

Chaos and Spatiotemporal Chaos in Convective Systems

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Much of early research on chaos from the viewpoint of physics was performed using spatially confined convective systems. In spatially extended convective systems, on the other hand, spatiotemporal chaos occurs. However, there is no unified definition for the term spatiotemporal chaos as for chaos. To unify definition, a property common to the three kinds of spatiotemporal chaos observed in electroconvection of nematic liquid crystals is presented.

Key words: Chaos, Spatiotemporal Chaos, Convection, Correlation Length

1. Chaos in Convective Systems

If fluid filled between two parallel plates is heated from below, thermal (Rayleigh–Bénard) convection occurs. On such convective systems much research has been done under the heading of nonlinear physics, chaos being one of the most important phenomena studied in nonlinear physics. Research on chaos in dissipative systems from the viewpoint of physics has been closely connected with convective systems. For example, the *Lorenz model* (Lorenz, 1963), in which important concepts about chaos such as the butterfly effect (sensitivity to initial conditions) and strange attractor were discovered (Bergé *et al.*, 1986), was derived as a dynamical system describing the time-development of amplitudes of a convective structure. The Lorenz model showed that ordinary coupled differential equations with only three variables can induce nonperiodic motion. The experimental verification of the Feigenbaum constant, discovered in period-doubling bifurcation to chaos in the logistic map, was done in convective systems (Libchaber *et al.*, 1983). A comparison between a map which exhibits intermittent scenarios to chaos and real physical systems was also performed for convective systems (Bergé *et al.*, 1980).

When the Lorenz model was derived, spatially incoherent motion was ignored. As a result, ordinary coupled differential equations with no spatial degrees of freedom were derived. However, real convective systems intrinsically have spatial degrees of freedom. Therefore, to experimentally observe chaos, a spatially confined system for which the aspect ratio Γ is small ($\sim \mathcal{O}(1)$) needs to be prepared. Here, the aspect ratio Γ is defined as the ratio of the width L of the plate to the distance d between the parallel plates; i.e., $\Gamma = L/d$. $\Gamma \sim \mathcal{O}(1)$ signifies that the number of roll pairs in the system is $\mathcal{O}(1)$. The spatial coherence of the motion of convective rolls can be maintained in such convective systems.

Furthermore, we need to determine whether a nonperiodic oscillation appearing in such a small convective system is chaos, namely induced by a few degrees of freedom. The technique and determining criterion were established as follows. If a small noninteger fractal dimension and at least one positive Lyapunov exponent are obtained from an orbit in phase space formed by the embedding technique, the dynamics can be classified as chaos. Furthermore, if the nonperiodic oscillation appears from a limit cycle via a characteristic bifurcation such as period doubling, torus collision or intermittency by increasing the control parameter (Rayleigh number), the classification becomes more definite.

We observe that a convective roll pattern shows nonperiodic oscillations after a characteristic bifurcation while maintaining spatial coherence only when the Rayleigh number is relatively small. Even in a system with the small Γ , spatial coherence is lost in developed turbulence for high Rayleigh number. Such nonperiodic oscillations at low Rayleigh number in convective systems constitute so-called *weak turbulence*.

2. Spatiotemporal Chaos

Weak turbulence in artificially prepared convective systems with small Γ has been studied within the context of chaos. In contrast, for systems with large aspect ratio, disorder appears spatially as well as temporally. Such weak turbulence in large systems is called *spatiotemporal chaos* (Cross and Hohenberg, 1993).

