

First and Second Nearest Distances in Archimedean Tilings

Masashi Miyagawa

Department of Regional Social Management, University of Yamanashi, 4-4-37 Takeda, Kofu, Yamanashi 400-8510, Japan
E-mail address: mmiyagawa@yamanashi.ac.jp

(Received October 4, 2013; Accepted February 21, 2014)

This paper provides the average and maximum distances to the first and second nearest vertices of Archimedean tilings. Distance is measured as the Euclidean distance. The distances in Archimedean tilings are useful for location analysis. The average distance can be used as a criterion of efficiency, whereas the maximum distance can be used as a criterion of equity. As an application to location analysis, we consider bi-objective problems where two distances are minimized. The result shows that tilings other than three regular tilings can be Pareto optimal.

Key words: Location, Euclidean Distance, Average Distance, Maximum Distance, Pareto Optimal

1. Introduction

The distance from a random point to its nearest point, which is called the nearest neighbor distance, provides fundamental characteristics of point patterns. Since the nearest neighbor distance method was introduced by Clark and Evans (1954), many statistics based on the nearest neighbor distance have been proposed for describing patterns for the distribution of various geographical objects (Cressie, 1993; Illian *et al.*, 2008). Although the nearest neighbor distance is the most important, the distance to the k th nearest point is necessary to deal with complicated patterns. Holgate (1965b) considered the distance to the second nearest point. Jones (1971) examined up to the sixth nearest neighbor distance. Ripley's K -function, which is one of the most frequently used tools for point pattern analysis, handles distances between all pairs of points (Ripley, 1976). The K -function method has been applied to the distribution of population (Getis, 1983), traffic accidents (Jones *et al.*, 1996), and trees (He and Duncan, 2000).

The nearest neighbor distance has also been used in location analysis. The distance from customers to their nearest facility represents the service level of facility location. Koshizuka and Ohsawa (1983) examined the location of schools using the distribution of the distance to the nearest school. The distance to the k th nearest facility is also important when facilities are closed or disrupted. In fact, facility location models incorporating a reliability aspect have considered service from the k th nearest facility. Weaver and Church (1985) addressed the vector assignment p -median problem, where a certain percentage of customers could be serviced by the k th nearest facility. Pirkul (1989) studied a similar problem in which customers are served by two facilities designated as primary and secondary facilities. Drezner (1987) formulated the unreliable p -median and p -center problems, and suggested heuristic solutions when the probability of facility failure is the same for all

facilities. In both models, customers are assigned to the k th nearest facility when closer facilities fail. Berman *et al.* (2007) extended Drezner's model by assuming that the probabilities of facility failure are not identical. Snyder and Daskin (2005) proposed two reliability models based on the p -median problem and the uncapacitated fixed-charge location problem. They made an ordered assignment of each customer to each facility. Lei and Church (2011) presented generalized closest assignment constraints in terms of multiple levels of closeness. Service from the k th nearest facility is also found in emergency vehicle location models, where the service availability is computed using queueing theory (Larson, 1974; Marianov and ReVelle, 1996; Sorensen and Church, 2010).

Analytical expressions for the k th nearest distance have been obtained for regular and random point patterns. The nearest Euclidean distance was derived by Clark and Evans (1954) for the random pattern, Persson (1964) for the square lattice, and Holgate (1965a) for the triangular lattice. The k th nearest Euclidean distance was derived by Thompson (1956) and Dacey (1968) for the random pattern, Koshizuka (1985) for $k = 1, 2, 3$ for the square lattice, and Miyagawa (2009) for $k = 1, 2, \dots, 7$ for the square, triangular, and hexagonal lattices.

In this paper, we obtain the distances to the first and second nearest vertices of Archimedean tilings, as shown in Fig. 1, where (3^6) means that each vertex is surrounded by six triangles. Distance is measured as the Euclidean distance. Archimedean tilings are edge-to-edge tilings by regular polygons such that all vertices are of the same type. They clearly include the three regular tilings (3^6) , (4^4) , (6^3) , which are the only edge-to-edge monohe-dral tilings by regular polygons (Grünbaum and Shephard, 1987). The first and second nearest distances in the three regular tilings were given by Miyagawa (2009). The present paper extends the analysis to Archimedean tilings.

Archimedean tilings are important for location analysis, because these dispersed patterns of facilities can provide

