How Many Loops Kolam Loop Pattern Consists of

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Kolam is a traditional loop pattern, in which a line goes around some dots in an array. In this paper, we reported on our study of the construction of Kolam, describing how many loops a drawn Kolam has. Considering Kolam as a knot-link pattern and a navigating line (N-line) of Kolam as a planar graph of the knot-link, we analyze the loop number as the component number using the Tutte polynomial and the invariant of it. In Appendix A, the author introduced also two matrix processes from the Kolam pattern as a Medial graph to obtain a loop number.
Key words: Kolam, N-line, Loop Number, Tutte Polynomial, Medial Graph

1. Introduction

Traditional loop patterns can be drawn on the sand such as occurs with the Sona in Angola or the Nitus of Vanuatu Island in the South Pacific: on some material, e.g. tombstones, rings and knits such as with Celtic knots in Ireland; and on the ground such as with the Kolam in South India. These are designed or drawn using some very simple common rules and in simple structures: the pattern has also very rich diversities in the form.

These artistic forms inspire some interesting issues. In this paper, the author discuses two questions: ***What conditions are possible to make a single loop or a multi-loop pattern?*** and ***How many loops does the given loop pattern consist of?*** In this study, we used Kolam as a representation of loop patterns.

2. Kolam Patterns

Kolam is drawn in such a way that at first a woman painter locates a set of dot arrays on the ground, and she then begins to draw a line around a dot. The line continues to go around the dot or another adjacent dot according to the following rules: When the line goes around the other adjacent dot, it must change the direction of right (clockwise) or left (anticlockwise) against the dot. When the line meets with itself, or another line at a point between two adjacent dots, the lines cross. At a crossing, the line goes straight (to cut through) and after crossing, the line changes a turning direction alternately. The line should be able to go back to the beginning point.

A painter seems to imagine a filling polygon (tile) in tessellation around each dot of the array, and also a crossing point between two adjacent dots surrounded by the polygons in her brain-inside-visual-field. Most Kolam patterns are drawn on a regular rectangle (square) dot array and crossing points between orthogonally arranged dots, which are regular crossings. She imagines also a line between two adjacent dots for setting the crossing point on the line, and this line is called a navigating line (N-line) by the authors.

There is an additional condition, in which instead of a crossing on the N-line between two adjacent dots (tiles), the lines become parallel at two points on the common edge between two tiles and connect the two tiles. This configuration is called a two-point-connection with non-crossing shown in Fig. 2(1) in Sec. 5. In topology, two adjacent tiles and two dots combine to form one expanded tile and one combined dot, maintaining the character of the graph.

3. How Many Loops does Kolam Consist of?

The Line around Dots - Swinging-Line Kolam pattern (LaD - Sikku Kolam in Tamil language) is studied on these questions: ***What conditions are possible to make a single loop or a multi-loop pattern?*** or ***How many loops does the given Kolam consist of?***

The mat (lattice) Kolam is a Kolam of a regular dot (tile) array of N $\times$ M, all edges of which have crossings. We know already that the loop number of the lattice Kolam on a N $\times$ M dot array is GCD(N,M)[1,2]. That fact came from the problem of how a ball in billiards reflects the edges of the table. Some other conditions were also analyzed using the N-line. An open line of the N-line results that the loop number is one, and one closed with even crossings results that the loop number is one (odd crossings make a multi loop Kolam).

However, when the N-line is a closed line (cycle), but not one circuit, the loop component number is not given easily. In graph theory, we know that the Tutte polynomial function is decided with a vertex number and a component number of a knot-link graph[3]. A Kolam corresponds to a knot-link. Therefore finding the loop number of a Kolam, the Tutte polynomial function will be useful.

4. Tutte Polynomial and the Invariant

In the mathematical discipline of graph theory, the medial graph of plane graph G is another graph M(G) that repre-

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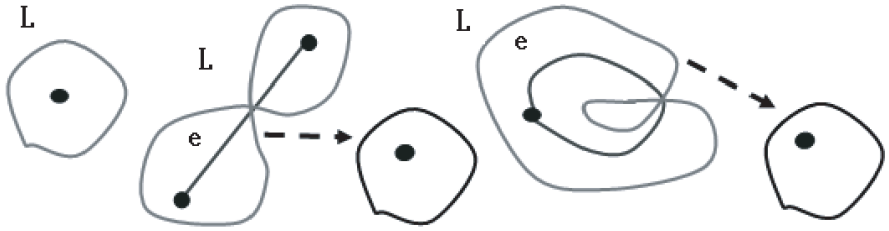
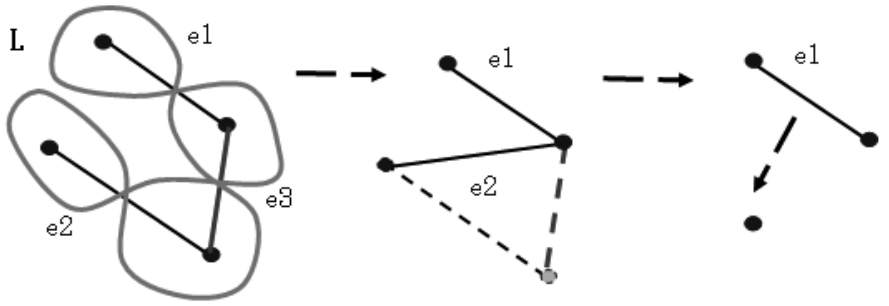
(Definition 0) Vertex T(G;X,Y) = 1,
(D1) Bridge T(G;X,Y) = XT(G/e) = X, contraction, T(G;−1,−1) =−1,
(D2) Loop T(G;X,Y) = YT(G-e) = Y, deletion, T(G;−1,−1) =−1