However, there is no criterion to determine whether spatiotemporal disorder appearing in a spatially extended system is spatiotemporal chaos. Indeed, how is spatiotemporal chaos distinguished from developed turbulence? From here, we shall present a unified perspective of spatiotemporal chaos based on the correlation length of spatially disordered patterns by considering electroconvection in a nematic liquid crystal (de Gennes and Prost, 1993) as an example. In the electroconvection of nematics, various types

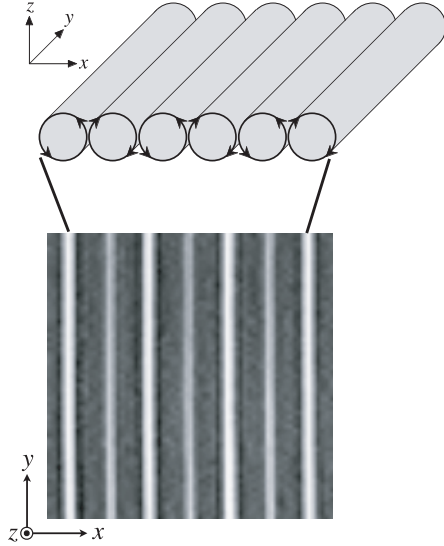


Fig. 1. Pattern in electroconvection of nematics. White lines correspond to convective rolls.

of spatiotemporal chaos appear in consequence of the interaction between convection and molecular alignment (Hidaka and Kai, 2009a).

3. Spatiotemporal Chaos in Electroconvection of Nematics

Electroconvection in planar alignment systems, for which the nematic director is parallel to the electrodes, has been under continual research since its discovery. The stripe pattern corresponding to the convective periodic structure appears after applying a voltage beyond a certain threshold (Fig. 1).

The stripe pattern becomes unstable and fluctuates with increasing applied voltage. These fluctuations induce the creation of defects and their motion as shown in Fig. 2(a). Because defect creation and motion occurs irregularly in space and time, this phenomenon is called *defect turbulence*.

As voltage increases, the defects assemble into a lattice pattern called a defect lattice (Oikawa *et al.*, 2004). By further increasing the applied voltage, the defect lattice breaks into developed turbulence via a type of spatiotemporal chaos called *spatiotemporal intermittency* (Fig. 2(b)). In spatiotemporal intermittency, turbulent areas partially appear, and disorder and order spatially coexist.

In contrast, the electroconvection in homeotropic alignment systems has been actively investigated over the past two decades. In this system, stationary stripe patterns become immediately unstable because of the Nambu-Goldstone mode for the nematic director and consequently spatiotemporal chaos appears (Hidaka and Kai, 2009b). In this type of spatiotemporal chaos, called *soft-mode turbulence* (Fig. 2(c)), local convective rolls assume any and every direction (Hidaka *et al.*, 2006).

These three types of spatiotemporal chaos are seemingly different. However, these have common properties by which they maintain local order corresponding to a convective roll pair despite the presence of global disorder. Here

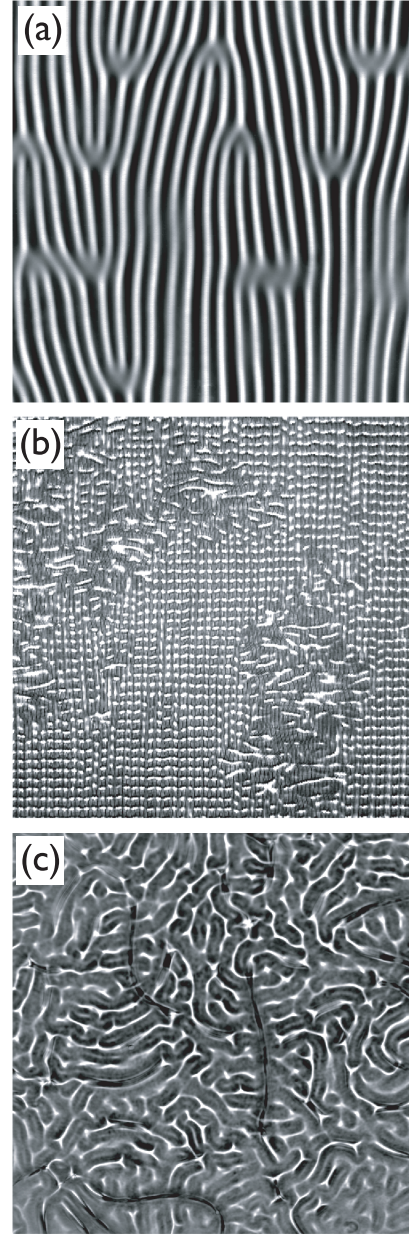


Fig. 2. Snapshots of patterns emerging as spatiotemporal chaos. (a) Defect turbulence. (b) Spatiotemporal intermittency. (c) Soft-mode turbulence.

we denote the width of the roll pair by λ ; λ corresponds to d mentioned above.