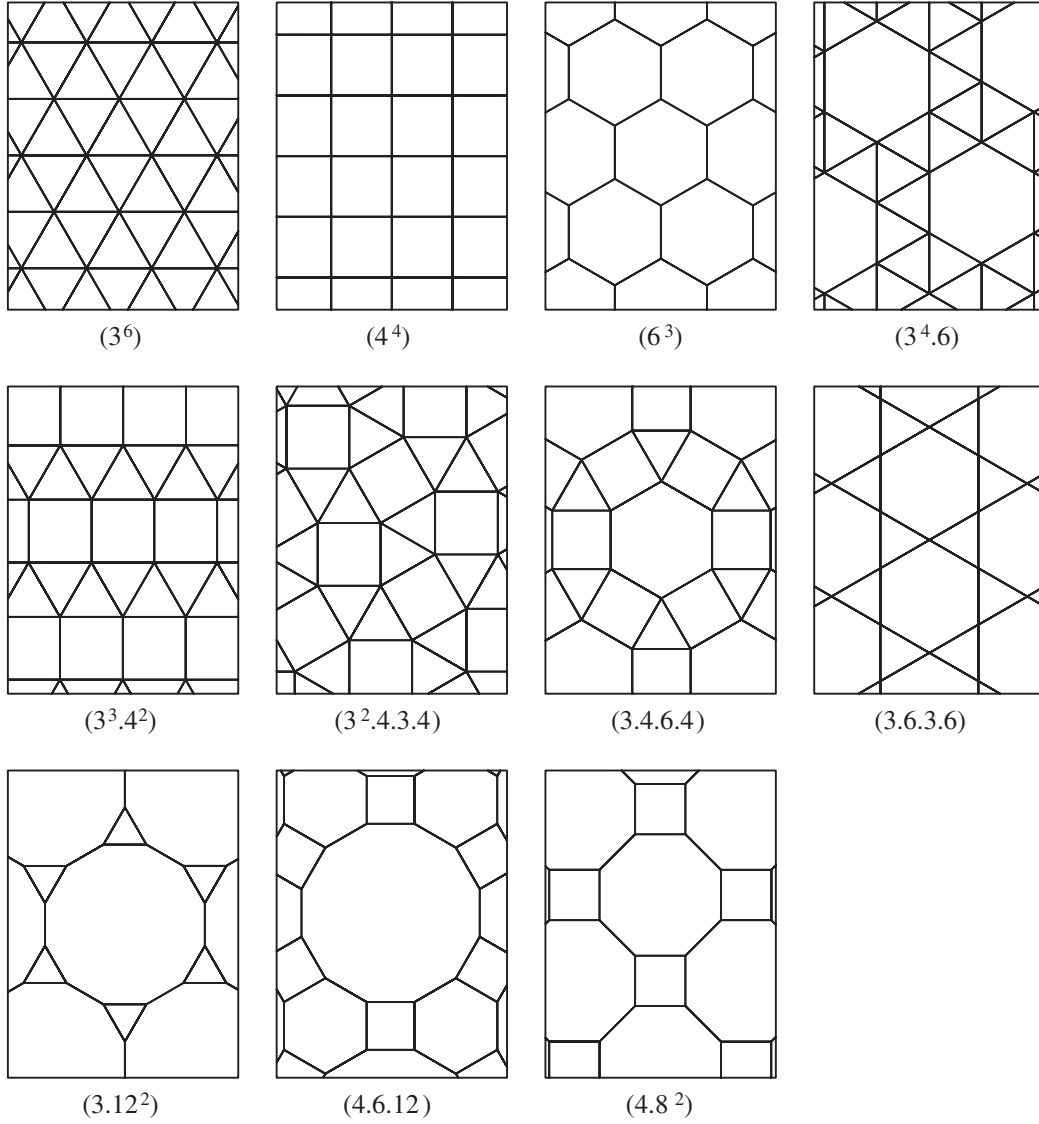


Fig. 1. Archimedean tilings.

close proximity to facilities. If customers are uniformly distributed and serviced by their nearest facility, the optimal facility location is (3^6) , as shown by Leamer (1968), Iri *et al.* (1984), and Du *et al.* (1999). If some of the existing facilities are closed and customers are serviced by their second nearest facility, however, other patterns of facilities can be optimal. The distances in Archimedean tilings will thus give an insight into facility location problems with closing of facilities. As an application to location analysis, we consider bi-objective problems where two distances are minimized. We then present Pareto optimal solutions for the problems. Pareto optimal solutions are such that no other solution is superior to them and have been used in multi-criteria facility location problems (Nickel *et al.*, 2005; Farahani *et al.*, 2010).

The remainder of this paper is organized as follows. The next section derives the average distances to the first and second nearest vertices of Archimedean tilings. The following section examines the maximum distances. The penultimate section provides an application to location analysis. The final section presents concluding remarks.

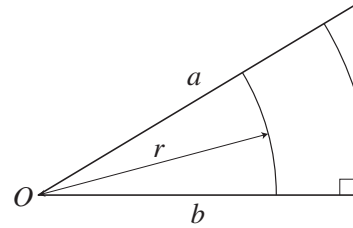


Fig. 2. Average distance in a right triangle.

2. Average Distance

Let $E(R_1)$ and $E(R_2)$ be the average distances from a random point on a plane to the first and second nearest vertices, respectively. In this section, we derive $E(R_1)$ and $E(R_2)$ in Archimedean tilings.

The average distance from a random point in a right triangle to a vertex was derived by Koshizuka and Ohsawa (1983). Let R be the distance from a random point in the right triangle with side lengths a and b ($a > b$) to the vertex O , as shown in Fig. 2. The sum of the distances $T(a, b)$ is

