# A Trapezium Generates Tiling of Concentric Regular Pentagons

Syed A. Jafar

*Flat Number 5-B, Whispering Meadows, Haralur Road, Bangalore 560 102, India*
*E-mail address: syeda_jafar@yahoo.com*

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A case of tiling using a single tile of golden trapezium is presented and each of the five pentagonal sectors when seen in isolation, reveals planes of translation suggesting periodicity. It is shown that this tile not only tessellates Euclidean plane, expanding to infinity, but also maintains perfect regular pentagonal outline at each completed generation of tiling. The entire tiling suggests a five-fold intergrowth structure of a twinned crystal, which should yield sum of five discrete and superimposed ordered periodic diffraction patterns. Clue to this tiling was primarily provided by a Coccolithophore species: *Braarudosphaera bigelowii*, which makes an elegant regular pentagonal dodecahedron of mineral Calcite, invisible to unaided human eye.
**Key words:** Golden Trapezium, Periodic Tiling, Concentric Regular Pentagons, Twinned Crystal, *Braarudosphaera bigelowii*

## 1. Introduction

A single tile tessellating Euclidean plane only nonperiodically continues to elude scientists, but a single tile of golden trapezium dealt herein could possibly be a potential candidate for generating forced aperiodic tiling when edges are marked, coloured or keyed. Congruent convex polygons generating nonperiodic tiling also tile periodically (Gardner, 1977). There are just three regular convex congruent polygons permitting complete tessellation of Euclidean plane in periodic patterns: equilateral Triangle, Square and regular Hexagon. Possibilities multiply when non-regular convex polygons are used (Grünbaum and Shephard, 1987). However, much confusion exists in mathematical literature about the terms periodic, aperiodic, nonperiodic and quasiperiodic. Penrose tilings are examples of aperiodic and quasiperiodic tiling, which require that the tile edges be coloured or keyed to force aperiodicity (Penrose, 1974). Non-repeating pattern is characteristic of aperiodic tiling, which lacks translational symmetry and serves as a model for quasicrystals (Schechtman *et al.*, 1984). In contrast, nonperiodic tiling tiles the plane in an irregular pattern but can also tile in a regular periodic manner.

## 2. Golden Trapezium and *Braarudosphaera bigelowii*

It was found that this unique single tile designated as "golden trapezium" could be carved out from the Pythagorean pentagram, which also contains familiar pairs of Penrose tiles (Fig. 3). Initial clue to this tiling was furnished by Scanning electron- and polarized light microscopic observation of a single cell fossil and extant marine haptophyte alga belonging to Coccolithophores: *Braarudosphaera bigelowii* (Gran and Braarud, 1935; Deflandre, 1947).



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The tests of *B. bigelowii* (Figs. 1 and 2) are made up of 12 regular pentagonal plates of finely laminated (*ca.* 0.1 $\mu$m) crystallites of mineral Calcite resulting in perfect regular pentagonal dodecahedral symmetry (Jafar, 1975, 1998; Brieskorn, 1983; Hagino *et al.*, 2013).

*B. bigelowii* which probably makes the most perfect regular pentagonal dodecahedral hollow test comprising of mineral Calcite ($CaCO_3$)—a unique combination of periodic lattice enveloping quasiperiodic dodecahedral frame in the world of microbiogenic particles (*ca.* 10–15 $\mu$m diameter). Three variants of dodecahedral tests are found matching with Euclidean-flat, Elliptic-convex and Hyperbolic-concave pentagonal faces (Thurston and Weeks, 1984). Each regular pentagonal plate displays minor variations but is occasionally subdivided into five "golden trapeziums" by five dextrally oriented radial sutures (Figs. 1 and 2). Another extinct Coccolithophorid species *Micrantholithus hoschulzii* (Reinhardt, 1966; Thierstein, 1971), flourished over 140 million years ago in ancient sea, also built perfect regular pentagonal dodecahedral tests with hollow interior, but owing to weak architectural design complete tests are not preserved but regular pentagonal plates with five radial sutures joining at vertex and creating five isosceles ($72^\circ$ - $54^\circ$ - $54^\circ$) triangular units of mineral Calcite ($CaCO_3$) are preserved. These triangular units can be used to generate similar concentric tiling as by golden trapezium in *B. bigelowii*, expanding to infinity.

