset\}.$$
$$\mathbf{G}_5 := \{(G, v_1, v_2) \mid v_1 \in N_g \cap N_3,\ v_2 \in N_g \cap N_4, N_g \cap N_1 = \emptyset\}.$$

For the sake of convenience, we set $\mathbf{G}_6 := \{(G, v_1, v_2) \mid v_1 \in N_g \cap N_1\}$. Then, all EE pentagonal tiles appear as $(G, v_1, v_2)$ in one of $\mathbf{G}_i$, $i = 1, 2, \ldots, 6$; the members in $\mathbf{G}_6$ are pentagonal tiles of type 1. Thus, some conditions (equations) on angles for an EE convex pentagonal tile not belonging to type 1 are obtained from $v_1, v_2$ of $(G, v_1, v_2)$ in $\mathbf{G}_1 - \mathbf{G}_5$.

In the subsequent sections, by considering edge fitting around tentative 3-valent nodes, we add edge conditions (equations) to the angle conditions derived from $v_1, v_2$.

Table 1. Sub-cases of tentative 3-valent nodes belonging to $N_2$.

| Tentative 3-valent nodes | | | |
|---|---|---|---|
| Sub-case | Edge conditions | Sub-case | Edge conditions |
| $ABD$-1 | $a = e, c = d$ | $DEB$-1 | $a = b, c = d$ |
| $ABD$-2 | $a = d, c = e$ | $DEB$-2 | $a = c, b = d$ |
| $ABD$-3 | $a = e, b = c = d$ | $DEB$-3 | $a = b = e, c = d$ |
| $ABD$-4 | $a = d, b = c = e$ | $DEB$-4 | $a = c = e, b = d$ |
| $ABD$-5 | $a = b = e, c = d$ | $DEB$-5 | $a = b, c = d = e$ |
| $ABD$-6 | $a = b = d, c = e$ | $DEB$-6 | $a = c, b = d = e$ |
| $ABD$-7 | $a = c, b = d = e$ | $DEB$-7 | $a = d, b = c = e$ |
| $BCE$-1 | $a = b, d = e$ | $EAC$-1 | $b = c, d = e$ |
| $BCE$-2 | $a = d, b = e$ | $EAC$-2 | $b = d, c = e$ |
| $BCE$-3 | $a = b, c = d = e$ | $EAC$-3 | $a = b = c, d = e$ |
| $BCE$-4 | $a = c = d, b = e$ | $EAC$-4 | $a = b = d, c = c$ |
| $BCE$-5 | $a = b = c, d = e$ | $EAC$-5 | $a = d = e, b = c$ |
| $BCE$-6 | $a = d, b = c = e$ | $EAC$-6 | $a = c = e, b = d$ |
| $BCE$-7 | $a = c = e, b = d$ | $EAC$-7 | $a = c = d, b = e$ |
| $CDA$-1 | $a = e, b = c$ | | |
| $CDA$-2 | $a = c, b = e$ | | |
| $CDA$-3 | $a = d = e, b = c$ | | |
| $CDA$-4 | $a = c, b = d = e$ | | |
| $CDA$-5 | $a = e, b = c = d$ | | |
| $CDA$-6 | $a = c = d, b = e$ | | |
| $CDA$-7 | $a = b = d, c = e$ | | |

Table 2. Sub-cases of tentative 3-valent nodes belonging to $N_3$ that satisfy condition under which the two vertices in $v$ are consecutive.

| Tentative 3-valent nodes | | | |
|---|---|---|---|
| Sub-case | Edge conditions | Sub-case | Edge conditions |
| $AAB$-1 | $a = b = c$ | $CCD$-1 | $c = d = e$ |
| $AAB$-2 | $b = c$ | $CCD$-2 | $d = e$ |
| $AAE$-1 | $a = e$ | $DDC$-1 | $c = d$ |
| $AAE$-2 | $a = b = e$ | $DDC$-2 | $c = d = e$ |
| $BBA$-1 | $a = b$ | $DDE$-1 | $a = d = e$ |
| $BBA$-2 | $a = b = c$ | $DDE$-2 | $a = e$ |
| $BBC$-1 | $b = c = d$ | $EEA$-1 | $a = b = e$ |
| $BBC$-2 | $c = d$ | $EEA$-2 | $a = b$ |
| $CCB$-1 | $b = c$ | $EED$-1 | $d = e$ |
| $CCB$-2 | $b = c = d$ | $EED$-2 | $a = d = e$ |

## 4. Edge Conditions

### 4.1 Edge fitting around a tentative 3-valent node

Given a label-set $v$ of a tentative 3-valent node, try to assemble pentagons around a point (3-valent node) according to the labels in the label-set $v$. Since each of three pentagons can be turned over, there are eight possible ways to assemble the three pentagons. Further, to fit edges, we have several conditions (equations) on edge-lengths.

