Percolation

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Percolation model is one of the most foundamental models holding important concept such as phase transition, growth phenomena, universality, and also it provides clues for studies of transport coefficients of porous media, forest fire, epidemics in an orchard, networks and so on. There are mainly two types of percolation models, discrete percolation model and continuum percolation model. The former is based on a lattice structure, where some exact solutions are found, while the latter on continuum space, where more experimental investigations are expected.

Key words: Discrete Percolation Model, Continuum Percolation Model, Critical Point, Critical Exponent, Fractal

1. Introduction

Elements such as particles, bonds etc. are placed onto lattice sites (or bonds) in a space one by one. If these elements occupy the nearest neighboring positions, then they form clusters. The cluster becomes larger and larger with increase of the elements, and at some quantity of elements, the cluster acquires a size extending from an end to opposite end in a finite space or becomes infinitely large in an infinite space. We call this phenomenon **percolation** (Essam, 1980; Grimmett, 1989; Stauffer and Aharony, 1994; Sahimi, 1994; Bunde and Havlin, 1996; Hunt, 2013) and this model is called **a the discrete percolation** model. If the element has electric conductivity, the whole media becomes conductive at the instance of percolation. This phenomenon shows critical behavior as shown in the text. Most important indications are critical point (p_c) and critical exponent (ν). The former depends on the details of lattice when the element occupies only lattice point or only bond between the nearest neighboring two points. The latter quantity does not depend on the species of lattice, but does on the space dimension (universality).

Most of investigations of percolation model are performed by computer simulation, but one of most simple and powerful theoretical approaches may be the real space renormalization group theory which provided the critical points and the critical exponents, but is limited to 2 dimension (Reynolds *et al.*, 1978). When we look at the pattern of percolation at the critical point, it is easy to see the fractal structure which provides the power law behaviors in several quantities such as probability that a site belongs to finite size cluster, percolation probability, average cluster size and so on. Recently the network problems are studied very actively, but these problems are profoundly related with percolation (Barabasi and Alber, 1999).

There are many modified percolation models such as the directed percolation model (Hinrichsen, 2000), kinetic growth model (Gould *et al.*, 2005), forest fire mode (Henley, 1993), and invasion percolation model (Wilkinson and Willemsen, 1983). In addition, a quantum percolation model (de Gennes *et al.*, 1959) is also investigated, but here we don't discuss these problems. Rather we stress on some percolation models with exact solutions. In the 2D the critical points and the critical exponent are calculated for some limited lattices. We are going to discuss some percolation models with exact critical values in higher dimensions.

In the above discussion we suppose that the elements are placed on the lattice point of regular lattices or the elements are replaced by other elements. But there are other families of percolation models where the elements are placed at arbitrary points in the space. We call this category of percolation models **continuum percolation model** (Meester, 1996). The most important point of this continuum percolation model is the critical exponents which are generally different from those in the discrete model, which means an existence of a different class of universality from that in the discrete percolation model.

In nature there are many media for which the continuum percolation model is adequate such as rocks, porous media, farm field, flow in vessel and so on (Sahimi, 1994; Hunt, 2013). Investigation of the continuum percolation model has many difficulties for the following reasons. (i) The simulation method does not work well, mainly because of difficulty in calculation of area or volume of overlapping elements. (ii) Also, natural samples are limited in variations of concentration of elements. (iii) Few experimental investigations are given in the text.

In the next section the discrete percolation model and some new percolation models with the exact value of characteristic quantity will be discussed. In Section 3, we will discuss the continuum percolation model.

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Table 1. The critical point of several lattices in 2D. ν is the ordination number (Sykes and Essam, 1964; Jacobsen, 2014).

Lattice	ν	p_c of bond model	p_c of site model
Honeycomb	3	$1 - 2\sin(\pi/18) \cong 0.6527^*$	0.6970
Square	4	1/2*	0.5927
Kagome	4	0.5244	0.6527*
Triangular	6	$2\sin(\pi/18) \cong 0.3473^*$	0.5*

*Means the exact critical point.

Table 2. The critical point of several lattices in the 3D. ν is the ordination number (Lorenz and Ziff, 1998a, b; Skvor and Nezbeda, 2009; Wang *et al.*, 2013).