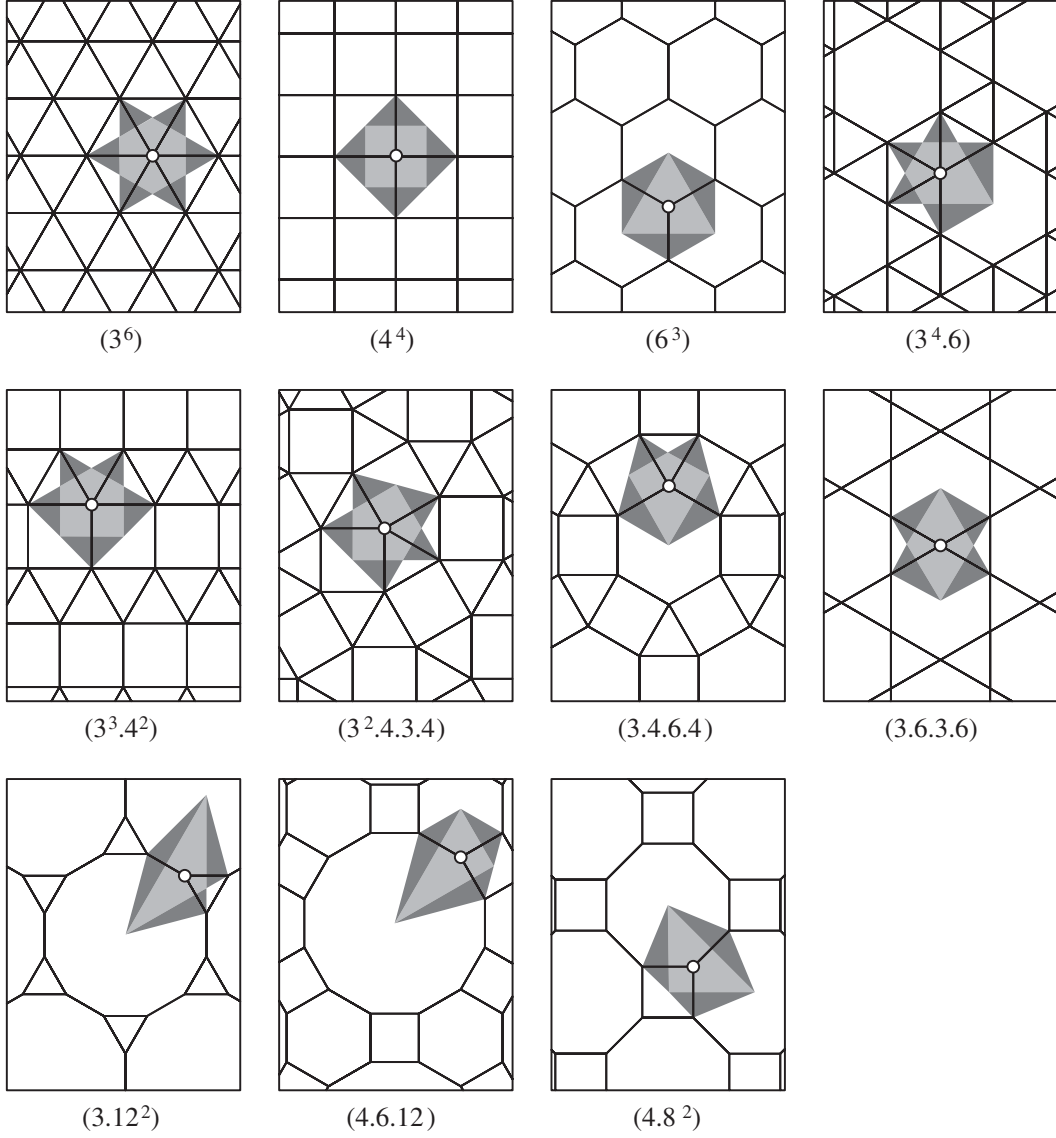


Fig. 3. First and second nearest regions.

given by

$$T(a, b) = \int_0^b \int_0^{\arccos b/a} r^2 d\theta dr + \int_b^a \int_{\arccos b/r}^{\arccos b/a} r^2 d\theta dr$$

$$= \frac{ab}{6} \sqrt{a^2 - b^2} + \frac{b^3}{6} \ln \frac{a + \sqrt{a^2 - b^2}}{b}. \quad (1)$$

Dividing $T(a, b)$ by the area of the triangle $S = b\sqrt{a^2 - b^2}/2$ yields the average distance $E(R)$ as

$$E(R) = \frac{T(a, b)}{S} = \frac{a}{3} + \frac{b^2}{3\sqrt{a^2 - b^2}} \ln \frac{a + \sqrt{a^2 - b^2}}{b}. \quad (2)$$

The average distances $E(R_1), E(R_2)$ in Archimedean tilings can be calculated by considering only one vertex, because all vertices are of the same type. Figure 3 shows the regions where the white point is the first and second nearest. We call these regions the first and second nearest regions, respectively. $E(R_1)$ and $E(R_2)$ are then the average distances from a random point in the first and second nearest

regions to the white point. $E(R_1)$ and $E(R_2)$ are obtained by partitioning the regions into right triangles. For example, the first nearest region for (3^6) is the hexagon centered at the white point with side length $a/\sqrt{3}$, where a is the side length of a tile. The region is partitioned into 12 right triangles with side lengths $a/\sqrt{3}$ and $a/2$. Using Eq. (1), we have

$$E(R_1) = \frac{12}{S} T\left(\frac{a}{\sqrt{3}}, \frac{a}{2}\right) = \frac{4 + 3 \ln 3}{6 \cdot 3^{3/4} \sqrt{2} \rho} \approx \frac{0.377}{\sqrt{\rho}}, \quad (3)$$

where $S = \sqrt{3}a^2/2$ is the area of the first nearest region and $\rho = 1/S$ is the density of vertices. Partitioning the second nearest region into right triangles, we have

$$E(R_2) = \frac{-4 + 6\sqrt{3} - 3 \ln(6 - 3\sqrt{3})}{3 \cdot 3^{3/4} \sqrt{2} \rho} \approx \frac{0.729}{\sqrt{\rho}}. \quad (4)$$