**(i)** The case $v \in N_2$.

There arise seven sub-cases for edge fitting around the 3-valent node. For example, if $v = \{A, B, D\}$, then, as shown in Fig. 5, we have the seven sub-cases *ABD*-1–*ABD*-7 (*ABD*-7 represents two sub-cases with the same edge conditions). Notice that the equations following the colons for each label in the figure give the conditions on edge-lengths. Among the sub-cases $ABD$-1$-$$ABD$-7, the three sub-cases $ABD$-2, $ABD$-4, and $ABD$-6 yield type 2 tiles, and hence, for our purpose, there are four sub-cases to consider.

For other cases see Table 1. From the relations on angles and edge-lengths, the cases *ABD*-2, *ABD*-4, *ABD*-6, *BCE*-2, *BCE*-4, *BCE*-6, *CDA*-2, *CDA*-4, *CDA*-6, *DEB*-2, *DEB*-4, *DEB*-6, *EAC*-2, *EAC*-4, and *EAC*-6 in Table 1 are judged to be convex pentagonal tiles of type 2, therefore these are excluded from further consideration.

Table 3. Sub-cases of tentative 3-valent nodes belonging to $N_3$ that satisfy condition under which the two vertices in $v$ are not consecutive.

| Tentative 3-valent nodes | | | |
|---|---|---|---|
| Sub-case | Edge conditions | Sub-case | Edge conditions |
| $AAC$-1 | $a = c = d$ | $CCE$-1 | $a = c = e$ |
| $AAC$-2 | $b = c = d$ | $CCE$-2 | $a = d = e$ |
| $AAC$-3 | $a = b = c = d$ | $CCE$-3 | $a = c = d = e$ |
| $AAD$-1 | $a = d = e$ | $DDA$-1 | $a = b = d$ |
| $AAD$-2 | $b = d = e$ | $DDA$-2 | $a = b = e$ |
| $AAD$-3 | $a = b = d = e$ | $DDA$-3 | $a = b = d = e$ |
| $BBD$-1 | $b = d = e$ | $DDB$-1 | $b = c = d$ |
| $BBD$-2 | $c = d = e$ | $DDB$-2 | $b = c = e$ |
| $BBD$-3 | $b = c = d = e$ | $DDB$-3 | $b = c = d = e$ |
| $BBE$-1 | $a = b = e$ | $EEB$-1 | $b = c = e$ |
| $BBE$-2 | $a = c = e$ | $EEB$-2 | $a = b = c$ |
| $BBE$-3 | $a = b = c = e$ | $EEB$-3 | $a = b = c = e$ |
| $CCA$-1 | $a = b = c$ | $EEC$-1 | $c = d = e$ |
| $CCA$-2 | $a = b = d$ | $EEC$-2 | $a = c = d$ |
| $CCA$-3 | $a = b = c = d$ | $EEC$-3 | $a = c = d = e$ |

Table 4. Sub-cases of tentative 3-valent nodes belonging to $N_4$.

| Tentative 3-valent nodes | |
|---|---|
| Sub-case | Edge conditions |
| $AAA$ | $a = b$ |
| $BBB$ | $b = c$ |
| $CCC$ | $c = d$ |
| $DDD$ | $d = e$ |
| $EEE$ | $a = e$ |

**(ii)-1** The case that $v \in N_3$ and two vertices in $v$ are consecutive.
The cases are possible as $v$, and for each of them, there arise two sub-cases for edge-fitting. See Table 2, and Fig. 6.

**(ii)-2** The case that $v \in N_3$ and the two vertices in $v$ are not consecutive.
The cases are possible as $v$, for each of them, there arise three sub-cases for edge-fitting. See Table 3, and Fig. 7.