(D1-2) T(G;X,Y) = XT(G/e3) = X XT(G/e3/e2) = X X X T(G/e3/e2/e1)
= X X X, T(G;−1,−1) =−1



(D2-2) T(G;X,Y) = Y T(G-e2)= X Y T(G-e2/e1) =X Y,
T(G−1,−1) =−1= (−1)*(−1) = 1, where e2 is a loop

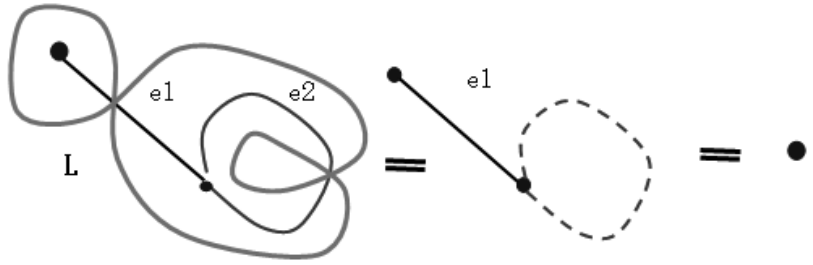


Fig. 1. These figures show how to obtain the invariants of a Tutte polynomial from a knot-link pattern L and a planar graph G of L. The Tutte polynomial of G top-down to a polynomial of X, Y, and vertexes, without any edges of T(G; X,Y) = 1 (non-resolvable graphs) using recurrence relation. Finally, we can calculate to obtain a real value of the invariant T(G; −1, −1).

sents the adjacencies between edges in the faces of G. M(G) corresponds to a Kolam pattern or a knot-link pattern L in this paper.

Here, the author introduces an analyzing way of a graph called the Tutte polynomial to get the component number of a knot-link, which corresponds to the loop number of a Kolam pattern. (Note this loop is different from a loop in (D2) of the Tutte polynominal, but a component in a knot-link.)

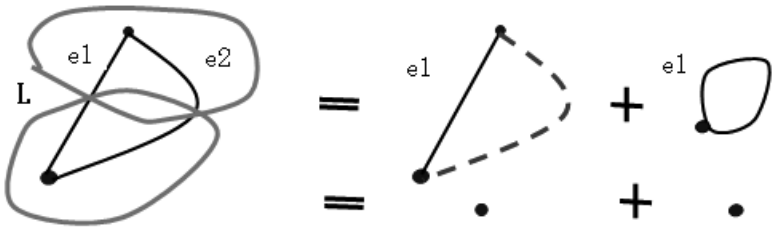
We must first introduce the definition of the Tutte polynomial. When G is a regular planar graph of a knot-link L - L is the medial graph of G -, the Tutte polynomial satisfies the following recurrence relations: Here an edge e ∈ E(G)

(D0) If G has no edges (only vertex, non-resolvable graph), then T(G; X,Y) = 1.

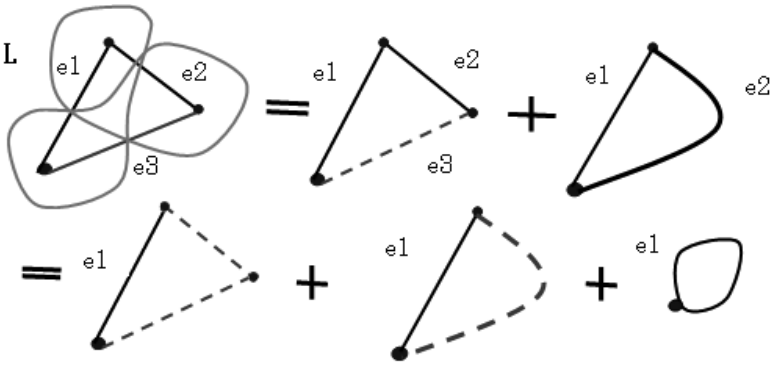
(D1) If G has a bridge e (open circuit), e can then be contracted to make the both of vertexes of e one vertex, T(G; X,Y) = XT(G/e; X,Y) (contraction).

If an edge is cut and then the connected (total) component number of G increments, the edge is a bridge (cut-edge). (If an edge connects two isolated Gs, the edge is a bridge.) If G itself is a bridge, T(G; X,Y) = X.