## 3. Concentric Regular Pentagonal Tiling

A paper of Bagley (1965) introduces an interesting tiling of concentric pentagons by using a plane of hard spheres, so that each pentagon side has an odd number of balls followed by construction of a plane with an even number of spheres per pentagon side. If both planes are placed in intimate contact with their five-fold axes coincident, there results a layer which can be stacked one upon another and packed to in-

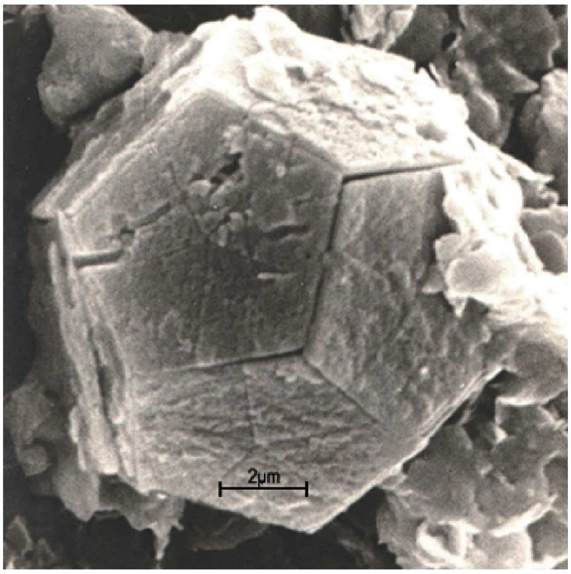


Fig. 1. Test of a complete regular pentagonal dodecahedron of *Braarudosphaera bigelowii* (Gran and Braarud, 1935) Deflandre, 1947, from *ca.* 15 million year old Miocene rocks of Walbersdorf, Austria. Note dextral rotation of five radial sutures on each pentagonal plate of mineral Calcite occasionally carving perfect tiles of "golden trapezium". A total of 60 such tiles (sinistrally oriented) make up the entire and hollow pentagonal dodecahedron (scanning Electron Micrograph: reproduced from Jafar,1975).


Fig. 2. Diagram of a complete regular pentagonal dodecahedral test of a Coccolithophore: *Braarudosphaera bigelowii* (Gran and Braarud, 1935) Deflandre, 1947, consisting of finely laminated pentagonal plates (six shown) of biogenic Calcite traversed by five dextrally rotated radial sutures carving six-sided Calcite units displaying outlines of tiles matching "Golden Trapezium" (blue).

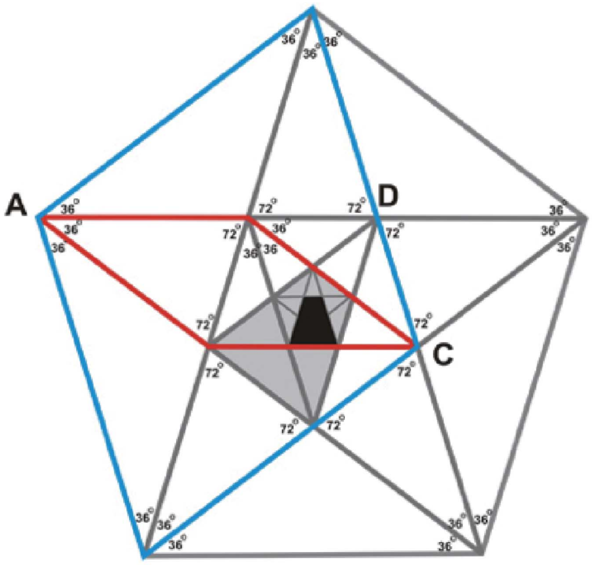


Fig. 3. Pythagorean Pentagram showing 3-generations of deflating "Golden Trapezium" (ABCD) tile in fractal dimension. Fat (blue) - Slim (red) rhombi and Kite-Dart pair of Penrose tiles are also marked.

finity. Thus an infinite structure can simply be generated, the nucleus of which is a pentagonal dipyramid of seven spheres. This structure bears striking resemblance to isosceles triangle tiling observed in *Micrantholithus hoschulzii* with interior angles of 72° - 54° - 54°. Length of the edge of pentagon: 1, 2, 3, 4, 5 $\cdots$. Area of the pentagon: 1, 4, 9, 16, 25 $\cdots$. These increasing ratios are remarkably the same as for golden trapezium tiling (Figs. 4 and 5), which essentially differs from aperiodic Penrose tiling in lacking nested properties, while Penrose tiling is a nested structure. The trapezium tiling has a solitary centre of global five-fold rotational symmetry. The Penrose tiling has both solitary centre of global five-fold rotational symmetry and several centres of the local five-fold rotational symmetry.