**(iii)** The case $v \in N_4$.
There arises one sub-case for each of five cases for $v$. See Table 4, and Fig. 8.

### 4.2 Number of patterns to examine

We now estimate the number of cases we need to examine. First, suppose $(G, v_1, v_2) \in \mathbf{G}_1$. Without loss of generality, we may suppose that $v_1 = \{A, B, D\}$. From this, by considering edge-fitting, we get four refined cases *ABD*-1, *ABD*-3, *ABD*-5, and *ABD*-7. On the other hand, since $v_2 \in N_2$, it can be one of five label-sets $\{A, B, D\}$, $\{B, C, E\}$, $\{C, D, A\}$, $\{D, E, B\}$, $\{E, A, C\}$, and from each, we get four refined cases. Therefore, from $\mathbf{G}_1$, we get $4 \times 5 \times 4 = 80$ cases (patterns) that should be examined.

If $v_1 = \{A, B, D\}$ is specified as $ABD$-1, and $v_2$ is specified as $CDA$-1, then we have

$$\text{Angle conditions:} \quad A + B + D = 360^\circ,\ B = C,$$
$$\text{Edge conditions:} \quad a = e,\ b = c = d.$$

Actually, what we care about is such [angle conditions + edge conditions]. Convex pentgons satisfying such conditions are referred to as *patterns* or *cases*. Thus, from $\mathbf{G}_1$ we have at most 80 patterns.

By similar consideration, from $\mathbf{G}_2$, $\mathbf{G}_3$, $\mathbf{G}_4$, and $\mathbf{G}_5$, we get 250, 100, 20, and 15 patterns, respectively. Therefore, the total number of patterns we need to examine is (at most) $80 + 250 + 100 + 20 + 15 = 465$. These 465 patterns include all patterns obtained from EE convex pentagonal tiles except those in type 1 or type 2.

## 5. Results of Our Examination

Each of 465 patterns in Subsection 4.2 was considered one by one, using Bagina's Proposition, the information on known convex pentagonal tiles, Theorem 1, geometric properties of a convex pentagon, etc. Let us show below just two examples.

**Example 5.1.** Case that $v_1$ is *ABD*-1 and $v_2$ is *CDA*-1.
Since *ABD*-1 implies $A + B + D = 360^\circ,\ a = e,\ c = d$, and *CDA*-1 implies $C + D + A = 360^\circ,\ a = e,\ b = c$, the pentagon $G$ satisfies that $A + B + D = 360^\circ,\ B = C,\ a = e,\ b = c = d$. A convex pentagon satisfying this condition is symmetric to itself with respect to the line through the vertex $E$ and the midpoint of the edge $BC$, and cannot have a spot of valence seven or more. Except the two special cases, this pentagon cannot generate an edge-to-edge tiling. The two special cases are the following: One case is $B = C = E = 90^\circ$ when the pentagon belongs to type 1 and type 4. The other case is $A = B = C = D = 120^\circ$. This can have a 6-valent node $6E = 360^\circ$, and the pentagon belongs to type 1, type 5, and type 6. (Though the pentagon can have a 5-valent tentative node, it cannot generate an edge-to-edge tiling.)

Table 5. Uncertain cases of whether a convex pentagon can generate an edge-to-edge tiling.