Lattice	ν	p_c of bond model	P_c of site model
Simple cubic	6	0.2488	0.3116
Body centered cubic	8	0.1803	0.2465
Face centered cubic	12	0.1202	0.1993

2. Discrete Percolation Model

In nature, at first there is a space, and when the space is completely filled by the elements, we call it as pure materials. If the element is a conducting lot, disk, or sphere, a cluster may appear and grow in the space as the number of elements increases. If the cluster spreads from one end to other end in the space, we call it **percolation**. When the element added is conductive, the bulk material in the space becomes conductive and the conductance increases with the number of elements. To the contrary, we can imagine that the initial state is conductive and insulating elements are added. In this case, as the insulator increases in number, the conductance decreases and it vanishes at the critical point. Here, the behavior how to decrease is also interesting. This is a kind of critical phenomena like that in ferromagnetism, where a typical model is the Ising Model (Bhattacharjee and Khare, 1995).

Percolation phenomena were studied in polymer science at first by Flory (1941), where rigid polymer cluster grows in liquid. In 1956, Broatbent and Hammersley introduced the mathematical concept of percolation discussing fluid flow in porous medium (Broatbent and Hammersley, 1957). They considered a regular lattice structure with random arrangement of open tube or closed tube at the bond of lattice. The fluid can flow in the open tube at the bond, then we call it open bond simply. On the other hand, the fluid cannot flow at the closed bond. Hereafter, we call this model as **the bond percolation model**.

Instead of open or closed bonds in a regular lattice, we can introduce a plug at the vertex in the lattice with all open bonds. When we image formation of an infinitely large cluster of polymer molecules, each molecule, which we call simply a particle, can occupy a vertex site. If two particles occupy the nearest neighbor vertices, these two particles are combined. When more particles are added at vertices, larger ensemble of particles may appear. We call this a cluster and this model **the site percolation model**. These site and bond percolation models have become the standard models.

2.1 The critical point of bond and site percolation models

As explained above, in initial state all bonds on a regular lattice are closed bonds. If $\nu Np/2$ bonds among all bonds are replaced by open bonds, where N, ν, p are the total vertex number, the ordination number of the lattice and the ratio of open bond to all bonds, respectively. A few smaller clusters appear and the size of cluster increases with p. As a regular 2D lattice there are the square, the triangular, the honeycomb and the Kagome lattice, while there are the simple cubic, the body-centered cubic and the face centered cubic lattice in 3D. At some value of p, the cluster becomes end-to-end cluster, where the cluster reaches both ends (from top to bottom, or from right to left). Or if the lattice is an infinitely large one, the cluster becomes infinitely large.

The threshold of the percolation model was initially calculated by simulation. The threshold value depends on the species of lattice and its dimension as shown in Tables 1 and 2.

In 2D, we have several exact values, one of those will be explained in the following subsection. As seen in Tables 1 and 2, if the ordination number ν increases, the critical point decreases both in the discrete and the continuum percolation models. On the other hand, the critical exponent ν of the correlation length is $4/3$ for 2D and 0.9 for 3D, which are independent of lattice species and dependent only on the space dimensions.

2.2 Real space renormalization group theory

The renormalization group theory was originally progressed in the elementary particle physics. The first application to the solid state physics was done by Wilson (1971), Amit (1984).

In 1975, Harris *et al.* applied this method to percolation in the Fourier space (Harris, 1974). He discussed the same problem in the real space (Harris *et al.*, 1975). Here we introduce the latter method because of its intuitive property.

Let us consider the site percolation model of the square

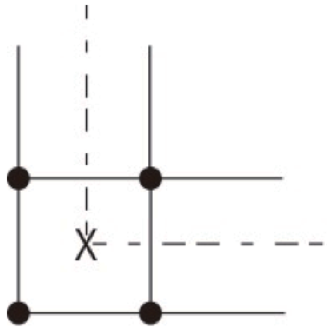

Fig. 1. 2×2 sites (dots) of the square lattice is renormalized into a renormalized site (X).

