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Distribution of the Angle between Directions of the First and Second Nearest Facilities

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This paper examines the angle between the directions of the first and second nearest facilities. An application of the angle can be found in location analysis of refuges where the direction of facilities is important. The angle represents the service level of facility location when customers are serviced by the first and second nearest facilities. The distribution of the angle is derived for regular and random patterns of facilities. The distribution shows how the angle is distributed in a study region, and is useful for location models using the direction of facilities. The distribution of the angle for actual facility location is also calculated.
Key words: Location, Facility Closing, Regular Pattern, Random Pattern, Average Angle

1. Introduction

Facility location problems have been addressed in a variety of fields such as geography, economics, and operations research. The most frequently used assumption of location models is that customers get service from their nearest facility. Facilities might, however, be closed or disrupted due to accidents, disasters, and intentional strikes. The possibility of closing should therefore be considered particularly when locating emergency facilities.

The service from the second nearest facility has been introduced into location models. Weaver and Church (1985) developed the vector assignment p-median problem, where a certain percentage of customers could be serviced by the kth nearest facility. The problem was extended by Lei and Tong (2013) to the expected median location problem and Lei and Church (2014) to the vector assignment ordered median problem. Pirkul (1989) studied a similar problem in which customers are served by two facilities designated as primary and secondary facilities. Drezner (1987) formulated the unreliable p-median problem, where customers are assigned to the kth nearest facility when closer facilities fail. Efficient heuristic solution methods for the problem were presented by Lee (2001). Berman *et al.* (2007) extended the unreliable p-median problem by relaxing the assumption that the probability of facility failure is the same for all facilities. Snyder and Daskin (2005) proposed the reliability p-median problem and the reliability fixed-charge location problem. They made an ordered assignment of customers to facilities. Lei and Church (2011) presented generalized closest assignment constraints in terms of multiple levels of closeness. Miyagawa (2008) and Miyagawa (2009) found the optimal location that minimizes the average distance to the nearest open facility when some of the existing facilities are closed. Miyagawa (2014) considered bi-objective problems where the distances to the first and second nearest facilities are minimized.

Most of the facility location models reviewed above examined the distance from customers to facilities. If customers are serviced by the first and second nearest facilities, not only the distance but also the direction of facilities is important. For example, when evacuating from disasters, the first and second nearest refuges should be in the opposite direction, because otherwise both the refuges might be disrupted. In fact, securing two-way evacuation routes is required in evacuation route planning (Mishima *et al.*, 2014). On the other hand, when purchasing shopping goods, it would be convenient if the first and second nearest shops are in the same direction. Thus, the angle between the directions of the first and second nearest facilities represents the service level of facility location.

In this paper, we derive the distribution of the angle between the directions of the first and second nearest facilities. The distribution shows how the angle is distributed in a study region, and will thus supply building blocks for facility location models using the direction of facilities. We focus on regular and random patterns of facilities to obtain analytical expressions for the distribution. The analytical expressions allow us to examine fundamental characteristics of the angle. The analytical expressions are also useful to interpret and comprehend numerical results.

The remainder of this paper is organized as follows. The next section derives the distribution of the angle for regular and random patterns of facilities. The following section examines the distribution of the angle for actual facility location. The final section presents concluding remarks.

2. Regular and Random Patterns

Facilities are represented as points of regular and random patterns on a continuous plane, as shown in Fig. 1. Since actual patterns of facilities can be regarded as intermediate between regular and random, the theoretical results of these