$E(R_1)$ and $E(R_2)$ for the other tilings are similarly obtained as follows:

(4⁴)

(3.6.3.6)

$$E(R_1) = \frac{\sqrt{2} + \ln(1 + \sqrt{2})}{6\sqrt{\rho}} \approx \frac{0.383}{\sqrt{\rho}} \quad (5)$$

$$E(R_2) = \frac{4 - \sqrt{2} - (1 - 2\sqrt{2}) \ln(1 + \sqrt{2})}{6\sqrt{\rho}} \approx \frac{0.700}{\sqrt{\rho}} \quad (6)$$

(6³)

$$E(R_1) = \frac{6 + \sqrt{3} \ln(2 + \sqrt{3})}{9 \cdot 3^{3/4} \sqrt{\rho}} \approx \frac{0.404}{\sqrt{\rho}} \quad (7) \quad (3.12^2)$$

$$E(R_2) = \frac{6 + 9 \ln 3 - \sqrt{3} \ln(2 + \sqrt{3})}{9 \cdot 3^{3/4} \sqrt{\rho}} \approx \frac{0.663}{\sqrt{\rho}} \quad (8)$$

(3⁴.6)

$$E(R_1) = \frac{18 + 8\sqrt{3} + 3\sqrt{3} \ln 9 (2 + \sqrt{3})}{21 \cdot 3^{3/4} \sqrt{7\rho}} \approx \frac{0.396}{\sqrt{\rho}} \quad (9)$$

$$E(R_2) = \frac{90 - 16\sqrt{3} + 3(9 - 4\sqrt{3}) \ln 3 + 9\sqrt{3} \ln(2 + \sqrt{3})}{21 \cdot 3^{3/4} \sqrt{7\rho}} \approx \frac{0.708}{\sqrt{\rho}} \quad (10)$$

(3³.4²), (3².4.3.4)

$$E(R_1) = \frac{4 + 4\sqrt{2} + 3 \ln 3 + 4 \ln(1 + \sqrt{2})}{6(2 + \sqrt{3})^{3/2} \sqrt{\rho}} \approx \frac{0.381}{\sqrt{\rho}} \quad (11)$$

$$E(R_2) = \frac{4 - 2\sqrt{2} + 6\sqrt{3} - 3 \ln(6 - 3\sqrt{3}) - 2(1 - 2\sqrt{2}) \ln(1 + \sqrt{2})}{3(2 + \sqrt{3})^{3/2} \sqrt{\rho}} \approx \frac{0.714}{\sqrt{\rho}} \quad (12)$$

(3.4.6.4)

$$E(R_1) = \frac{1}{12 \cdot 3^{3/4} (2 + \sqrt{3})^{5/2} \sqrt{\rho}} \cdot \left\{ 36 + 42\sqrt{2} + 24\sqrt{3} + 22\sqrt{6} + 3\sqrt{2} (3 + 2\sqrt{3}) \ln \sqrt{3} (3 + 2\sqrt{2}) (2 + \sqrt{3}) \right\} \approx \frac{0.395}{\sqrt{\rho}} \quad (13)$$

$$E(R_2) = \frac{1}{6 \cdot 3^{3/4} (2 + \sqrt{3})^{5/2} \sqrt{\rho}} \cdot \left\{ -18 + 66\sqrt{2} - 12\sqrt{3} + 38\sqrt{6} + \frac{3(15 + 7\sqrt{3})}{\sqrt{2}} \cdot \ln 3 + 3(4 - \sqrt{2}) (3 + 2\sqrt{3}) \ln(1 + \sqrt{2}) \right\} \approx \frac{0.693}{\sqrt{\rho}} \quad (14)$$

$$E(R_1) = \frac{36 + 4\sqrt{3} + 3\sqrt{3} \ln 3 (7 + 4\sqrt{3})}{48\sqrt{2} \cdot 3^{3/4} \sqrt{\rho}} \approx \frac{0.403}{\sqrt{\rho}} \quad (15)$$

$$E(R_2) = \frac{(9 - \sqrt{3}) (4 + 3 \ln 3)}{24\sqrt{2} \cdot 3^{3/4} \sqrt{\rho}} \approx \frac{0.685}{\sqrt{\rho}} \quad (16)$$

$$E(R_1) = \frac{7 + 4\sqrt{3}}{(12 + 7\sqrt{3})^{5/2} \sqrt{(2 - \sqrt{3}) \rho}} \cdot \left[24 - \sqrt{2} + 12\sqrt{3} + \sqrt{6} + \frac{3\sqrt{2 - \sqrt{3}}}{2} \cdot \left\{ \ln 3 + 4 \ln(2 + \sqrt{3} + 2\sqrt{2 + \sqrt{3}}) \right\} \right] \approx \frac{0.499}{\sqrt{\rho}} \quad (17)$$

