

Vortex Dynamics

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Dynamics of vortices of filament type is introduced for the following three cases: deformation of a vortex filament by self-induction, entanglement of two vortex filaments by self- and mutual inductions and reconnection of two vortex filaments. The first two cases are treated by the use of vorticity equation without the viscous term, while the third one requires a treatment with the effect of viscosity.

Key words: Vorticity, Self-Induction, Mutual Induction, Entanglement, Reconnection

1. What is Vortex?

The vortex is a swirling motion of fluid, where the angular velocity of the motion is largest in its central region and decreases with the distance from the center. The vortex has been familiar to us since many years, and has been often expressed as artworks such as “Study of water falling into still water” by Leonardo da Vinci and “Panel with red and white apricot flowers” by Korin Ogata (Japanese artist in 18–19 c.). Recently, we often see satellite photos of typhoon, which is a large scale vortex with large angular velocity in the central part, called “eye of a typhoon”.

In most vortices including the typhoon the fluid velocity $u(r)$ vanishes at the center, increases linearly with the radius r and then decreases, as is shown in Fig. 1. The angular velocity $u(r)/r$ is nearly uniform in the central region, which corresponds to the eye of a typhoon. On the other hand, the vorticity, which is defined below, vanishes out of this central region. Of course, vortices appearing in the nature receive various disturbances from the environment, and the distributions shown in Fig. 1 are much distorted.

The vorticity $\omega(x)$ is obtained from a velocity vector $u(x) = (u, v, w)$ by the following differential operation:

$$\begin{aligned}\omega(x) &= \text{rot } u(x) = \nabla \times u(x) \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right),\end{aligned}\quad (1)$$

where $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. The physical meaning of the vorticity is a strength of rotation of a small fluid element around its center.

The equation governing the vorticity can be derived from the Navier-Stokes equation (a basic equation governing the dynamics of viscous fluid), and the result is given below (refer some textbooks for its derivation, such as Batchelor (1967) and Imai (1973)).

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u = \frac{\mu}{\rho} \Delta \omega. \quad (2)$$

The second and the third terms in the left-hand side indicate, respectively, convection of the vorticity by the flow field u and increase of vorticity by stretching of vortex by u . The right-hand side indicates the diffusion of the vorticity due to the viscosity.

If the fluid is inviscid (non-viscous), there are some conserved quantities in the vortex dynamics. In order to understand the conserved quantities, the Stokes’ theorem given below is necessary. Consider a closed smooth surface in the 3D space with its edge denoted by C , then the following equation is satisfied by a vector field u defined in this space:

$$\iint (\text{rot } u) \cdot n dS = \oint_C u \cdot d\ell, \quad (3)$$

where n is a normal vector on the surface and $d\ell$ is a line element of the edge C . If $\text{rot } u$ is replaced by ω , it is rewritten as

$$\Gamma = \iint \omega \cdot n dS = \oint_C u \cdot d\ell, \quad (4)$$

where Γ is called a circulation and indicates a strength of the part of vorticity distribution within the closed curve C , as shown in Fig. 2. If the effect of viscosity is neglected, the circulation is conserved, i.e. it does not change with time.

A vorticity distribution confined within a thin and long column is sometimes called a vortex filament, and its part with vorticity is called a core.

Other two conserved quantities are introduced. One is the kinetic energy T of fluid motion including vortices, which is given by the following equation (for precise, see chapter 7 of Batchelor (1967)):

$$T = \frac{\rho}{2} \int u^2 dV = \rho \int u \cdot (x \times \omega) dV. \quad (5)$$

The other is the helicity H , which is proposed by Moffatt (1969) and defined by

$$H = \int u \cdot \omega dV. \quad (6)$$

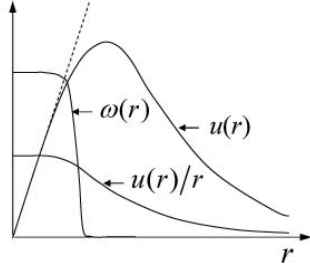


Fig. 1. Typical distributions of the velocity $u(r)$, the angular velocity $u(r)/r$ and the vorticity $\omega(r)$ in a vortex.

If some vortices with circulations Γ_i are linked with each other, as shown in Fig. 3, the helicity is given by

$$H = \sum_{i \neq j} \alpha_{ij} \Gamma_i \Gamma_j, \quad (7)$$

where α_{ij} is a constant called a winding number, which indicates the degree of winding between the i -th and j -th vortices. Examples of values of α_{ij} are shown in Fig. 3.

2. Dynamics of a Single Vortex Filament

The basic formula for dynamics of curved vortex filament is the Biot-Savart law to give a flow velocity induced by the filament. This law is known as that giving an effect of electric current on the magnetic field. In the electromagnetism a magnetic field $d\mathbf{H}$ produced at point Q by a part of an electric current I at point P with length $d\ell$ is given by the following equation (see Fig. 4(a)):

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\ell \times \mathbf{r}}{r^3}. \quad (8)$$

It is noted here that $\mathbf{H}$ is related to the current density $\mathbf{i}$ by a law, $\mathbf{i} = \text{rot } \mathbf{H}$, which is similar to the definition of the vorticity, $\boldsymbol{\omega} = \text{rot } \mathbf{u}$. Since the current I is an integration of the current density and an integration of vorticity gives the circulation Γ , the law for the velocity $\mathbf{u}$ induced by a part of vortex filament is also called a Bio-Savart law in the vortex dynamics and is expressed as follows (see Fig. 4(b)):

$$d\mathbf{u} = \frac{\Gamma}{4\pi} \frac{d\ell \times \mathbf{r}}{r^3}. \quad (9)$$

Integration of the right-hand side of Eq. (9) along the vortex filament gives the velocity at the position Q induced by the whole filament. This formula is applied to both cases with mutual and self-induction. In the case of the self-induction the integration of Eq. (9) is applied to obtain velocity at point Q which is located on the filament itself (see Fig. 4(c)).