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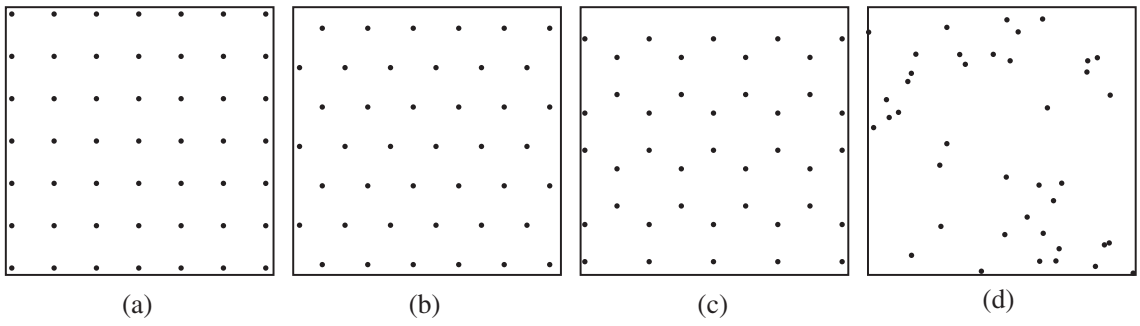

Fig. 1. Regular and random patterns of facilities: (a) Square lattice; (b) Triangular lattice; (c) Hexagonal lattice; (d) Random.

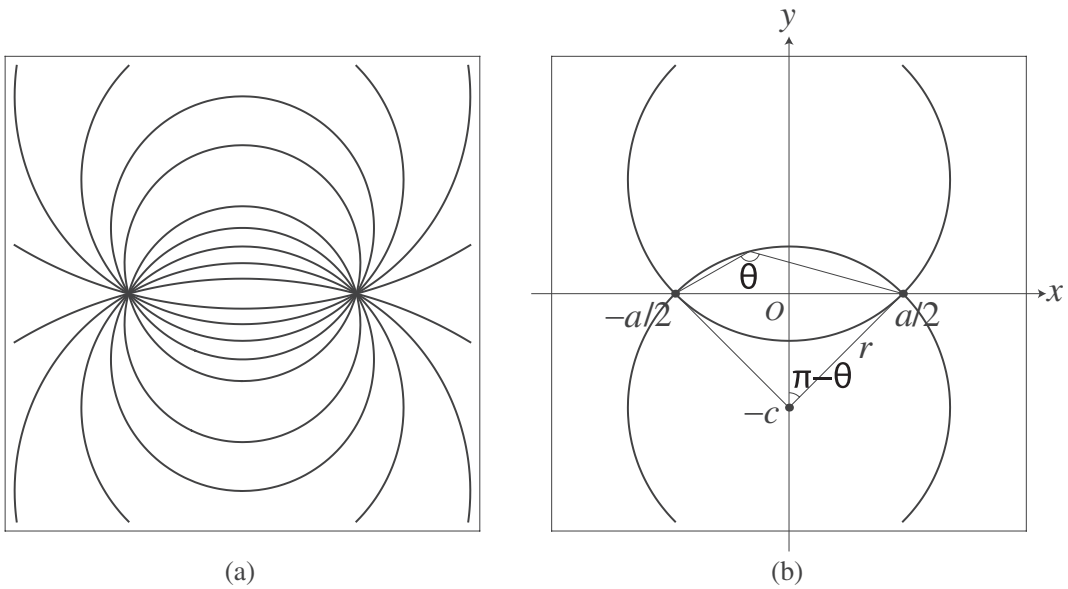

Fig. 2. (a) Contour of the angle; (b) Locus such that $\Theta = \theta$.

extremes serve as a basis for empirical analysis of actual patterns. In fact, the regular and random patterns have frequently been used in location analysis (O'Kelly and Murray, 2004; Sadahiro, 2005; Miyagawa, 2009). If customers are uniformly distributed, the optimal location that minimizes the average distance to the nearest facility is the triangular lattice (Fig. 1b) (Leamer, 1968; Iri *et al.*, 1984; Du *et al.*, 1999).