4. Correlation Lengths of Spatiotemporal Chaos

Defect turbulence is recognized as a phenomenon where the position of rolls in x is fluctuating. A snapshot $u(\mathbf{r})$ ($\mathbf{r} = (x, y)$) of the defect turbulence is described as

$$u(\mathbf{r}) = R(\mathbf{r}) \exp[i(qx + \alpha(\mathbf{r}))] + \text{c.c.}, \quad (1)$$

where $q = 2\pi/\lambda$. Thus defect turbulence can be described by the phase α in the reduced form. Figure 3(a) shows $\sin \alpha(\mathbf{r})$ obtained from Fig. 2(a) using an image processing technique. A defect corresponds to a singular point of $\alpha(\mathbf{r})$. Because $\alpha(\mathbf{r})$ can be recognized as a kind of two-dimensional XY field, a two-point correlation function

$C(r)$ of $\alpha(\mathbf{r})$ can be introduced

$$C(r) = \langle \cos[\alpha(\mathbf{r} + \mathbf{r}_0) - \alpha(\mathbf{r}_0)] \rangle_{\mathbf{r}_0}, \quad (2)$$

where $r = |\mathbf{r}|$ denotes the distance between any two points. By analogy with the two-dimensional XY model (Käser *et al.*, 2007), it is conjectured that $C(r)$ is expressed as $\exp(-r/\xi)$ and that the correlation length ξ corresponds to the average distance between the nearest defects. As observed in Fig. 3(a), ξ is sufficiently larger than λ .

Each point in the image of a snapshot of spatiotemporal intermittency is classified as either turbulent or ordered. Figure 3(b) is obtained from Fig. 2(b) by this classification. In this view, spatiotemporal intermittency is similar to an Ising spin system or a percolation system. In the percolation model, the two-point correlation function is defined as the probability that two specified points are included in the same cluster of a state. If each cite is in the state with probability p ,

$$C(r) = p^r = \exp\left(-\frac{r}{\xi}\right), \quad (3)$$

where $\xi = -1/\ln p$ corresponds to the average diameter of the cluster (Stauffer and Aharony, 1994). If an analogy between the percolation model and spatiotemporal intermittency is assumed, p corresponds to the areal fraction of the turbulent state, and ξ corresponds to the average diameter of the turbulent cluster. Also in this case, ξ is sufficiently larger than λ .

In soft-mode turbulence, with the local convective roll assuming any direction, a snapshot $u(\mathbf{r})$ is expressed as

$$u(\mathbf{r}) = R_0 \exp(i\mathbf{q}(\mathbf{r}) \cdot \mathbf{r}) + \text{c.c.}, \quad (4)$$

where R_0 is constant. The wavevector $\mathbf{q}(\mathbf{r})$ can be described only by the azimuthal angle ψ , namely,

$$\mathbf{q}(\mathbf{r}) = (q \cos \psi(\mathbf{r}), q \sin \psi(\mathbf{r})), \quad (5)$$

where $q = |\mathbf{q}(\mathbf{r})| = 2\pi/\lambda$ is constant. Figure 3(c) is a realization of $\psi(\mathbf{r})$ obtained from Fig. 2(c). This image shows that the spatial pattern consists of patches over which the direction of the local convective roll is uniform. The two-point correlation function for $\psi(r)$ decays as $\exp(-r/\xi)$ (Anugraha *et al.*, 2008) where the correlation length ξ corresponds to the average diameter of patches.