| Type of Set | Sub-case of $v_1$ | Sub-case of $v_2$ | Conditions of convex pentagon† Angle conditions | Edge conditions | Cyclic-edge-type | The simplest node condition |
|---|---|---|---|---|---|---|
| $\mathbf{G}_1$ | $ABD$-1 | $ABD$-1 | $A+B+D=360^\circ$ | $a=e, c=d$ | [11223] | |
| | $ABD$-1 | $ABD$-3 | $A+B+D=360^\circ$ | $a=e, b=c=d$ | [11122] | |
| $\mathbf{G}_2$ | $AAB$-1 | $BBC$-1 | $2A+B=2B+C=360^\circ$ | $a=b=c=d$ | [11112] | N |
| | $AAB$-1 | $BBE$-1 | $2A+B=2B+E=360^\circ$ | $a=b=c=e$ | [11112] | N |
| | $AAB$-1 | $CCD$-2 | $2A+B=2C+D=360^\circ$ | $a=b=c, d=e$ | [11122] | |
| | $AAB$-1 | $DDA$-1 | $2A+B=2D+A=360^\circ$ | $a=b=c=d$ | [11112] | N |
| | $AAB$-1 | $DDA$-2 | $2A+B=2D+A=360^\circ$ | $a=b=c=e$ | [11112] | N |
| | $AAB$-1 | $DDE$-2 | $2A+B=2D+E=360^\circ$ | $a=b=c=e$ | [11112] | |
| | $AAB$-1 | $EEA$-1 | $2A+B=2E+A=360^\circ$ | $a=b=c=e$ | [11112] | N |
| | $AAB$-1 | $EEA$-2 | $2A+B=2E+A=360^\circ$ | $a=b=c$ | [11123] | N |
| | $AAB$-2 | $AAD$-1 | $2A+B=360^\circ, B=D$ | $a=d=e, b=c$ | [11122] | N |
| | $AAB$-2 | $BBD$-1 | $2A+B=2B+D=360^\circ$ | $b=c=d=e$ | [11112] | N |
| | $AAB$-2 | $CCD$-2 | $2A+B=2C+D=360^\circ$ | $b=c, d=e$ | [11223] | |
| | $AAB$-2 | $DDB$-1 | $A+B+D=360^\circ, A=D$ | $b=c=d$ | [11123] | |
| | $AAB$-2 | $DDC$-1 | $2A+B=2D+C=360^\circ$ | $b=c=d$ | [11123] | |
| | $AAC$-1 | $BBA$-1 | $2A+C=2B+A=360^\circ$ | $a=b=c=d$ | [11112] | N |
| | $AAC$-1 | $BBD$-2 | $2A+C=2B+D=360^\circ$ | $a=c=d=e$ | [11112] | |
| | $AAC$-1 | $DDB$-1 | $2A+C=2D+B=360^\circ$ | $a=b=c=d$ | [11112] | |
| | $AAC$-2 | $DDB$-1 | $2A+C=2D+B=360^\circ$ | $b=c=d$ | [11123] | |
| $\mathbf{G}_3$ | $AAB$-1 | $EAC$-1 | $2A+B=E+A+C=360^\circ$ | $a=b=c, d=e$ | [11122] | N |
| | $AAB$-2 | $ABD$-1 | $A+B+D=360^\circ, A=D$ | $a=e, b=c=d$ | [11122] | N |
| | $AAB$-2 | $CDA$-1 | $2A+B=C+D+A=360^\circ$ | $a=e, b=c$ | [11223] | N |
| | $AAB$-2 | $CDA$-3 | $2A+B=C+D+A=360^\circ$ | $a=d=e, b=c$ | [11122] | N |
| | $AAB$-2 | $CDA$-5 | $2A+B=C+D+A=360^\circ$ | $a=e, b=c=d$ | [11122] | N |
| | $AAB$-2 | $DEB$-7 | $2A+B=D+E+B=360^\circ$ | $a=d, b=c=e$ | [11212] | |
| | $AAB$-2 | $EAC$-1 | $2A+B=E+A+C=360^\circ$ | $b=c, d=e$ | [11223] | N |
| $\mathbf{G}_4$ | $ABD$-1 | $AAA$ | $A=120^\circ, A+B+D=360^\circ$ | $a=b=e, c=d$ | [11122] | N |
| $\mathbf{G}_5$ | $AAB$-1 | $DDD$ | $D=120^\circ, 2A+B=360^\circ$ | $a=b=c, d=e$ | [11122] | N |
| | $AAB$-1 | $EEE$ | $E=120^\circ, 2A+B=360^\circ$ | $a=b=c=e$ | [11112] | N |
| | $AAB$-2 | $DDD$ | $D=120^\circ, 2A+B=360^\circ$ | $b=c, d=e$ | [11223] | N |
| | $AAB$-2 | $EEE$ | $E=120^\circ, 2A+B=360^\circ$ | $a=e, b=c$ | [11223] | N |
| | $AAC$-1 | $BBB$ | $B=120^\circ, 2A+C=360^\circ$ | $a=b=c=d$ | [11112] | N |
| | $AAC$-2 | $BBB$ | $B=120^\circ, 2A+C=360^\circ$ | $b=c=d$ | [11123] | N |
| | $AAC$-2 | $EEE$ | $E=120^\circ, 2A+C=360^\circ$ | $a=e, b=c=d$ | [11122] | N |

†: The notation of the conditions follows the present classification rule.

**Example 5.2.** Case that $v_1$ is *AAB*-1 and $v_2$ is *EED*-1. This is the case that $2A+B=2E+D=360^\circ$, $a=b=c$, $d=e$. Suppose that such convex pentagon exists. Let $M$ be the midpoint of the diagonal $AC$. Then, since $a=b$, we have $BM \perp AC$, and since $2A+B=360^\circ$, $BM \parallel AE$. Hence $AE \perp AC$. Similarly, we have $AE \perp CE$. Therefore, $\angle ACE = 0^\circ$, a contradiction.