(D3-1) T(G;X.Y) = T(G-e2) deletion + T(G/e2) contraction
= X T(G-e2/e1) + Y T(G/e2-e1)= X + Y, T(G;−1,−1) =−1−1=−2



(D3-2) T(G;X.Y) = T(G-e3) deletion + T(G/e3) contraction
= X T(G-e3/e2) + (T(G/e3-e2) + T(G/e3/e2))
= X X T(G-e3/e2/e1) + (X T(G/e3-e2/e1) + Y T(G/e3/e2-e1))
= X X + X + Y, T(G;−1,−1) = +1−1−1 =−1



(D3-3) T(G;X,Y) = T(G-e4) deletion + T(G/e4) contraction
= X X X + (X X + X + Y), T(G;−1,−1) =−1+1−1−1 =−2

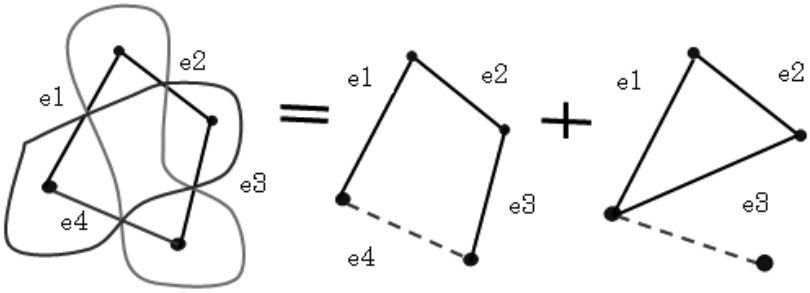


Fig. 1. (continued).

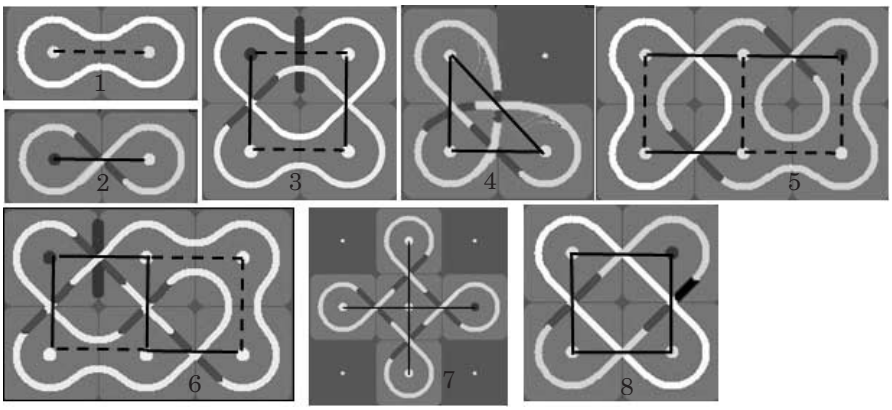
(D2) If G has a loop e and e can be deleted, T(G; X,Y) = YT(G-e; X,Y) (deletion). If both ends of an edge are the same vertex, the edge is a loop. If G itself is a loop, T(G; X,Y) = Y. (Note this loop is different from a loop of the description of Kolam.)

(D3) If otherwise a closed circuit. e is neither a loop nor a bridge, then T(G; X,Y) = T(G/e; X,Y) contraction + T(G-e; X,Y) deletion.

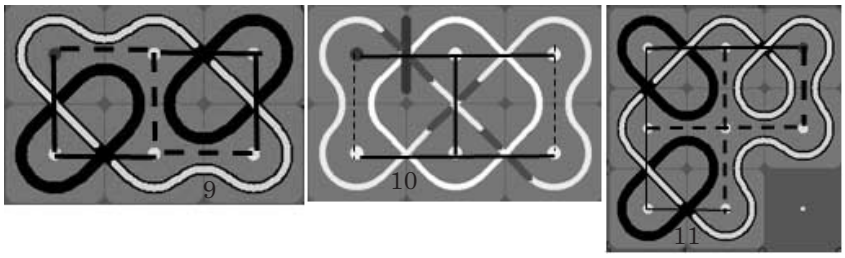
We analyzed the Tutte polynomial of G top-down to a polynomial of X, Y, and vertexes, without any edges of T(G; X,Y) = 1 (non-resolvable graphs) using recurrence relation. Finally, we can calculate to obtain a real value of the invariant T(G; −1,−1). Figure 1 shows how to obtain the invariants of the Tutte polynomial from a knot-link pattern L and a planar graph G of L.

Schwarzler and Welsh[4] introduced the relation between the component number of L and invariant T(G; −1,−1) of the Tutte polynomial of G, which was proofed by Martin[6].

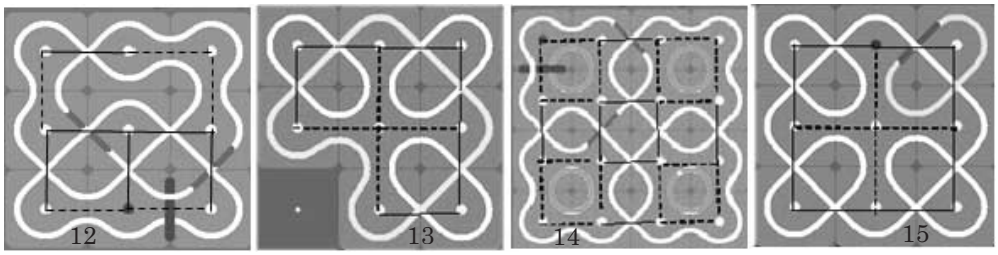
Here, when T(G; X,Y) is the Tutte polynomial of a graph G (E an edge set, V a vertex set), |E(G)| is the edge number



1: Unknot-with no edge (no crossing. Two-point connection Kolam) E = 0,
2: The infinity symbol, unknot E = 1, The edge in the graph G (N-line of this Kolam) is contracted to the graph (1). It corresponds to an untwisting or uncrossing Kolam.
3: Hopf Link E = 2, C = 2, 4: Trefoil Knot using diagonal crossings E = 3,
5: Link E = 3, C = 2, 6: Figure Eight Knot E = 4, 7: Unknot E = 4, 8: Link E = 4, C = 2



9: Link E = 4, C = 3, 10:Whitehead Link E = 4, C = 2, 11:Link E = 5. C = 3



12: Borromean Rings E=6, C=2, 13; Link E=6, C=4, 14: Link E=8, C=4, 15: Link E=8, C=5.