Golden trapezium by definition is an isosceles trapezium whose two Legs and larger Base are of identical length and Base angles are equal (108° - 108° - 72° - 72°). An intercept equal to the length of smaller Base made on any of the equal line segments of Legs and larger Base results in golden ratio phi: $1 + \sqrt{5}/2 = 1.618\ldots$ (Mario, 2002). This tiling is indeed unique and could be taken to imply that slicing a regular pentagon into five golden trapeziums and concentric expansion of the pattern using a single tile could actually yield larger and larger regular pentagons in a perfect and infinite manner.

A Pythagorean pentagram containing any inscribed figure can inflate or deflate to infinity as every line segment in relation to the next smaller one maintains golden ratio. Familiar Penrose pair of Kite-Dart and Fat-Slim rhombi including "golden trapezium" described herein are demarcated (Fig. 3). Here it is shown that this single tile not only tessellates in periodic pattern but also retains conspicuous regular pentagonal outline at each completed generation of tiling. There is more than one elegant way to arrange five "golden trapeziums" around a vertex and let the pattern be inflated concentrically by adding 10 extra tiles in each completed generation of tiling: 5 - 15 - 25 - 35 - 45 ... (Figs. 4 and 5). The tiling has been named as "Cobweb" pattern. Figure 5 is generated by arranging the tiles slightly differently: first layer sinistral (black), second layer dextral (blue), third layer sinistral (red), fourth layer sinistral (green).

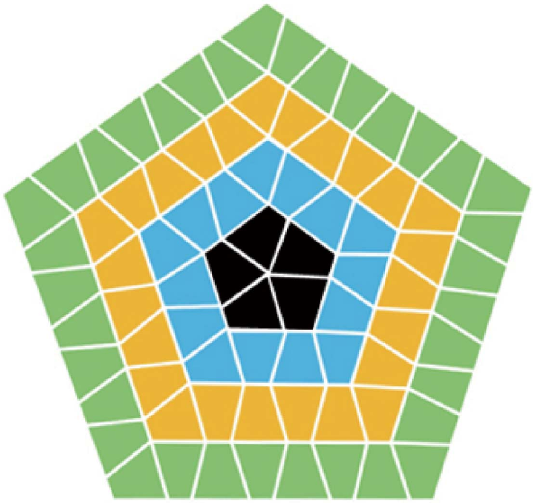

Fig. 4. The most fundamental concentric tiling of regular pentagons is generated by arranging five “golden trapeziums” (108° - 108° - 72° - 72°) dextrally around a vertex in the central area and the pattern is concentrically inflated by adding more tiles around it as shown.

Fig. 5. A concentric tiling of regular pentagon is generated by arranging five “golden trapeziums” differently than in Fig. 4.

## 4. Tiling and Twinned Crystal

Senechal (1995) commented: “$\cdots$ while from a crystallographic perspective, the structure of Fig. 4 is just a twinned crystal for which the twin boundaries can be filled nicely with the same trapezoidal shape as is used to make the crystals”. Closer examination of the areas between adjoining vertices of pentagon and laying a lattice reveals period parallelograms containing identical pieces of the tile matching fundamental domains having twice the area of golden trapezium and displaying translational symmetry, characteristic of periodic tiling. The central region displaying pentagonal-chiral symmetry could be interpreted as a 5-fold intergrowth structure of a twinned crystal. If subjected to laser beam experiment, such a tiling would reveal pseudo-fivefold rotational symmetry with periodic bright spots. The diffraction pattern would thus be a sum of discrete periodic diffraction patterns.

However, Bagley (1965) citing experimental evidence, suggested that it is unlikely that twinning could produce such a structure in small sized nanoparticles, instead, formation of pentagonal dipyramid nucleus and its subsequent growth is a simpler and more probable mechanism.

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