$$E(R_2) = \frac{1}{(12 + 7\sqrt{3})^{5/2} \sqrt{(2 - \sqrt{3}) \rho}} \cdot \left\{ 228 + 47\sqrt{2} + 132\sqrt{3} + 27\sqrt{6} + 3(26 + 15\sqrt{3} + \sqrt{26 + 15\sqrt{3}}) \ln(2 + \sqrt{3}) + 6(26 + 15\sqrt{3}) \ln(2 + \sqrt{2 - \sqrt{3}}) - 3\sqrt{2} (5 + 3\sqrt{3}) \ln \sqrt{3} (2 + \sqrt{3} + 2\sqrt{2 + \sqrt{3}}) \right\} \approx \frac{0.676}{\sqrt{\rho}} \quad (18)$$

$$E(R_1) = \frac{1}{3(3 + 2\sqrt{3})^{5/2} \sqrt{(2 - \sqrt{3}) \rho}} \cdot \left[3(9 + \sqrt{2} + 5\sqrt{3} + \sqrt{6}) + (3 + 2\sqrt{3}) \cdot \sqrt{2 - \sqrt{3}} \left\{ \ln(1 + \sqrt{2}) (2 + \sqrt{3}) \cdot \sqrt{3 + 2\sqrt{3}} (2 + \sqrt{2 + \sqrt{3}}) - \frac{1}{4} \ln 3 \right\} \right] \approx \frac{0.477}{\sqrt{\rho}} \quad (19)$$

$$\begin{aligned}
E(R_2) &= \frac{1}{3(3+2\sqrt{3})^{5/2}\sqrt{(2-\sqrt{3})\rho}} \\
&\cdot \left[3(5+4\sqrt{2}+3\sqrt{3}+2\sqrt{6}) + \frac{3(11+13\sqrt{3})}{4\sqrt{2}} \right. \\
&\cdot \ln 3 + \frac{(4-\sqrt{2})(3+\sqrt{3})}{2} \ln(1+\sqrt{2}) \\
&+ \left(6-3\sqrt{2} + \frac{7\sqrt{3}}{2} + \sqrt{6} \right) \ln(2+\sqrt{3}) \\
&+ (12+7\sqrt{3}) \ln\left(2+\sqrt{2-\sqrt{3}}\right) - \frac{3\sqrt{2-\sqrt{3}}}{2} \\
&\cdot \left\{ \ln(3+2\sqrt{3}) + 2\ln(2+\sqrt{3}) \left(2+\sqrt{2+\sqrt{3}}\right) \right\} \\
&\cdot \left. \left[-\sqrt{2}(3-\sqrt{3}) \ln(2+\sqrt{2}+\sqrt{3}+\sqrt{6}) \right] \right\} \\
&\approx \frac{0.689}{\sqrt{\rho}} \quad (20)
\end{aligned}$$

(4.8²)

$$\begin{aligned}
E(R_1) &= \frac{1}{3(3+2\sqrt{2})^{5/2}\sqrt{(2-\sqrt{2})\rho}} \\
&\cdot \left[28+20\sqrt{2} + \sqrt{20+14\sqrt{2}} + \sqrt{10+7\sqrt{2}} \right. \\
&\cdot \left. \left\{ \ln(1+\sqrt{2}) + 2\ln\left(1+\sqrt{2}+\sqrt{4+2\sqrt{2}}\right) \right\} \right] \\
&\approx \frac{0.430}{\sqrt{\rho}} \quad (21)
\end{aligned}$$

$$\begin{aligned}
E(R_2) &= \frac{1}{3(3+2\sqrt{2})^{5/2}\sqrt{(2-\sqrt{2})\rho}} \\
&\cdot \left[12+8\sqrt{2} + \sqrt{676+478\sqrt{2}} + \sqrt{2-\sqrt{2}} \right. \\
&\cdot \left\{ (5+4\sqrt{2}) \ln(1+\sqrt{2}) - 2(3+2\sqrt{2}) \right. \\
&\cdot \ln\left(1+\sqrt{2}+\sqrt{4+2\sqrt{2}}\right) \left. \right\} - 4(7+5\sqrt{2}) \\
&\cdot \left. \ln\left(-3+2\sqrt{2}+\sqrt{20-14\sqrt{2}}\right) \right] \\
&\approx \frac{0.665}{\sqrt{\rho}}. \quad (22)
\end{aligned}$$

$E(R_1)$ and $E(R_2)$ in Archimedean tilings are shown in Table 1, where the density of vertices is normalized at $\rho = 1$ for comparison. Note that (3⁶) has the smallest $E(R_1)$. This is the reason why the optimal facility location is (3⁶) when customers are serviced by their nearest facility. Note also that (6³) has the smallest $E(R_2)$.

3. Maximum Distance

Let $U(R_1)$ and $U(R_2)$ be the maximum distances from a random point on a plane to the first and second nearest vertices, respectively. In this section, we derive $U(R_1)$ and $U(R_2)$ in Archimedean tilings.