Fig. 2. Samples of relations between Kolam (L) and N-lines (G). Some patterns of the cases of the graph 1–19 are analyzed by the Tutte polynomial. Refer to Table 1 for the cases 1–15. The case 20 will be studied more for getting a formula in the future.

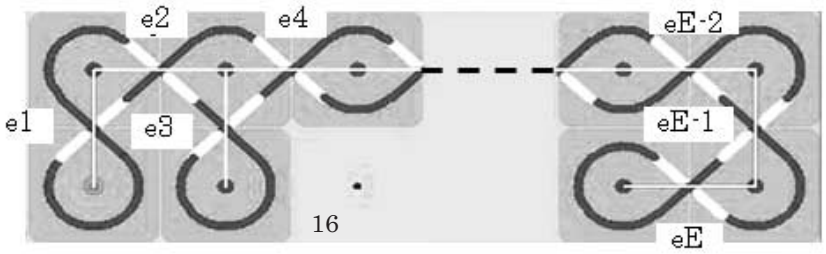
of the graph G, and C(L) is the component number of a knot-link L, the following relational equation is known:

$$T(G;-1,-1) = (-1)**|E(G)| * (-2)**(C(L)-1),$$

where a medial graph of G is a knot-link L (a planar regular four-edge graph), as each vertex of the crossing points of Kolam has four edges of lines- and C(L) is the component number of L. |E| and C correspond to the crossing number and the loop number of a Kolam respectively.

According to the previous equation, the author calculated and confirmed the Tutte polynomial invariants T(G; −1,−1) of some Kolam patterns shown in Table 1. The component number C(L) is derived by reverse lookup from this table or it is derived from the following reserve equation.

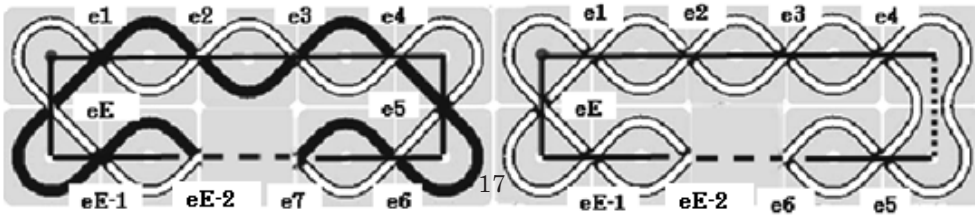
$$\begin{aligned} C &= Log_{-2}(T(G;-1,-1)/(-1)^{|E(G)|}) + 1 \\ &= Log_2(|T(G;-1,-1)|) + 1. \end{aligned}$$



16: When the N-line of a Kolam is an open circuit, it looks like the branches of a tree. Imagine N-lines to some branches, around which a vine twines or a snake winds at the crossings. When the snake bites its tail, or the initial vine is combined with the last, it makes a loop. When the twining line is untwisted at the crossing on the initial edge e1, the number of the edges decreases by one. This operation, a contraction in the Tutte polynomial, continues until the terminal edge eE and finally, we can obtain only one loop (C=1).

T(G; X,Y) = XT(G/e1) = XXT(G/e1/e2) = XXXT(G/e1/e2/e3)

= X**E T(G/e1/e2/e3…/eE) = … = X**E, T(G;−1,−1) = (−1)**E, then C = 1.



17: When the N-Line is one closed circuit. Delete and contract the final edge eE, and the deleted G becomes open and the contracted G becomes closed, then again contract the e(E−1) edge of the closed G, and continue until only the initial edge e1 (Refer D3-1, D3-2 and D3-3 in the previous Sec. 4)

$T(G; X,Y) = T(G\text{-}e_E) + T(G/e_E) = X^{**}(E-1) + T(G\text{-}e_E\text{-}e_{E-1}) + T(G/e_E/e_{E-1})$

$= X^{**}(E-1) + X^{**}(E-2) + T(G\text{-}e_E\text{-}e_{E-1}\text{-}e_{E-2}) + T(G/e_E/e_{E-1}/e_{E-2})$

$= \ldots = X^{**}(E-1) + X^{**}(E-2) + X^{**}(E-3) + \ldots + X^{**}2 + X + Y$

$= \mathrm{Sum}(n=1 \text{ to } E)\ (X^{**}(n-1)) + Y,$

$T(G;-1,-1) = \mathrm{Sum}(n=1 \text{ to } E)\ ((-1)^{**}(n-1)) - 1, = \{-2 \text{ for E even}, -1 \text{ for E odd}\}$

Then C= {2 loops for E even of Figure left, 1 loop for odd of Figure right}

Fig. 2. (continued).