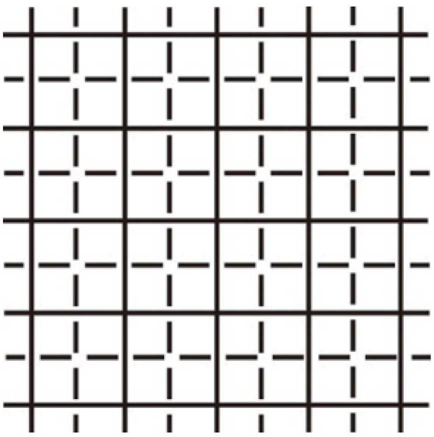

Fig. 2. The solid line shows the real square lattice and the dashed line the dual one.

lattice, and we choose 2×2 sites (small black circles) and its renormalized site (denoted by X) as shown in Fig. 1. The renormalized site is occupied by renormalized particle if more than one site are occupied. On the other hand, if less than two sites are occupied, the renormalized site is unoccupied. In the case that two sites are occupied, we share two cases between occupation and un-occupation by renormalization (Stauffer and Aharony, 1994). p' is renormalized concentration of particles is expressed as

$$p' = p^4 + 4p^3q + 2p^2q^2,$$

where $q = 1 - p$. At the threshold point we can expect $p' = p = p_c$. Therefore, we obtain

$$p_c = 0.6180,$$

which is very near to the value $p_c = 0.592746$.

For $p \cong p_c$, the correlation length is expressed as $\xi \approx (p - p_c)^{-\nu}$. If the renomalization process is applied once, the bond length is renormalized by the factor 2 and the correlation length is also renormalized as $\xi = b\xi' \approx b(p' - p_c)^{-\nu} \approx (p - p_c)^{-\nu}$, where $b = 2$. Therefore

$$\nu = \frac{\log b}{\lim_{p \to p_c} \log \dfrac{p' - p_c}{p - p_c}} = \frac{\log b}{\log \dfrac{dp'}{dp}}.$$

If we use the critical value $p_c = 0.6180$, we obtain $\nu = 1.63$, which is very near to the exact value $\nu = 4/3$.

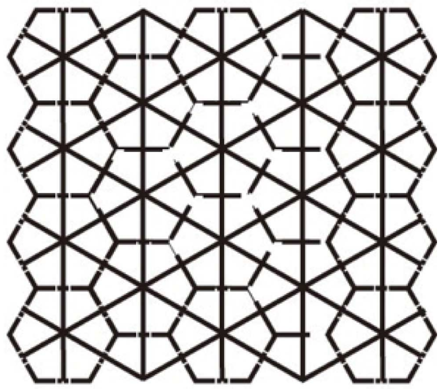

Fig. 3. The solid line shows the triangle lattice and the dashed the honeycomb lattice.

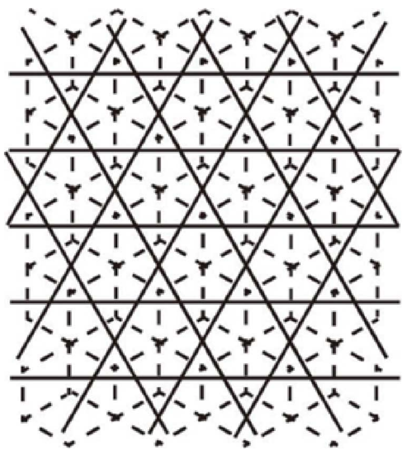

Fig. 4. The solid line shows the real Kagome lattice and the dashed line the dual, which is the diced lattice.

In the same way, we can consider the bond problem as is done by Stauffer and Aharony (1994). We adopt 2×2 bonds as shown in Fig. 1. We consider only the horizontal renomalized bond, then we obtain

$$p' = p^5 + 5p^4q + 8p^3q^2 + 2p^2q^3.$$

The details of each term in the above equation are given in Stauffer and Aharony (1994). From this equation, we obtain

$$p_c = 1/2 \text{ and } \nu = 1.428, \text{ where } b = 2.$$

The threshold value is the same as the exact one and the correlation exponent is also very near to the exact one.

There exist several trials to extend 2D real space renormalization group theory to the 3D, however, the results were not good enough and a better technique is expected. Monte Carlo renormalization method is along this direction.