Let Θ be the angle between the directions from a randomly selected location in a study region to the first and second nearest facilities. The contour of the angle Θ is given by a circle passing through facilities, as depicted in Fig. 2a. Recall that angles subtended at the circumference by the same arc of a circle are equal. The locus such that $\Theta = \theta$ is obtained as follows. Set the coordinate system as shown in Fig. 2b, where facilities are at $(-a/2, 0)$, $(a/2, 0)$. Note that θ is the angle subtended at the circumference by the facilities. Let $(0, -c)$ and r be the center and radius of the circle, respectively. Since

$$\begin{aligned} c &= \frac{a}{2\tan(\pi-\theta)} = -\frac{a}{2\tan\theta}, \\ r &= \frac{a}{2\sin(\pi-\theta)} = \frac{a}{2\sin\theta}, \end{aligned} \tag{1}$$

the locus such that $\Theta = \theta$ is the circles expressed as

$$x^2 + \left(y \pm \frac{a}{2\tan\theta}\right)^2 = \frac{a^2}{4\sin^2\theta}. \tag{2}$$

2.1 Square lattice

Suppose that facilities are regularly distributed on a square lattice with spacing a. Let $F(\theta)$ be the cumulative distribution function of Θ, that is, the probability that $\Theta \leq \theta$. $F(\theta)$ is given by

$$F(\theta) = \frac{S(\theta)}{S}, \tag{3}$$

where S and $S(\theta)$ are the area of the study region and the area of the region such that $\Theta \leq \theta$ in the study region, respectively. The study region can be confined to the region where two facilities are the first and second nearest, which is the square centered at the midpoint of the facilities with side length $a/\sqrt{2}$, as shown in Fig. 3. The area of the study region is then $S = a^2/2$. The region such that $\Theta \leq \theta$ is given by the dark gray region in Fig. 3. Thus, $S(\theta)$ is obtained by subtracting the area of the intersection of the

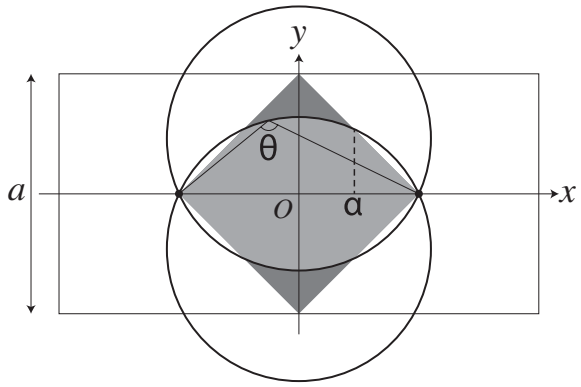

Fig. 3. Region such that $\Theta \le \theta$ for the square lattice.

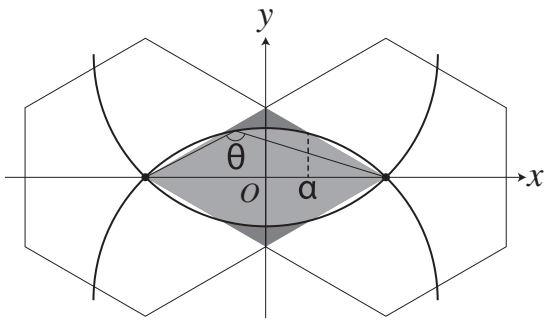

Fig. 4. Region such that $\Theta \le \theta$ for the triangular lattice.

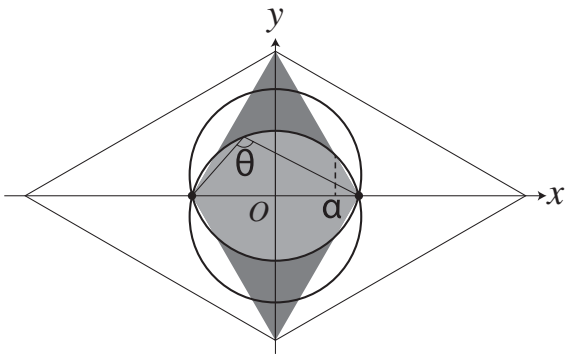

Fig. 5. Region such that $\Theta \le \theta$ for the hexagonal lattice.

two circles (2) and the square from the area of the square as

$$S(\theta) = \alpha\left(2\alpha + \sqrt{\frac{a^2}{\sin^2\theta} - 4\alpha^2}\right) - \frac{a^2}{2\sin^2\theta}\arcsin\frac{2\alpha\sin\theta}{a}, \quad \frac{\pi}{2} \le \theta \le \pi, \tag{4}$$

where

$$\alpha = \frac{a}{4}\left(1 - \frac{1}{\tan\theta} - \sqrt{\frac{2}{\tan\theta} + \frac{1}{\sin^2\theta}}\right). \tag{5}$$