$U(R_1)$ and $U(R_2)$ are the distances from the farthest point in the first and second nearest regions to the white

point in Fig. 3, and obtained as follows:
(3⁶)

$$U(R_1) = \frac{\sqrt{2}}{3^{3/4}\sqrt{\rho}} \approx \frac{0.620}{\sqrt{\rho}} \quad (23)$$

$$U(R_2) = \frac{\sqrt{2}}{3^{1/4}\sqrt{\rho}} \approx \frac{1.075}{\sqrt{\rho}} \quad (24)$$

$$U(R_1) = \frac{1}{\sqrt{2}\rho} \approx \frac{0.707}{\sqrt{\rho}} \quad (25)$$

$$U(R_2) = \frac{1}{\sqrt{\rho}} \quad (26)$$

$$U(R_1) = U(R_2) = \frac{2}{3^{3/4}\sqrt{\rho}} \approx \frac{0.877}{\sqrt{\rho}} \quad (27)$$

(3⁴.6)

$$U(R_1) = U(R_2) = \frac{2 \cdot 3^{1/4}}{\sqrt{7}\rho} \approx \frac{0.995}{\sqrt{\rho}} \quad (28)$$

(3³.4²), (3².4.3.4)

$$U(R_1) = \frac{-1+\sqrt{3}}{\sqrt{\rho}} \approx \frac{0.732}{\sqrt{\rho}} \quad (29)$$

$$U(R_2) = \frac{2}{\sqrt{(2+\sqrt{3})\rho}} \approx \frac{1.035}{\sqrt{\rho}} \quad (30)$$

(3.4.6.4)

$$U(R_1) = U(R_2) = \frac{3^{1/4}(-1+\sqrt{3})}{\sqrt{\rho}} \approx \frac{0.963}{\sqrt{\rho}} \quad (31)$$

(3.6.3.6)

$$U(R_1) = U(R_2) = \frac{3^{1/4}}{\sqrt{2}\rho} \approx \frac{0.931}{\sqrt{\rho}} \quad (32)$$

(3.12²)

$$U(R_1) = U(R_2) = \frac{2\sqrt{-3+2\sqrt{3}}}{\sqrt{\rho}} \approx \frac{1.363}{\sqrt{\rho}} \quad (33)$$

(4.6.12)

$$U(R_1) = U(R_2) = \frac{2}{3^{1/4}\sqrt{\rho}} \approx \frac{1.520}{\sqrt{\rho}} \quad (34)$$

(4.8²)

$$U(R_1) = U(R_2) = \frac{2}{\sqrt{(2+\sqrt{2})\rho}} \approx \frac{1.082}{\sqrt{\rho}}. \quad (35)$$

$U(R_1)$ and $U(R_2)$ in Archimedean tilings are shown in Table 2. Note that (3⁶) has the smallest $U(R_1)$ and that $U(R_1) = U(R_2)$ for the tilings with hexagons, octagons, and dodecagons.

Table 1. Average distance.

	(3^6)	(4^4)	(6^3)	$(3^4.6)$	$(3^3.4^2)$	$(3.4.6.4)$	$(3.6.3.6)$	(3.12^2)	$(4.6.12)$	(4.8^2)
$E(R_1)$	0.377	0.383	0.404	0.396	0.381	0.395	0.403	0.499	0.477	0.430
$E(R_2)$	0.729	0.700	0.663	0.708	0.714	0.693	0.685	0.676	0.689	0.665

Table 2. Maximum distance.

	(3^6)	(4^4)	(6^3)	$(3^4.6)$	$(3^3.4^2)$	$(3.4.6.4)$	$(3.6.3.6)$	(3.12^2)	$(4.6.12)$	(4.8^2)
$U(R_1)$	0.620	0.707	0.877	0.995	0.732	0.963	0.931	1.363	1.520	1.082
$U(R_2)$	1.075	1.000	0.877	0.995	1.035	0.963	0.931	1.363	1.520	1.082

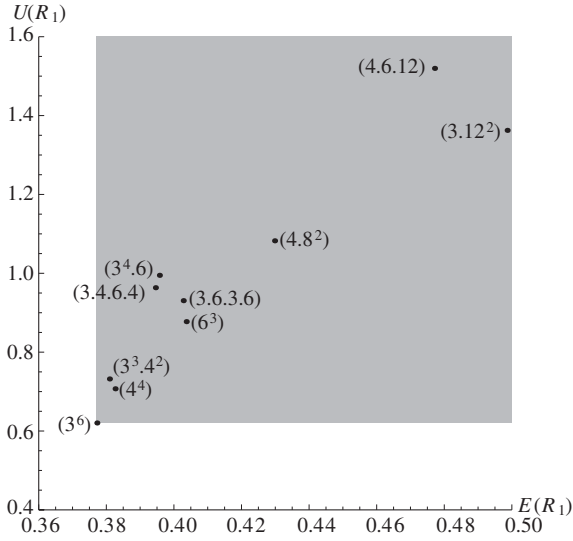


Fig. 4. Average and maximum distances.

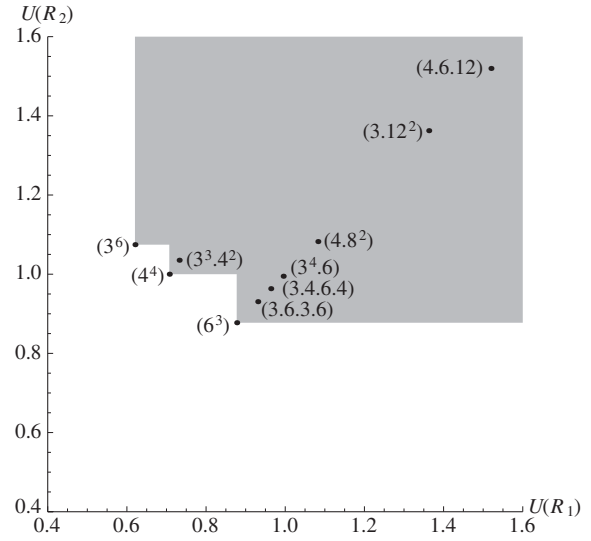


Fig. 6. Maximum distance.