2.3 An exact method

2.3.1 Dual transformation An argument was given by Krammers and Wannier to obtain the critical temperature of the Ising model (Kramers and Wannier, 1941) and Harris applied it to the percolation model (Harris, 1960). The

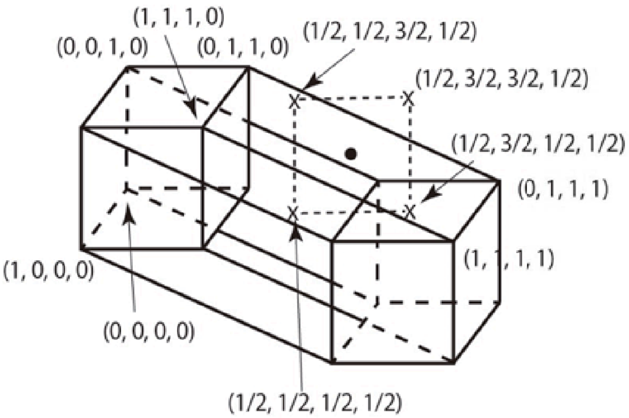

Fig. 5. The unit cell of a 4D hyper-cubic lattice is shown by the solid lines. The big dot is a common center of two squares. One of them is the square $(0, 1, 1, 0) - (0, 1, 1, 1) - (1, 1, 1, 1) - (1, 1, 1, 0)$ in the original lattice. The other one is denoted by dotted lines in the dual lattice. These two squares are orthogonal each other.

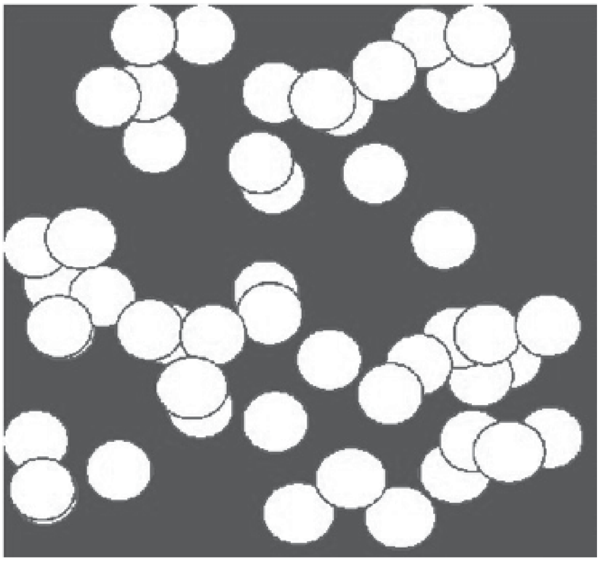

Fig. 6. 2D Swiss-cheese model. Many holes are punched out on aluminum foil. This picture shows a situation a little before beginning of percolation from the left side to the right, where the conductivity from the top to the bottom vanishes.

partition function of Ising model with N spins is

$$Z\left(\frac{2J}{kT}\right) = \sum_{\{\mu\}} \exp \sum_{i,j} \left\{\frac{J\mu_i\mu_j}{kT}\right\}$$
$$= \cosh^N \left\{\frac{J}{kT}\right\} \sum_{\{\mu\}} \prod_{i,j} \left[1 + \tanh\left\{\frac{J\mu_i\mu_j}{kT}\right\}\right].$$

Here, if we make a bond of the dual lattice correspond to a term "$\tanh\{J\mu_i\mu_j/kT\}$" and no bond to "1" in the above equation and sum over all the spin arrangement ($\mu = \pm 1$), we can easily understand that only closed circuits made of the bonds contribute to the partition function. This closed circuit means an important equivalence between the spin arrangement of the original lattice so that a new spin arrangement of $\sigma = -1$ corresponds inside the closed circuit and that of $\sigma = +1$ to outside of the closed circuit, where these new spins are located at the center of each square of the original lattice. We call a set of this new site for new spin a dual lattice which is discussed below. Furthermore we put

$$e^{-\frac{J'}{kT'}} = \tanh\frac{J}{kT},$$

then we can rewrite the above partition function as

$$Z\left(\frac{2J}{kT}\right) = \left(\sinh\frac{2J}{kT}\right)^{N/2} Z\left(\frac{2J'}{kT}\right).$$

If the original lattice is the square lattice, then the dual lattice is also the square lattice. Therefore, if they have critical temperatures, they must be the same, and we have $2J/kT = 0.2441$.