Substituting S and $S(\theta)$ into Eq. (3) yields

$$F(\theta) = 2\alpha\left(2\alpha + \sqrt{\frac{1}{\sin^2\theta} - 4\alpha^2}\right) - \frac{1}{\sin^2\theta}\arcsin(2\alpha\sin\theta), \quad \frac{\pi}{2} \le \theta \le \pi, \tag{6}$$

where

$$\alpha = \frac{1}{4}\left(1 - \frac{1}{\tan\theta} - \sqrt{\frac{2}{\tan\theta} + \frac{1}{\sin^2\theta}}\right). \tag{7}$$

Note that $F(\theta)$ is independent of the facility spacing a. Differentiating $F(\theta)$ with respect to θ yields the probability density function of Θ as

$$f(\theta) = \frac{\mathrm{d}F(\theta)}{\mathrm{d}\theta}. \tag{8}$$

The distribution of the angle $f(\theta)$ is shown in Fig. 7.

2.2 Triangular lattice

Suppose that facilities are regularly distributed on a triangular lattice. The study region can be confined to the rhombus where two facilities are the first and second nearest, as shown in Fig. 4. The cumulative distribution function is similarly obtained by calculating the area of the intersection of the two circles (2) and the rhombus as

$$F(\theta) = \sqrt{3}\alpha\left(2\alpha + \sqrt{\frac{2\sqrt{3}}{\sin^2\theta} - 12\alpha^2}\right) - \frac{\sqrt{3}}{\sin^2\theta}\arcsin\left(3^{1/4}\sqrt{2}\alpha\sin\theta\right), \quad \frac{2}{3}\pi \le \theta \le \pi, \tag{9}$$

where

$$\alpha = \frac{1}{4\cdot 3^{3/4}\sqrt{2}} \cdot \left(\sqrt{3} - \frac{3}{\tan\theta} - \frac{3}{\sin\theta}\sqrt{2 + \sqrt{3}\sin 2\theta - \cos 2\theta}\right). \tag{10}$$

The distribution of the angle $f(\theta)$ is obtained from Eq. (8) and shown in Fig. 7.

2.3 Hexagonal lattice

Suppose that facilities are regularly distributed on a hexagonal lattice. The study region can be confined to the rhombus where two facilities are the first and second nearest, as shown in Fig. 5. The cumulative distribution function is similarly obtained by calculating the area of the intersection of the two circles (2) and the rhombus as

$$F(\theta) = \alpha\left(3\sqrt{3}\alpha + \sqrt{\frac{\sqrt{3}}{\sin^2\theta} - 9\alpha^2}\right) - \frac{1}{\sqrt{3}\sin^2\theta}\arcsin\left(3^{3/4}\alpha\sin\theta\right), \quad \frac{\pi}{3} \le \theta \le \pi, \tag{11}$$

Fig. 6. Hospitals and nodes such that $\Theta < \pi/4$ in Setagaya, Japan.

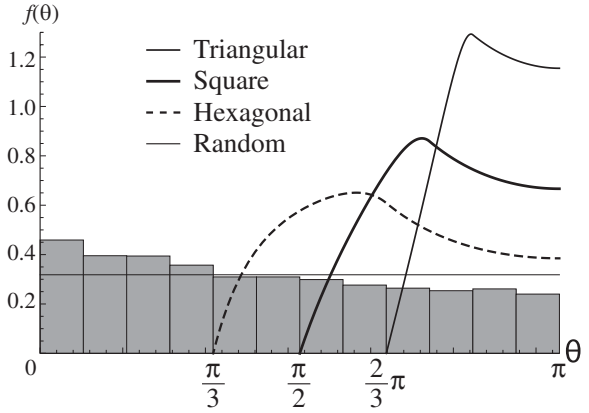

Fig. 7. Distribution of the angle.