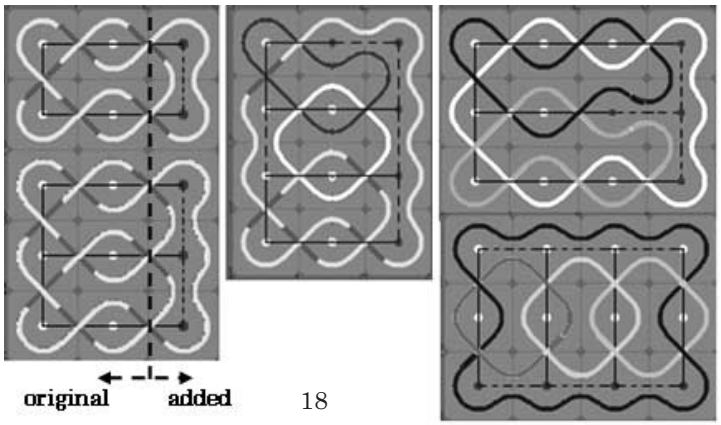
5. Application of the Tutte Polynomial to Kolam Patterns

In this section, a Kolam corresponds to a graph L of a knot-link projected onto a plane, and the N-line of Kolam is also a planar graph G - a segment of the N-line is an edge of G and a dot is a vertex of G. - L is the medial graph of G consisting of vertexes (at crossing points) with four degrees of edges in L. There should be one crossing of L on each edge e of G. The edge number E of G is the same as the crossing number on the N-line of the Kolam.

The following figures of Fig. 2 show samples of relations between Kolam (L) and N-lines (G). Some figures are analyzed by the Tutte polynomial. Refer to Table 1.

6. How to Modify from a Multi-Loop Kolam to a Single Loop Kolam and the Reverse

For reducing the loop number of a multiple loop pattern, one crossing of two different loops will be uncrossed to join the different strings of the loops (x → > <), or a new crossing at the near points of two different loops will be made a combined loop (> < → x); this reduction process



18 A closed N-line Kolam pattern from combining an open N-line Kolam pattern with some additional vertexes and edges. Let us consider an original pattern, which consists of one component loop and an open N-line. Then connect an additional common vertex (black dots with dotted lines) to all of the terminal vertexes belonging to the original pattern through each edge respectively. In a new pattern, a closed cycle - N-Line consists of two new edges from the original terminal vertexes and the additional vertex, and some other original edges (left-upper).

When the number of the edges, or crossings, on each cycle of N-lines is odd, the loop number of the new pattern is kept the same (one loop) as the original (left-lower).

When the number of the edges on a cycle of N-lines is even, the loop number of the N-line cycle increases by one. In this case (center), the edge numbers on two cycles of the N-lines are two and two (even), and the edge number of another cycle of the N-line is five (odd), and then the total loop number of the Kolam pattern becomes three. The maximum possible loop number becomes the original terminal-vertex number, so that all of the numbers of the edges on each cycle of N-lines are even (right). This result implicates "Proposition 1: For a tree T, the link component number of the suspended tree ST is at most the number of the leaves of T" reported by Endo [5-1,5-2].

Fig. 2. (continued).

will continue until the loop number becomes only one.

The proof is the following: Imagine a crossing of two different loops, and four lines A, B, C and D of the crossing. A line from B goes to A out of the crossing, and another line from C goes to D. When the crossing is made uncrossed, and A and C, and B and D should be connected respectively. On the result, the line goes to A-C-D-B-A in one loop. The proof of the reversal case making a crossing of a loop is made on the reversal.

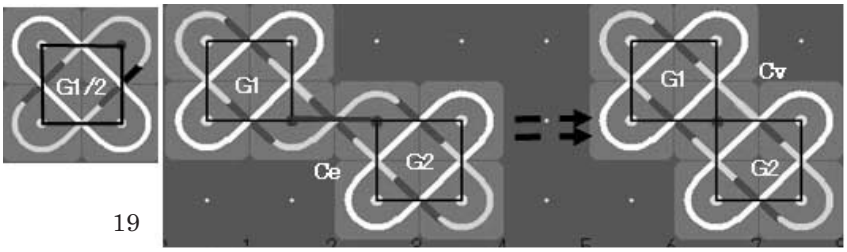
7. Summary and Future Works

This paper described the process of making a Kolam pattern, which is a knot-link consisting of loops. When L is a knot-link, and G is a planar graph of L, which corresponds to the N-line of the Kolam pattern, the component number C of the knot-link L or the loop number C of the Kolam pattern L is calculated from the invariant T(G; −1,−1) of the Tutte polynomial of G and the edge number E of G, or the crossing number E of L.

However, it is very difficult to calculate this T(G; −1,−1), as a Tutte polynomial cannot be obtained from the Kolam pattern directly; it is also very difficult to analyze the first polynomial to the final polynomial in recurrence relation.

The loop numbers of some Kolam were obtained practically using actual Kolam patterns, and the Tutte polynomial T(G; X,Y) was not used for them.