The property common to the three types of spatiotemporal chaos is that the correlation length ξ of disorder is sufficiently larger than the size λ corresponding to local order. This can be a universal criterion for spatiotemporal chaos in convective systems.

5. Discussion

By adopting this criterion, spatiotemporal chaos can be distinguished from developed turbulence where ξ is much smaller than λ . Indeed, because convective rolls are broken in developed turbulence, we should express it as $\xi \ll d$.

In a system where chaos can be observed, $\Gamma \sim \mathcal{O}(1)$ implies $L \sim \lambda$, because $\lambda \sim d$. The fact that ξ is sufficiently larger than λ means that L is sufficiently smaller than ξ . Because ξ corresponds to the size of coherent motion, global

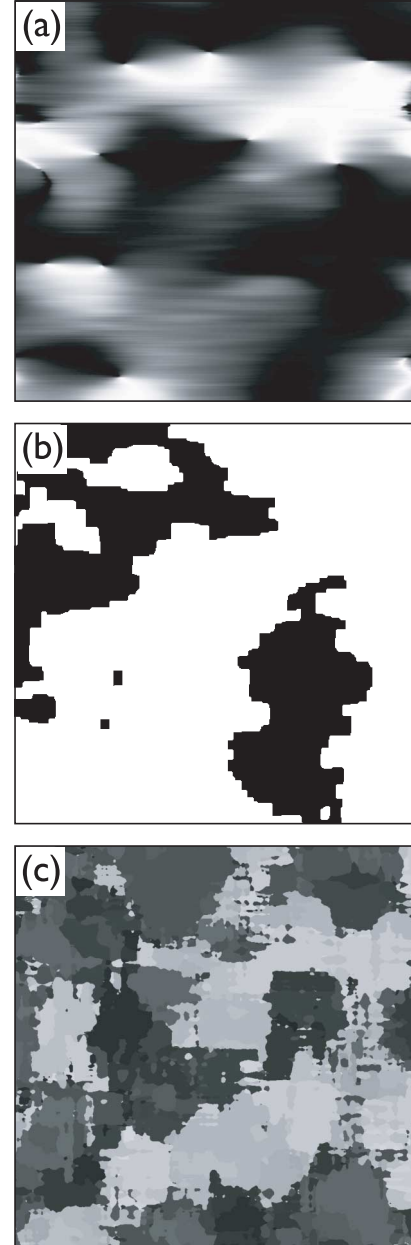


Fig. 3. Reduction patterns for the three types of spatiotemporal chaos. Each image corresponds to Fig. 2. (a) Gray scale indicates $\sin \alpha$. White and black correspond to $\sin \alpha = 1$ and -1 , respectively. (b) White and black indicate ordered and turbulent states, respectively. (c) Gray-scale plot of ψ . White and black correspond to $\psi = \pi/2$ and $-\pi/2$, respectively.

coherence is kept in the system with L sufficiently smaller than ξ .

The order of convective structures remains in the scale range between λ and ξ in soft-mode turbulence and defect turbulence. This means order and disorder coexist. Recently the form of the temporal correlation function was found to change with the time range (Narumi *et al.*, 2013). It is thought that this “dual structure” reflects this coexistence.

The coexistence is explicit in spatiotemporal intermittency. The spatial correlation functions for defect turbulence and spatiotemporal intermittency have not been

sufficiently-well investigated. In this article, $C(r)$ was assumed exponential for the sake of simplicity. In the Ising and directed-percolation models the distribution functions exhibit power-law behavior near the critical point. For some nonequilibrium open systems, it was found that the distribution function exhibits power-law behavior over a wide range for the control parameter (Roberts *et al.*, 1996). This property which is called “generic scale invariance” (Roberts *et al.*, 1996) may appear for actually obtained $C(r)$ for defect turbulence and spatiotemporal intermittency. If $C(r)$ exhibits power-law behavior, the correlation length cannot be defined specifically. Nevertheless, it can be concluded that the correlation size is larger than that of the local order in spatiotemporal chaos.

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