Table 1. Average and standard deviation of the angle.

	Average	Standard deviation
Hospital	1.39 ($\approx 80°$)	0.91 ($\approx 52°$)
Square	2.45 ($\approx 140°$)	0.39 ($\approx 22°$)
Triangular	2.71 ($\approx 155°$)	0.25 ($\approx 14°$)
Hexagonal	2.09 ($\approx 120°$)	0.54 ($\approx 31°$)
Random	1.57 ($\pi/2 = 90°$)	0.91 $\left(\pi/\left(2\sqrt{3}\right) \approx 52°\right)$

where

$$\alpha = \frac{1}{4 \cdot 3^{3/4}} \left(3 - \frac{\sqrt{3}}{\tan\theta} - \frac{1}{\sin\theta} \sqrt{2 + \sqrt{3}\sin 2\theta + \cos 2\theta} \right). \tag{12}$$

The distribution of the angle $f(\theta)$ is obtained from Eq. (8) and shown in Fig. 7.

2.4 Random

Suppose that facilities are uniformly and randomly distributed. The probability that a region of area S contains exactly x facilities, denoted by $P(x, S)$, is given by the Poisson distribution as

$$P(x, S) = \frac{(\rho S)^x}{x!} \exp(-\rho S), \tag{13}$$

where ρ is the density of facilities (Clark and Evans, 1954). The probability $P(x, S)$ is independent of the location and shape of the region. The angle Θ is then uniformly distributed over the interval $[0, \pi]$ as

$$f(\theta) = \frac{1}{\pi}, \quad 0 \le \theta \le \pi. \tag{14}$$

The distribution of the angle $f(\theta)$ is shown in Fig. 7.

3. Actual Facility Location

In this section, we examine the distribution of the angle for actual facility location to discuss whether the model of the regular and random patterns can be applied to actual patterns. As an example, we consider 32 hospitals in Setagaya, Japan, as shown in Fig. 6, where black circles represent hospitals.

Let Θ be the angle between the directions from a node to the first and second nearest hospitals. The angle between the directions from a node $\mathbf{q}$ to hospitals $\mathbf{p_1}$, $\mathbf{p_2}$ is given by

$$\Theta = \arccos \frac{(\mathbf{p_1} - \mathbf{q}) \cdot (\mathbf{p_2} - \mathbf{q})}{|\mathbf{p_1} - \mathbf{q}||\mathbf{p_2} - \mathbf{q}|}. \tag{15}$$

Dark gray circles in Fig. 6 represent nodes such that $\Theta < \pi/4$. It can be seen that the boundary of the set of the nodes forms a circle, as shown in Fig. 2. The normalized histogram of the angle for all nodes is shown in Fig. 7. The distribution for the actual pattern is similar to that for the random pattern. The average and standard deviation of the angle are summarized in Table 1. The average angle for the actual pattern is smaller than that for the regular and random patterns, and the standard deviation is as large as that for the random pattern. Note that the average angle for the triangular lattice is the largest among three regular patterns. It follows that the triangular lattice is suitable for the location of refuges. Note also that the standard deviation for the triangular lattice is the smallest, which leads to a small disparity in service level among customers.

4. Conclusions

This paper has derived the distribution of the angle between the directions of the first and second nearest facilities. The analytical expressions for the distribution for regular and random patterns are useful for location models using the direction of facilities as follows. First, they give an estimate for the service level of actual facility location. By comparing distributions, we can evaluate the efficiency of actual patterns. For example, if the angle for the location of refuges is much smaller than that for the regular patterns, relocating some refuges should be considered. Second, they have all the information about the angle. The minimum, average, and standard deviation of the angle, which can be

used as objective functions, are obtained from the distribution. Finally, they lead to a better understanding of the optimal facility location. Since the average angle for the triangular lattice is the largest among three regular patterns, the triangular lattice would be the most suitable for the location of refuges.

Although the focus of this paper is on the angle between the directions of the first and second nearest facilities, only the angle is insufficient for evaluating the service level of facility location. Future research should address facility location problems that simultaneously consider the distance and angle.

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