In Appendix A, two ways to get the loop number of Kolam pattern using matrices of the graph of Kolam were introduced. However the larger the size of Kolam is, the larger the sizes of the matrixes are. And it is not easy to get the first matrix of a large size Kolam. A new mathematical



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19: When two closed N-lines are combined,

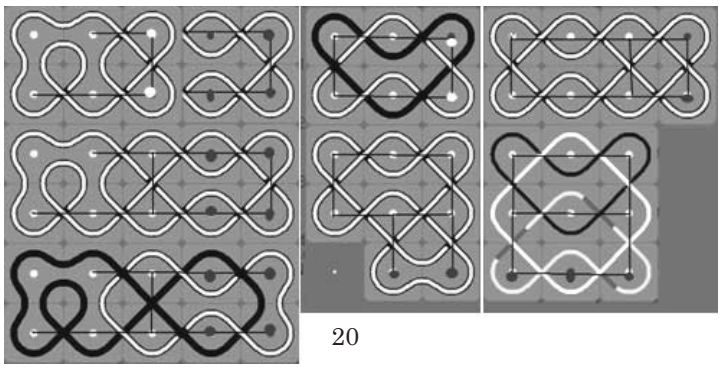
The graphs of G1 and G2 have the same closed N-line (left). At first, combine these two graphs of G1 and G2 with a cut edge (Ce). Refer to Definition (1) in Section 4, or a bridge. The combined G is expressed G = G1 U G2 (center). Each string at the crossing on the Ce enters and exits to/from G1 and G2, and then joins to each one of the loops of G1 and G2. Therefore, C(G = G1 U G2, Ce) = C(G1)+C(G2) −1. In the figure, C = 2 + 2−1 = 3. In this case, when Ce of the graph G (center) is contracted, the graph G becomes the graph with a common vertex Cv (left). The edge number decrements one.

T(G=G1 U G2,Ce;X,Y) = XT(G/Ce) = XT(G = G1 U G2,Cv;X,Y),

T(G1 U G2,Cv;X,Y) = T(G1 U G2,Ce;X,Y) /X

Although the edge number decrements by one, and the sign of T(G;−1,−1) is changed by X=−1, the component number does not change in Table 1. therefore, we can obtain the following:

C(G1 U G2,Cv) = C(G1 U G2,Ce) = C(G1) + C(G2)−1.



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20: A double closed N-line loop pattern from a closed N-line loop pattern (white dots) added vertexes (or edges, black dots)

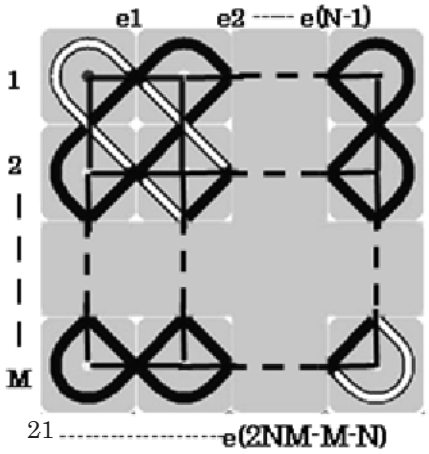
Left-upper: The original, the loop number Co=1. right: Additional edges Ea=5 (odd). center: The added loop pattern keeps the original loop number Ca=1. lower: By additional Ea=4 (even), the loop number increases Ca=2.

Center-upper: The original, the loop number Co=2. lower: By additional edges Ea=2 (even), the loop number decrease Ca=1.

Right-upper: By additional edges Ea=3 (odd), the loop number decrease Ca=1. lower: By additional edges Ea=4 (even), the loop number increases Ca=3.

A formula of the above will be studied in the future.

Fig. 2. (continued).



21: A lattice Kolam such that the Kolam pattern is drawn around all dots in the N x M matrix dot array is called a mat Kolam. In this Kolam, the loop number C is GCD(N x M). The author cannot find the proof of this result using the Tutte polynomial and also the invariant of T(G;−1,−1).

Fig. 2. (continued).

Table 1. Invariant values of the Tutte polynomial T(G; −1,−1) for E(G) and C(L); $T(G;-1,-1) = (-1)**|E(G)|*(-2)**(C(L)-1)$, where C(L) is the link component number and |E(G)| is the edge number of the planar graph G without any isolated vertexes of a knot-link L. In this paper, the medial graph L of the graph G corresponds to the Kolam loop pattern, a regular planar graph with four degrees of a knot-link diagram. In the Kolam loop pattern, |E(G)| is the same number as the crossing number of L. G is the same graph as the N-line circuit of L. C(L) is obtained from the reverse lookup value of Table 1 or the following reversal equation; $C = Log_{-2}(T(G;-1,-1)/(-1)^{|E(G)|}) + 1 = Log_2(|T(G;-1,-1)|) + 1$, * for any |E(G)|, which should be larger than 2C-3. Each case indexed with the number is shown in Fig. 2 of the next section 5.

	E=0	1	2	3	4	5	6	7	8	9	10
C=1	T=1	−1	1	−1	1	−1	1	−1	1	−1	1
case	1	2		4	6, 7						
2	*	*	−2	2	−2	2	−2	2	−2	2	−2
case			3	5	8	10					
3	*	*	*	*	4	−4	4	−4	4	−4	4
case					9	11	12				
4	*	*	*	*	*	*	−8	8	−8	8	−8
case							13		14		
5	*	*	*	*	*	*	*	*	16	−16	16
case									15		
6	*	*	*	*	*	*	*	*	*	*	−32

method with computer software (program) will one day solve this problem.