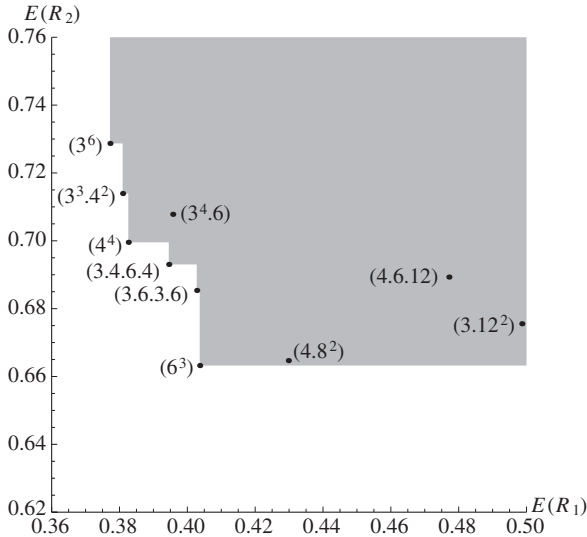


Fig. 5. Average distance.

4. Application

The distances in Archimedean tilings are useful for location analysis. The average distance can be used as a criterion of efficiency, whereas the maximum distance can be

used as a criterion of equity. In this section, we present an application to location analysis.

Suppose that facilities are located on vertices of Archimedean tilings and that customers are uniformly distributed on a plane. We examine three bi-objective problems where two distances are minimized. For any two solutions x and y , x dominates y , if each criterion for x is as good as that for y and at least one criterion for x is strictly better than that for y . The solution x is called Pareto optimal, if no feasible solution that dominates x exists.

First, we consider the following problem:

$$\min. (E(R_1), U(R_1)). \quad (36)$$

$E(R_1)$ and $U(R_1)$ in Archimedean tilings are plotted in Fig. 4. Since (3^6) has the smallest $E(R_1)$ and $U(R_1)$, (3^6) dominates the other tilings and is Pareto optimal.

Next, we consider the following problem:

$$\min. (E(R_1), E(R_2)). \quad (37)$$

$E(R_1)$ and $E(R_2)$ in Archimedean tilings are plotted in Fig. 5. It can be seen that (4^4) and $(3.4.6.4)$ dominate $(3^4.6)$, $(3.6.3.6)$ dominates $(4.6.12)$, and (6^3) dominates (4.8^2) , $(4.6.12)$, (3.12^2) . Note that

$(3^3.4^2)$, $(3.4.6.4)$, $(3.6.3.6)$ as well as the three regular tilings (3^6) , (4^4) , (6^3) are Pareto optimal.

Finally, we consider the following problem:

$$\min. (U(R_1), U(R_2)). \quad (38)$$

$U(R_1)$ and $U(R_2)$ in Archimedean tilings are plotted in Fig. 6. In this case, the three regular tilings (3^6) , (4^4) , (6^3) are Pareto optimal.

5. Conclusion

This paper has obtained the average and maximum distances to the first and second nearest vertices of Archimedean tilings. The analytical expressions for the distances are useful for location analysis as follows. First, they give an estimate for the service level of actual facility patterns. Although actual patterns of facilities are not always regular, the regular patterns are important as a typical dispersed pattern. By comparing the distances, we can evaluate the efficiency of actual patterns. Second, they demonstrate how the density of facilities affects the distances. This relationship helps planners to estimate the number of facilities required to achieve a certain level of service. The estimated number of facilities can be used as an input in location models. Note that finding the relationship between the number of facilities and the distances by using network location models requires computations for each number of facilities. Finally, the result that tilings other than three regular tilings can be Pareto optimal gives an insight into the understanding of optimal facility location.

Although the Euclidean distance is a good approximation for the actual travel distance, the rectilinear distance is more suitable for cities with a grid road network (Love and Morris, 1979; Brimberg *et al.*, 2007; Griffith *et al.*, 2012). In fact, the rectilinear distance has frequently been used in facility location models (Francis *et al.*, 1992; Aras *et al.*, 2008; O'Kelly, 2009). Examining the rectilinear distance in Archimedean tilings could be an interesting issue for future research.

Acknowledgments. This research was supported by JSPS Grant-in-Aid for Scientific Research. I am grateful to anonymous reviewers for their helpful comments and suggestions.