As an example of the percolation model, we treat the square lattice. At the center of each cell, we assume another site (we call it as a dual site (or vertex)). Combining all dual sites with dual bonds, we find another square lattice (dual lattice) as shown in Fig. 2. In this case, we obtain the same square lattice as a dual lattice (self dual). Applying the same technique to the triangular lattice, we obtain the honeycomb lattice. And we obtain triangular lattice from the honeycomb lattice (Fig. 3). The diced lattice and the Kagome lattice are mutually dual (Fig. 4).

This dual relation was used to percolation model in the square lattice which is the self-dual as seen in Fig. 2. Initially we assume that the real lattice is fully occupied with bonds and when we remove a bond randomly from the real square lattice, say,

$$(i, j) - (i, j+1), \text{ or } (i, j) - (i+1, j)$$

then immediately we place a dual bond

$$(i + 1/2, j + 1/2) - (i - 1/2, j + 1/2) \text{ or}$$
$$(i + 1/2, j + 1/2) - (i + 1/2, j - 1/2),$$

respectively. Therefore, the total number of bonds in both the real and the dual square lattices are $2N$. If we introduce p and p' as the concentrations of the real and the dual lattices, respectively, we have

$$p + p' = 1.$$

Here, the removed and replaced bonds are orthogonal each other and the replaced bond is determined without ambiguity. If the percolation model of the square has a phase transition at a concentration value, the other dual lattice must have the similar transition at the same value, therefore p_c and p'_c should be the same. Therefore,

$$p_c = p'_c = \frac{1}{2}.$$

Further mathematically exact treatment was given by Kesten (1980).

2.3.2 Application of dual transformation to 3D space In this section we will apply the dual transformation to cubic lattice in 3D. We choose dual site at the center of each cubic cell in the real 3D space. These dual sites form

Table 3. The critical exponent of several transport coefficients of the continuum percolation models. S.C. and I.S.C. are Swiss-cheese model and Inverse Swiss-cheese model, respectively. The figures with asterisk are different from those of the discrete percolation model (Feng *et al.*, 1987).

	2D S.C.	2D I.S. C	3D S.C.	3D I.S. C
Conductivity	1.3	1.3	2.4*	1.9
Elasticity	5.1*	1.3	4.4*	2.4*
Permeability	5.1*	1.3	6.2*	4.2*

the same cubic lattice in the dual space. In this lattice a corresponding element to a bond in the real space is a square (or parquet) in the dual lattice. First we assume that the dual cubic lattice is occupied by parquets. When we choose a bond in the real cubic lattice, say, $(i, j, k) - (i + 1, j, k)$, the center of this bond is located at $(i + \frac{1}{2}, j, k)$. The corresponding parquet in the dual space $(i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2})-(i+\frac{1}{2}, j-\frac{1}{2}, k+\frac{1}{2})-(i+\frac{1}{2}, j-\frac{1}{2}, k-\frac{1}{2})-(i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2})$, is deleted. This process will be repeated untill an infinite chain is accomplished in the real space. Then, if we inject the water from the top to bottom along the infinite chain in real space, then we find the water flow in the dual lattice, although at initial there is no such infinite chain of hole, which means disappearance of infinite surface. The infinite surface prohibited at initial the flow from top to bottom in the dual space.

At the same time, the water flow starts from the top to the bottom in the original cubic lattice, the tube for water in the dual cubic lattice is formed along the infinite chain of bonds in the original lattice. Therefore, the threshold value in the cubic lattice with distribution of square is 0.7512 $\cdots$. Unfortunately we don't know the exact value of the threshold value in any 3D lattices. 0.2488 ... is obtained from the numeric method, therefore 0.7512 ... is also not exact one.

In the case of site percolation problem, if the nearest neighboring two sites are occupied, we assume that these two sites are combined forming a bond. In a similar manner, if four sites at the four corners of a parquet are occupied, let us assume they form plane which prohibits water flow. As particles are distributed onto the cubic lattice, initially we find single particles, and then two-particle clusters, and so on. They grow with particle concentration. At $p_{c1} = 0.2173\ldots$, we find an infinite chain. Furthermore as the concentration of particles increases, we will find many parquets. At $p_{c2} = 1 - 0.217\ldots$, ensemble of parquets form an infinitely wide film in the cubic lattice. Finally we can say that there are two phase transitions in the cubic lattice, if we permit formation a bond by two particles and a parquet by four particles (Miyazima, 2005a).