By examining each of 465 patterns analogously to the above, the 465 cases are classified into four categories: (i) a convex pentagon cannot exist; (ii) a convex pentagon cannot generate an edge-to-edge tiling (even if it exists); (iii) a convex pentagon belongs to at least one of type 1, type 2, type 4, type 5, type 6, type 7, type 8, or type 9 (if it exists); and (iv) uncertain case (unknown whether a convex pentagon can generate an edge-to-edge tiling). At present, there are 34 uncertain cases remained. These 34 uncertain cases are listed in Table 5. I am working on these 34 patterns now, and they will be settled in near future.

## 6. Proofs of Theorems 2, 3, 4

### 6.1 Proof of Theorem 2

Pentagons can be classified by the number of equal edges and their positions. In the following, the edges are designated symbolically in $1, 2, \ldots$ in cyclic (anticlockwise) order, with the same symbol for equal edges. Mirror-reflections are excluded. Beginning with equilateral pentagons, followed by those with four equal edges, etc., they are classified into 12 cyclic-edge-types: [11111], [11112], [11122], [11212], [11123], [11213], [11223], [11232], [12123], [11234], [12134], [12345] (Sugimoto and Ogawa, 2006).

The convex pentagons of cyclic-edge-types in the 465

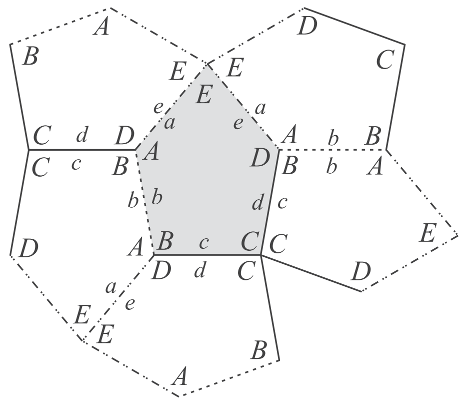

Fig. 9. Case that $v_1$ is *ABD*-1 and $v_2$ is *ABD*-1.

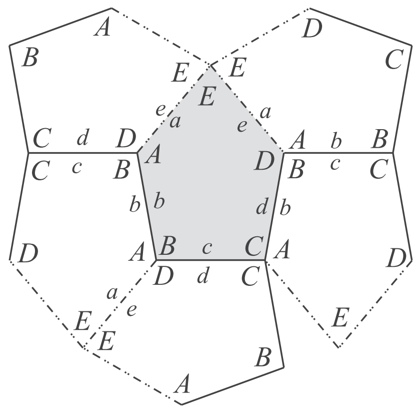

Fig. 11. Case that $v_1$ is *ABD*-1 and $v_2$ is *ABD*-3, and convex pentagon has two *ABD*-1 and one *ABD*-3.

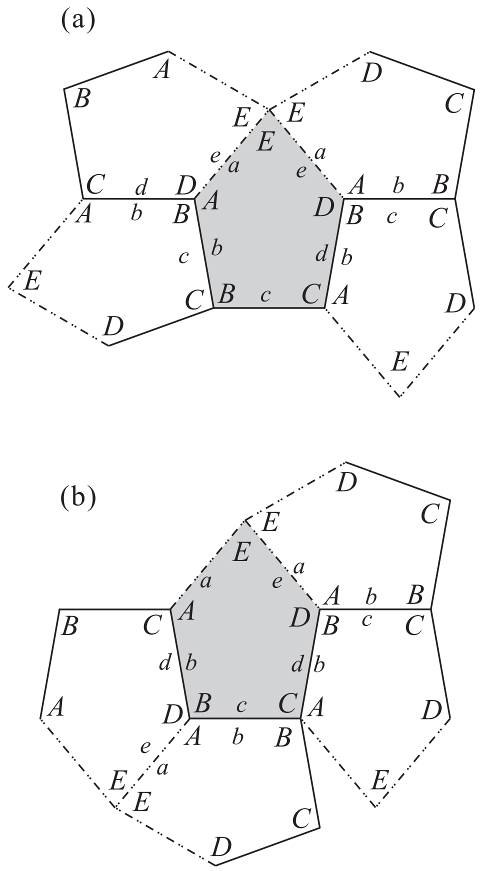


Fig. 10. Case that $v_1$ is *ABD*-1 and $v_2$ is *ABD*-3, and convex pentagon has two *ABD*-3.

patterns are [11111], [11112], [11122], [11212], [11123], [11213], [11223], or [11234]. The convex pentagons of cyclic-edge-types that are judged to type 2 in $\mathbf{G}_1$ are [12123] and [11212] from Table 1. The pentagons of type [12134] belongs to none of $\mathbf{G}_i$, $i = 1, 2, 3, 4, 5$. Since every EE convex pentagonal tile that does not belong to type 1 is contained in at least one of $\mathbf{G}_i$, $i = 1, 2, \ldots, 5$, it is not a convex pentagon of type [12134]. If a new condition on edges or angles is added to a pentagon, then the number of equal edges may increase but never decreases. If the number of equal edges increases, then the pentagon no longer satisfies the condition of Theorem 2. This proves Theorem 2.