Acknowledgments. The author would like to thank Prof. Ryo Nikkuni, Tokyo Women Christian Univ. for introducing the paper by Schwarzler and Welsh and the paper by Martin, which described the proof of the relation of T(G; −1,−1), the numbers of the edges of G and the component umber C of the medial graph of G, and teaching recurrence equations of the Tutte polynomial. He would like to also thank Prof. Toshiki Endo, Jiyu-Gakuen College for introducing his paper.

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Appendix A.

Other ways how to get the component or loop number using matrixes.

After submitting the paper, the author was introduced by Prof. Nikkuni to the paper “On the Component Number of Links from Plane Graphs” by Daniel S. Silver and Susan G. Williams[8].

The author is introducing a way using matrix because the matrix way to obtain the loop number of a given Kolam pattern might have simpler formula than the way of the Tutte polynomial T(G; −1,−1) and it might be more programmable. Silver and Williams gave a short, elementary

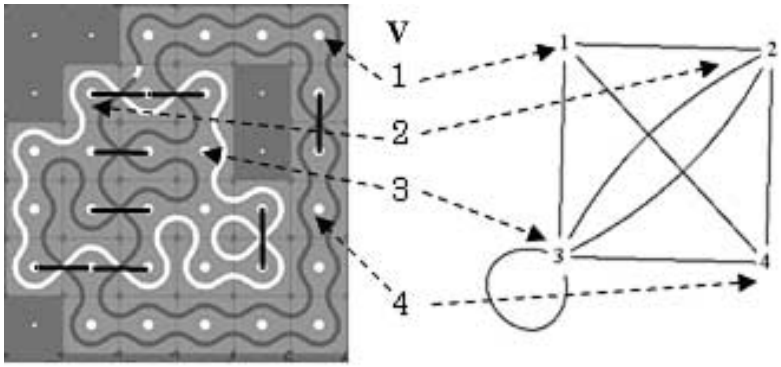

Fig. A.1. In Kolam (left), white dots represent vertices, but some dots surrounded by the lines with two-point-connections are combined to one vertex, the N-lines of black linear lines represent the planar graph (right), and then the lines belonging crossings at the medial positions between two adjacent and combined dots are the medial graph. This Kolam becomes a knot-link with up-down crossings. Components of the medial graph are called left-right cycles sometimes. The numbers in the center are assigned to the vertices representing combined dots of Kolam. Note one vertex represents some combined dots.

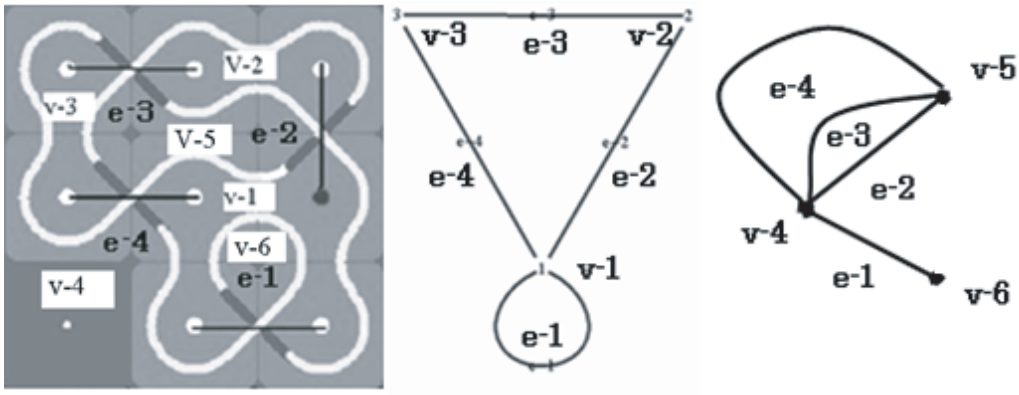

Fig. A.2. A sample Kolam (left. the medial graph of G, or the diagram D) and N-lines (G: black lines), the Tait graph G (center. the planar graph) and the dual graph (right).

and self-contained proof of the following:

THEOREM A.1 (Silver and Williams). Let L be a link arising from a medial graph M(Γ) by resolving vertices. The number μ(L) of components of L is the nullity of the mod-2 Laplacian matrix Q2(Γ). This Theorem was first given in the paper by C. Godsil and G. Royle[8].

The process for getting the component number is the following: Start with the graph G and number the vertices with the maximum vn. Make the adjacency matrix A (entry in ith row and jth column is number of edges from i to j, with loops counted twice). Obtain the Laplacian matrix Q = D − A, where D is a diagonal matrix of degrees of vertices from A. Finally, calculate the nullity of the mod-2 Laplacian matrix Q2(G). and then the result is the component number of the medial graph of G. The following calculation example is of the sample Kolam (Fig. A.1) using Mathematica code by the author;

Each matrix of the process for Fig. A.1 is the following respectively: Adjacent matrix ma with vertex number vn of four, Diagonal matrix md of degrees of vertices, md[i=j] = sum{ma[I,1] + ma[I,2] + ⋯ + ma[I,vn]}, md[i!=j] = 0. Laplacian matrix mq = md - ma, modular matix m2 = Mod[mq,2], and Mod-2 row reduced matrix mq2 from m2.