2.3.3 Application of dual transformation to 4D space—An exact threshold value in 4D space— In the similar manner, we consider a dual lattice of a 4D hyper-cubic lattice. It is easy to see that the dual lattice of a 4-dimension hyper-cubic lattice is also a 4D hyper-cubic lattice. We choose an arbitrary square in the original hyper-cubic lattice with lattice sites expressed as $(i, j, k, l) - (i + 1, j, k, l) - (i, j + 1, k, l) - (i + 1, j + 1, k, l)$, for example, and also other 5 different squares with different directions, where i, j, k, l, are integer. Then, the center of this square is located at $(i+\frac{1}{2}, j+\frac{1}{2}, k, l)$ and the corresponding square in the dual 4D hyper-cubic lattice is $(i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}, l+\frac{1}{2}) - (i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}, l-\frac{1}{2}) - (i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}, l+\frac{1}{2}) - (i+\frac{1}{2}, j+\frac{1}{2}, k-\frac{1}{2}, l-\frac{1}{2})$. If we choose a parquet as an example, i.e. $(0, 1, 1, 0) - (0, 1, 1, 1) - (1, 1, 1, 1) - (1, 1, 1, 0)$, is in the $x - \xi$ plane, where ξ is the fourth axis, then the corresponding parquet in the dual lattice, $(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2})-(\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{2})-(\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, \frac{1}{2})-(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, is in the $y - z$ plane. These two squares are orthogonal each other and the square in the dual lattice is definitely determined.

Here, the dual lattice is a 4D hyper-cubic which is equivalent to the original lattice. Therefore, if the latter has a critical value, then the former also has the critical point at the same concentration as in the original lattice. As the same manner as before, we assume a dual 4D hyper-cubic lattice filled with squares. If we delete a square in the dual lattice, we place a corresponding square onto the original lattice. At critical concentration of square of the original lattice, we can expect an infinite surface. At the same time, an infinite surface in the orthogonal direction disappears in the dual 4D hyper-cubic lattice.

Since the original lattice and the dual lattice are equivalent, they should have the same threshold value at $Pc = 0.5$. This result is easily extended to all even $2n$D space, where we distribute a unit with half dimension n. For example, in the case of 6D space, the unit is 3D, i.e. a cube (Miyazima, 2005b).

3. Continuum Percolation Model

The percolation model is divided into two types, one is the discrete percolation model discussed in the previous paragraph. The other is the continuum percolation model where components are placed on arbitrary position in the space. In nature, however, most of cases, such as rocks with conducting parts dispersed in insulating part, or vice versa, should be analyzed by adopting the continuum percolation model, where the components occupy arbitrary positions in the 3D space.

Even if we adopt a discrete percolation model with very fine lattice, we cannot obtain critical exponents which are obtained by experiments and observations. One of the simplest models with this direction is Swiss-cheese model, where in the 2D case we open many holes at arbitrary positions in a conducting thin film (see Fig. 6). A modification of the Swiss-cheese model is an inverse Swiss-cheese model, where many conductive circles (or spheres) are placed on insulating plate (or into bulk). Mostly electricity, elasticity and permeability are discussed by observation of rocks, and theoretical and experimental studies.

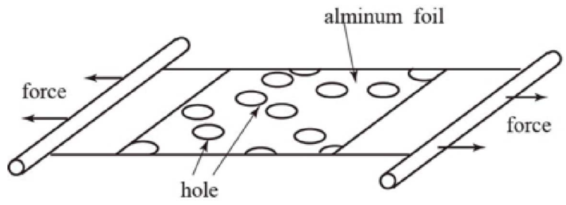

Fig. 7. The elasticity of aluminum foil is measured as increase of the number of holes.

3.1 Theoretical work

The characteristic points of Swiss-cheese model are

1. The center of circles, spheres and so on, can occupy arbitrary point in the space.

2. Overlapping area between circles in 2D space or overlapping volume of spheres in 3D can change continuously from zero.

3. The number of the neighboring elements can change.

Most important property can be found in the critical exponents of the transport coefficients, especially in the 3D space. Estimated values of the critical exponent coefficients such as the electric conductivity, the elasticity and the permeability are given in the Table 3 (Feng *et al.*, 1987).