Note also that we proved that there is not an EE convex pentagonal tile of [12345] (Sugimoto and Ogawa, 2006). □

### 6.2 Proof of Theorem 3

If each tile of an edge-to-edge tiling by convex pentagonal tile has one 4-valent node and at most two 3-valent nodes, the tiling is called to satisfy *the simplest node condition* (Sugimoto and Ogawa, 2005, 2009a). Note that, under the simplest node condition, the number of the same labels in the label-set of one 4-valent node and two 3-valent nodes must be only two.

As for the 34 uncertain patterns in Table 5, it seems, from the information on $v_1$ and $v_2$, that 23 patterns in the 34 patterns do not satisfy the simplest node condition. For example, the pattern in which $v_1$ is *AAB*-1 and $v_2$ is *BBC*-1 does not satisfy the simplest node condition since the number of $B$ in the label-sets is three. In Table 5, the 23 patterns that do not satisfy the simplest node condition are indicated by "N" in the last column. On the other hand, it can be proved that the remaining 11 patterns cannot generate an edge-to-edge tiling that satisfies the simplest node condition. For example, the pattern in which $v_1$ is *ABD*-1 and $v_2$ is *ABD*-1 cannot generate an edge-to-edge tiling with $A + B + D = 360^\circ$ and $2C + 2E = 360^\circ$ since the spot of $\{C, C, E, E\}$ is not a tentative node. As another example, in the pattern in which $v_1$ is *AAB*-2 and $v_2$ is *CCD*-2, there is a place where the vertices $B$ and $C$ meet when *AAB*-2 is formed (see Fig. 6). However, in a tiling with the simplest node condition for this pattern does not have a node where the vertices $B$ and $C$ meet. Therefore, this pattern can not generate an edge-to-edge tiling with only nodes $\{A, A, B\}$, $\{C, C, D\}$, and $\{E, E, B, D\}$. On the other hand, EE convex pentagonal tiles outside the 34 uncertain cases belong to at least one of type 1, type 2, type 4, type 5, type 6, type 7, type 8, or type 9, and the tilings of type 4 and type 5 do not satisfy the simplest node condition. Thus, we obtain Theorem 3. □

### 6.3 Proof of Theorem 4

Let $G$ be a convex pentagon that generates the tiling $T$.

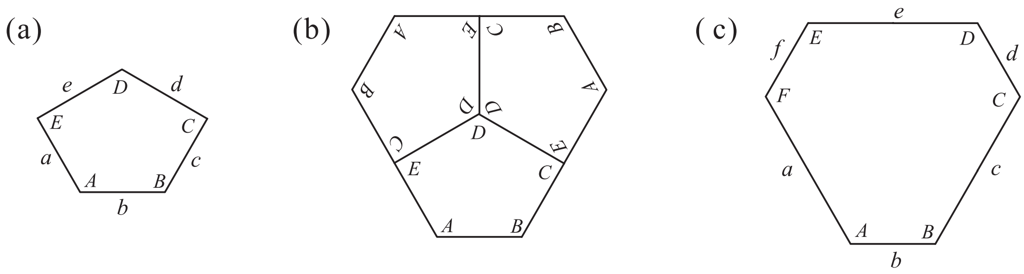


Fig. A1. Convex pentagon that satisfies the conditions "$A = B = D = 120^{\circ}$, $a = b = c$, $d = e$."

Since $T$ has 3-valent nodes of size 3, we have $N_t \cap N_1 \neq \emptyset$ or $N_t \cap N_2 \neq \emptyset$. Since we may take $v_1, v_2$ as $v_2 = v_1$, $G$ belongs to $\mathbf{G}_1$ or $\mathbf{G}_6$. If $G$ belongs to $\mathbf{G}_6$, then since $G$ generates an edge-to-edge tiling $T$, $G$ belongs to type 1. Therefore, we may assume that $G$ is a pentagon in $\mathbf{G}_1$. $\mathbf{G}_1$ has patterns in which a convex pentagon is equilateral, two (solved) patterns in which a convex pentagon that is not equilateral exists, two uncertain patterns (see Table 5), and patterns in which a convex pentagon cannot exist.