$$ma = \begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \end{array} \begin{array}{c} \begin{array}{cccc} v1 & v2 & v3 & v4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array},$$

$$md = \begin{bmatrix} 3&0&0&0 \\ 0&4&0&0 \\ 0&0&6&0 \\ 0&0&0&3 \end{bmatrix}, \quad mq = \begin{bmatrix} 3&-1&-1&-1 \\ -1&4&-2&-1 \\ -1&-2&4&-1 \\ -1&-1&-1&3 \end{bmatrix},$$

$$m2 = \begin{bmatrix} 1&1&1&1 \\ 1&0&0&1 \\ 1&0&0&1 \\ 1&1&1&1 \end{bmatrix}, \quad mq2 = \begin{bmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}.$$

The nullity is the number of zero rows in the matrix mq2. From the rank-nullity theorem, Nullity = vn − Matrix-Rank[mq2] = 2. Finally, we obtain the output 2, which means that the component (loop in Kolam) number is two. Note the rank of [mq2] is calculated using arithmetic modulo 2 as an example of ma = [[0,1,1][1,0,1][1,1,0]], mq2 = [[0,1,1][1,0,1][1,1,0]], the rank of mq2 = 2, and then the

nullity (=3−2) means 1 loop of ma.

The author was introduced also to another paper "On Region Crossing Change and Incidence Matrix" by Cheng Zhiyun and Gao Hongzhu[9]. In that paper they prove that a signed planar graph G represents an n-component link diagram D if and only if the rank of the associated incidence matrix M(D) equals to c−n+1, here M(D) denotes the incidence matrix of the diagram D from G, and c denotes the size of the graph G. This way called Cheng's process to obtain the component number - loop number in Kolam- is the following;

In Fig. A.2, in Cheng's process, we must set some vertices representing every region bounded by Kolam or knot-link lines including regions without dots, e.g. the outside region. In Godsil-Susan's process, however, we are enough to set only vertices representing every region by Kolam or knot-link lines, which have vertex-dots, crossings of lines, and edges connecting the crossings. The later is simpler rather than the former.

Two first matrixes of Incidence matrix mi in Cheng's process and Adjacency matrix ma in Godsil-Susan's process are the followings:

For a loop of e1 though v1, mi[e1,v1] = mod(1+1,2) = 0, the graph of ma is undirected, and ma[v1,v1] = 2 on counting twice. The Tait graph is the same as the planar graph.

$$mi = \begin{array}{c} \\ v1 \\ v2 \\ v2 \\ v4 \\ v4 \\ v6 \end{array} \begin{array}{c} \begin{array}{cccc} e1 & e2 & e3 & e4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$ma = \begin{array}{c} \\ v1 \\ v2 \\ v3 \end{array} \begin{array}{c} \begin{array}{ccc} v1 & v2 & v3 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}.$$

In Cheng's process, the rank of the incidence matrix rmi is 4, we obtain the component number n=c+1−rmi=4+1−4=1, where c is the edge, or crossing, number. In Godsil-Susan's process, the rank of the mod-2 Laplacian matrix rmq2 is two, and we then obtain the component number n=nullity=vn−rmq2=3−2=1, where vn is the vertex, or dot, number, which connects together with the edge.

Comparing the two, setting the first matrix from a given graph in Godsil-Susan's process is simpler, but matrix calculation of Cheng's is simpler than the other: this is maybe a trade-off. For programming, the setting of the first matrix from the given Kolam should be simple.

Notes

About the reference[4], the page 125 claimed the following:

When M is the cycle matroid of a planar graph G, we know that $T(M; -1, -1) = (-1) * *|E(G)| \times (-2) * *(C(m(G)) - 1)$, where m(G) is the medial graph of G. G = (V, E) is a finite undirected graph with a vertex v set V and an edge e set E.

The author understands that M means the same as G, and C(m(G)) is the same as C (knot-link L component number or Kolam loop pattern) in this paper. Here |E(G)| is the edge number of an edge set of E consisting of G.

About the reference[6], the page 321 claimed the following:

THEOREM A.2 (Martin). For any planar multi-graph Γ, with m edges,

$$\chi(\Gamma, -1, -1) = (-1) * *m(-2) * *k,$$

where $0 < k < m/2$. Furthermore, if Γ is connected, s = k + 1 is the number of strings of the medial graph corresponding to any planar representation of Γ.

Proof. First, we notice that it is sufficient to prove the proposition in the connected case: As a matter of fact, if Γ is not connected, the polynomial $\chi(\Gamma)$ is equal to the product $\pi i \chi(\Gamma i)$ over all the connected components Γi of Γ. Thus, as we check that m is the sum of mi and that the relation $k <= m/2$ is preserved by summation, the proposition reduces to the connected case. The proof of the connected case can be performed by induction. There are two one-edge multi-graphs. Each of them verifies that $\chi(\Gamma,-1,-1)=-1$ and has a medial composed of one string (refer two graphs Γ of an edge and a loop of Definition 1 and 2 in Fig. 1). If $m >= 2$, the polynomial $\chi(\Gamma, x, y)$ may be obtained by addition (Case 1) or multiplication (Case 2). The rest is omitted.

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