3.1.1 Conductivity and elasticity of 2D Swiss-cheese model The critical exponents 1.1 for conductivity for a foil of Al and 1.2 for a foil of Cu, and for elasticity 3.3 (Al) and 3.7 (Cu) were obtained in the experiment shown in Fig. 7, where many holes are punched out in a sheet of Al and Cu, respectively. The exponents for the conductivity show good agreement with estimated values in Table 3, but this exponent is the same as in the discrete model. The exponents for the elasticity lie between the discrete one and expected continuum one in Table 3 (Benguigui, 1984).

3.1.2 3D experiment It is difficult to build up apparatus for 3D experiment of the Swiss-cheese model for conductivity, elasticity and permeability. However, experimental apparatus for the Inverse Swiss-cheese model is easier and was approximately built up by Miyazima *et al.*, two of them are shown in Figs. 8 and 9 (Miyazima *et al.*, 1991, 1992; Okazaki *et al.*, 1996).

We pushed many rubber balls into pure water in the acrylic tube as shown in Fig. 8 and measured the electric current from the top to the bottom through the water. If we press the rubber balls by a piston, the gap among the balls decreases together with current. If we measure the flow rate of water, we can obtain the permeability. The exponents of the conductivity and the permeability were 2.4 and 4.4, respectively, which show very good agreements with the expected values in Table 3.

Figure 9 shows the experiment for elasticity. First, we put enamel wire into acrylic tube which has a very small elasticity. Between the enamel wire we inserted rubber balls softly. Initially the rubber balls had no contact each other. When we push the system from the top slowly, then the rubber balls began to contact and at some instant the rubber ball system began to have elasticity between the top and the bottom. In this way we obtained the critical exponent of elasticity 5.4 which agrees reasonably with expectation, where the exponent of the discrete percolation model is 3.7 (Maruyama *et al.*, 1993).

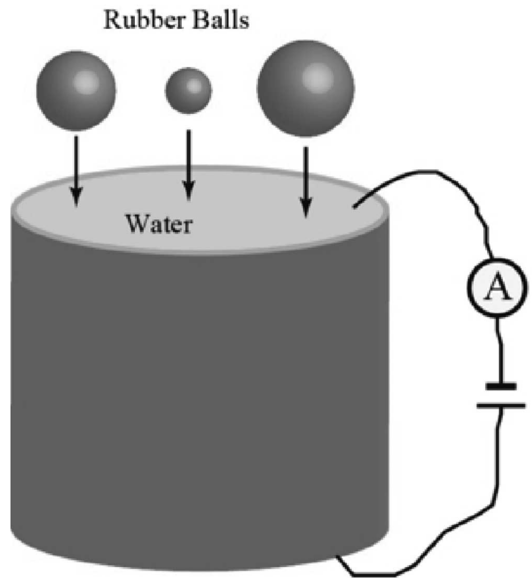

Fig. 8. Measurement of electric current in the pure water, where a large amount of rubber balls were inserted and the water flow through the space of rubber balls was measured.

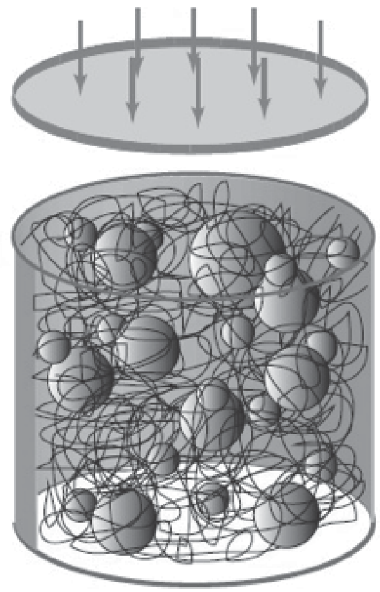

Fig. 9. Measurement of elasticity of ensemble of rubber balls. The thin thread in the figure indicates an enamel wire which has much smaller elasticity than rubber.

4. Concluding Remarks

The percolation model is the simplest one among models appearing in physics. It is, however, a very important model for a quantitative description of various phenomena which appear in nature. The covering area of the percolation model is not only materials but also social, economic, biological problems and so on. Especially the universality is very important concept for estimating what the crucial properties is in various phenomena. But, we know only few properties in the percolation model. We have to investigate the percolation model from the both standpoints, actual applicability and theoretical interest.

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