(i) Patterns in which a convex pentagon is equilateral.
If $G$ is equilateral, then it belongs to type 1 or type 2 by Theorem 1. Note that the convex pentagon with $A \approx 70.88^{\circ}$, $B \approx 144.56^{\circ}$, $C \approx 89.26^{\circ}$, $D \approx 99.93^{\circ}$, $E \approx 135.37^{\circ}$ in Theorem 1 belongs to type 7 and its tiling is not $T$.

(ii) Two (solved) patterns in which a convex pentagon that is not equilateral exists.

(ii)-1 Case that $v_1$ is $ABD$-1 and $v_2$ is $CDA$-1: If $G$ satisfies either $B = C = E = 90^{\circ}$ or $A = B = C = D = 120^{\circ}$, then it belongs to type 1 (we note that other pentagons of this case cannot generate an edge-to-edge tiling).

(ii)-2 Case that $v_1$ is $ABD$-7 and $v_2$ is $ABD$-7: If a convex pentagon has a node with label-set ($\neq \{A, B, D\}$) in $N_2$ and a 3-valent node with other label-set, then the pentagon belongs to type 1 or type 2.

(iii) The pentagon $G$ is none of the two uncertain patterns in $\mathbf{G}_1$.

(iii)-1 Consider the case that $v_1$ is $ABD$-1 and $v_2$ is $ABD$-1. In this case, $A + B + D = 360^{\circ}$, $a = e$, $c = d$. By Bagina's Proposition, in the edge-to-edge tiling $T$, there is a pentagon (copy of $G$) with at least three 3-valent nodes as vertices. If these nodes are all $ABD$-1 case, then the pentagon (copy) can be regarded as the gray pentagon in Fig. 9. In this gray pentagon, both vertices $C$, $E$ must be 4-valent nodes, and from $a = e$, $c = d$, we must have $4C = 4E = 360^{\circ}$. Therefore, $C = E = 90^{\circ}$ and $a = e$, $c = d$, and hence $G$ belongs to type 4.

(iii)-2 Consider the case that $v_1$ is $ABD$-1 and $v_2$ is $ABD$-3 (angle and edge conditions are $A + B + D = 360^{\circ}$, $a = e$, $b = c = d$). If all 3-valent nodes in $T$ are $ABD$-1, then $G$ belongs to type 4 as seen in (iii)-1. The convex pentagon with two 3-valent nodes of $ABD$-3 cannot have other 3-valent nodes, see Fig. 10. The convex pentagon with two $ABD$-1 and one $ABD$-3 is unique as in Fig. 11. In this case we have $4E = 360^{\circ}$ and the convex pentagon has the relation $C = E = 90^{\circ}$, $a = e$, $b = c = d$. Thus it belongs to type 4. □

**Acknowledgments.** The author is very grateful to Emeritus Professor Hiroshi Maehara, University of the Ryukyus, for discussions about the paper. His numerous suggestions and helpful comments led to the significant improvements to the paper. The author would like to thank Professor Nikolai Dolbilin, Steklov Mathematical Institute, for his helpful comments.

## Appendix A.

We show that there is no EE convex pentagonal tile with three 3-valent nodes of size 1 such that three different vertices in the three label-sets with size 1 are not consecutive.

Let $G$ be a convex pentagon satisfying $A = B = D = 120^{\circ}$. Then we have

$$a = b = c = d = e \quad \text{or} \quad a = b = c \neq d = e.$$

The case $a = b = c = d = e$ is impossible, because the length of the base $CE$ of the trapezoid $ABCE$ with $A = B = 120^{\circ}$ is longer than the base $CE$ of the isosceles triangle $CDE$ with $D = 120^{\circ}$. So, we have $a = b = c \neq d = e$. Then, since $C + E = 180^{\circ}$ and since $G$ is symmetric to itself with respect to the line through $D$ and the midpoint of $AB$, we must have $C = E = 90^{\circ}$ (see Fig. A.1(a)). If this pentagon generates an edge-to-edge tiling, the vertex $D$ becomes a 3-valent node with label-set $\{D, D, D\}$, and the convex hexagon as shown in Fig. A.1(b) will be used in the tiling. By using the new notation as in Fig. A.1(c), this convex hexagon must satisfy $A = B = C = D = E = F = 120^{\circ}$, $a = c = e$, $b = d = f$. Such convex hexagon does not belong to the three types of convex hexagonal tiles in Fig. A.2 (Kershner, 1968; Gardner, 1975; Grünbaum and Shephard, 1987). Thus, the convex pentagon $G$ cannot generate an edge-to-edge tiling.