References

- Aras, N., Orbay, M. and Altinel, I. (2008) Efficient heuristics for the rectilinear distance capacitated multi-facility Weber problem, *Journal of the Operational Research Society*, **59**, 64–79.
- Berman, O., Krass, D. and Menezes, M. (2007) Facility reliability issues in network p -median problems: Strategic centralization and co-location effects, *Operations Research*, **55**, 332–350.
- Brimberg, J., Walker, J. and Love, R. (2007) Estimation of travel distances with the weighted ℓ_p norm: Some empirical results, *Journal of Transport Geography*, **15**, 62–72.
- Clark, P. and Evans, F. (1954) Distance to nearest neighbor as a measure of spatial relationships in populations, *Ecology*, **35**, 85–90.
- Cressie, N. (1993) *Statistics for Spatial Data*, John Wiley & Sons, New York.
- Dacey, M. (1968) Two-dimensional random point patterns: A review and an interpretation, *Papers of the Regional Science Association*, **13**, 41–55.
- Drezner, Z. (1987) Heuristic solution methods for two location problems with unreliable facilities, *Journal of the Operational Research Society*, **38**, 509–514.
- Du, Q., Faber, V. and Gunzburger, M. (1999) Centroidal Voronoi tessellations: Applications and algorithms, *SIAM Review*, **41**, 637–676.
- Farahani, R., SteadieSeifi, M. and Asgari, N. (2010) Multiple criteria facility location problems: A survey, *Applied Mathematical Modelling*, **34**, 1689–1709.
- Francis, R., McGinnis, F. and White, J. (1992) *Facility Layout and Location: An Analytical Approach*, Prentice Hall, Upper Saddle River.
- Getis, A. (1983) Second-order analysis of point patterns: The case of Chicago as a multi-center urban region, *The Professional Geographer*, **35**, 73–80.
- Griffith, D., Vojnovic, I. and Messina, J. (2012) Distances in residential space: Implications from estimated metric functions for minimum path distances, *GIScience & Remote Sensing*, **49**, 1–30.
- Grünbaum, B. and Shephard, G. (1987) *Tilings and Patterns*, Freeman, New York.
- He, F. and Duncan, R. (2000) Density-dependent effects on tree survival in an old-growth douglas fir forest, *Journal of Ecology*, **88**, 676–688.
- Holgate, P. (1965a) The distance from a random point to the nearest point of a closely packed lattice, *Biometrika*, **52**, 261–263.
- Holgate, P. (1965b) Some new tests of randomness, *Journal of Ecology*, **53**, 261–266.
- Illian, J., Penttinen, A., Stoyan, H. and Stoyan, D. (2008) *Statistical Analysis and Modeling of Spatial Point Patterns*, John Wiley & Sons, Chichester.
- Iri, M., Murota, K. and Ohya, T. (1984) A fast voronoi diagram algorithm with applications to geographical optimization problems, in *Proceedings of the 11th IFIP Conference on System Modelling and Optimization* (ed. P. Throft-Christensen), pp. 273–288, Springer-Verlag, Berlin.
- Jones, A. (1971) An order neighbor approach to random disturbances on regular point lattices, *Geographical Analysis*, **3**, 361–378.
- Jones, A., Langford, I. and Bentham, G. (1996) The application of K -function analysis to the geographical distribution of road traffic accident outcomes in Norfolk, England, *Social Science & Medicine*, **42**, 879–885.
- Koshizuka, T. (1985) On the relation between the density of urban facilities and the distance to the nearest facility from a point in a given area, *Journal of the City Planning Institute of Japan*, **20**, 85–90 (in Japanese).
- Koshizuka, T. and Ohsawa, Y. (1983) Probability density function of the distance in urban facility planning, *Journal of the City Planning Institute of Japan*, **18**, 25–30 (in Japanese).
- Larson, R. (1974) A hypercube queuing model for facility location and redistricting in urban emergency services, *Computers & Operations Research*, **1**, 67–95.
- Leamer, E. (1968) Locational equilibria, *Journal of Regional Science*, **8**, 229–242.
- Lei, T. and Church, R. (2011) Constructs for multilevel closest assignment in location modeling, *International Regional Science Review*, **34**, 339–367.
- Love, R. and Morris, J. (1979) Mathematical models of road travel distances, *Management Science*, **25**, 130–139.
- Marianov, V. and ReVelle, C. (1996) The queueing maximal availability location problem: A model for the siting of emergency vehicles, *European Journal of Operational Research*, **93**, 110–120.
- Miyagawa, M. (2009) Order distance in regular point patterns, *Geographical Analysis*, **41**, 252–262.
- Nickel, S., Puerto, J. and Rodríguez-Chía, A. (2005) MCDM location problems, in *Multiple Criteria Decision Analysis: State of the Art Surveys* (eds. J. Figuiera, S. Greco and M. Ehrgott), pp. 761–795, Springer, New York.
- O'Kelly, M. (2009) Rectilinear minimax hub location problems, *Journal of Geographical Systems*, **11**, 227–241.
- Persson, O. (1964) Distance methods: The use of distance measurements in the estimation of seedling density and open space frequency, *Studia Forestalia Suecica*, **15**, 1–68.
- Pirkul, H. (1989) The uncapacitated facility location problem with primary and secondary facility requirements, *IIE Transactions*, **21**, 337–348.
- Ripley, B. (1976) The second-order analysis of stationary point processes, *Journal of Applied Probability*, **13**, 255–266.
- Snyder, L. and Daskin, M. (2005) Reliability models for facility location: The expected failure cost case, *Transportation Science*, **39**, 400–416.
- Sorensen, P. and Church, R. (2010) Integrating expected coverage and local reliability for emergency medical services location problems, *Socio-Economic Planning Sciences*, **44**, 8–18.
- Thompson, H. (1956) Distribution of distance to n th neighbour in a population of randomly distributed individuals, *Ecology*, **37**, 391–394.
- Weaver, J. and Church, R. (1985) A median location model with nonclosest facility service, *Transportation Science*, **19**, 58–74.