### References

Bagina, O. (2004) Tiling the plane with congruent equilateral convex pentagons, *Journal of Combinatorial Theory*, Series A **105**, 221–232.

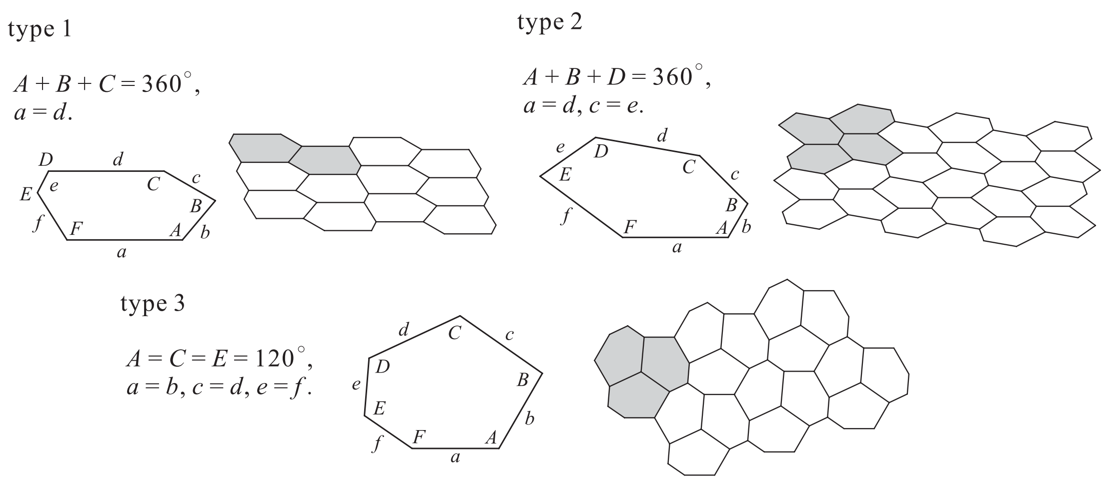


Fig. A2. Convex hexagonal tiles of 3 types. If a convex hexagon is tileable, it belongs to at least one of types 1–3. The pale gray hexagons in each tiling indicate the fundamental region.

Bagina, O. (2011) Tilings of the plane with convex pentagons, *Vestnik KemGU*, **4**(48), 63–73 (in Russian).

Gardner, M. (1975) On tessellating the plane with convex polygon tiles, *Scientific American*, July, 112–117.

Grünbaum, B. and Shephard, G. C. (1987) *Tilings and Patterns*, pp. 15–35 (Chapter 1), pp. 113–134 (Chapter 3), pp. 471–487, pp. 492–497 and pp. 517–518 (Chapter 9), W. H. Freeman and Company, New York.

Hallard, T. C., Kenneth, J. F. and Richard, K. G. (1991) *Unsolved Problems in Geometry*, pp. 95–96 (C14), Springer-Verlag, New York.

Hirschhorn, M. D. and Hunt, D. C. (1985) Equilateral convex pentagons which tile the plane, *Journal of Combinatorial Theory*, Series A **39**, 1–18.

Kershner, R. B. (1968) On paving the plane, *American Mathematical Monthly*, **75**, 839–844.

Sugimoto, T. (2012) Classification of convex pentagons that can generate edge-to-edge monohedral tilings of the plane, *Forma* (to be submitted).

Sugimoto, T. and Ogawa, T. (2005) Systematic study of convex pentagonal tilings, I: Case of convex pentagons with four equal-length edges, *Forma*, **20**, 1–18.

Sugimoto, T. and Ogawa, T. (2006) Properties of tilings by convex pentagons, *Forma*, **21**, 113–128.

Sugimoto, T. and Ogawa, T. (2009a) Systematic study of convex pentagonal tilings, II: Tilings by convex pentagons with four equal-length edges, *Forma*, **24**, 93–109.

Sugimoto, T. and Ogawa, T. (2009b) Properties of nodes in pentagonal tilings, *Forma*, **24**, 117–121.

Wells, D. (1991) *The Penguin Dictionary of Curious and Interesting Geometry*, pp. 177–179, Penguin Books, London.