The estimation of the effect of self-induction is first obtained by Hama (1962), which is explained in a well-formulated way by Batchelor (1967). Let the circulation and the local curvature of a filament are denoted by Γ and c , respectively. Then, this part of the filament moves to the direction perpendicular to a contact plane of the filament at this part with the following velocity:

$$\mathbf{u} \cong A\mathbf{c}, \text{ where } A = \frac{\Gamma}{4\pi} \log \frac{L}{\sigma}, \quad (10)$$

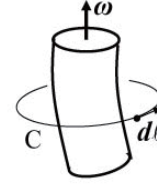


Fig. 2. Definitions of the circulation.

where L is a length of a part of the filament which contributes to induce the velocity and σ is the radius of the filament core. The length L can not be defined precisely, and its value might not be constant. However, since $L/\sigma \gg 1$ and the quantity $\log L/\sigma$ will not change much, this factor is often assumed constant. Equation (10) can be expressed also as a vector form as follows, by introducing a coordinate $\mathbf{x}(\ell, t)$ of a point on the filament:

$$\frac{\partial \mathbf{x}}{\partial t} = A \frac{\partial \mathbf{x}}{\partial \ell} \times \frac{\partial^2 \mathbf{x}}{\partial \ell^2}. \quad (11)$$

Let us consider a vortex filament which is directed on the average to the z -axis and deformed to a sinusoidal shape with a wave number k and a small amplitude a . When the amplitude is small enough, z can be replaced with ℓ , and it is easily confirmed that the following shape of filament satisfies Eq. (11):

$$\mathbf{x} = (a \sin k\ell \sin \omega t, a \sin k\ell \cos \omega t, \ell), \quad (12)$$

where $\omega = k^2 A$. This filament rotates around the z -axis with the angular velocity ω without change of shape.

When the amplitude a is larger, solution of Eq. (11) with a wavy shape periodic in the z -direction is, in general, difficult to obtain analytically. A computer simulation was made by Takaki (1975) for deformation of a vortex filament with a periodic initial shape, $\mathbf{x} = (a \sin kzs, 0, z)$, with a large amplitude. Figure 5 is one of its results, which shows variations of Fourier components of the filament shape with time-wise periodicity along with instantaneous shapes of the filament. This result reminds us of the behavior of a lattice of nonlinear springs computed by Fermi *et al.* (1955), which shows a recurrence of initial state after complicated deformations.

Another example of nonlinear behavior of a single vortex filament is a helical shape of vortex filament obtained by Betchov (1965). Here, it is expressed by the use of new variables and another expression of Eq. (11), which is also applied to other problems in the next section. First, the length and time are normalized by L (a representative scale of filament in the z -direction) and L^2/A . Let us express the filament shape by the two functions, $W(z, t) = X(z, t) + iY(z, t)$, $Z(z, t)$, where (X, Y, Z) denotes the position of the fluid element within the filament which existed initially ($t = 0$) at the height z . The initial condition is specified by giving a set of functions $(W(z, 0), Z(z, 0))$, where $Z(z, 0) = z$. Then, the following equations for $W(z, t)$ and

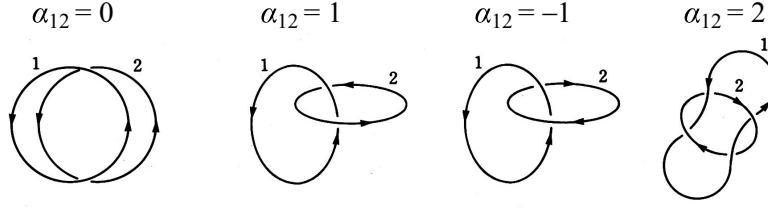


Fig. 3. Examples of linking of two ring vortices (reproduced from a review by Takaki (1988)).

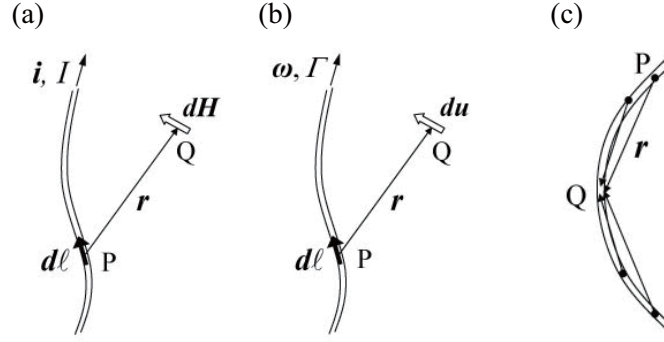


Fig. 4. Biot-Savart law for (a) the magnetic field induced by the current I , and (b) the velocity field induced by the vortex filament. (c) Induction of the velocity at point Q in the self-induction case.

$Z(z, t)$ are obtained (Takaki, 1975):

$$P\dot{W} = i(Z'W'' - Z''W'), \quad P\dot{Z} = \text{Im}(\bar{W}'W''),$$

$$\text{where } P = \left(\frac{d\ell}{dz}\right)^2 = [(1 + W''\bar{W}')^{3/2}]_{t=0}. \quad (13)$$

This set of equation has an exact solution,

$$W = \epsilon \exp i(\pi z - \tau), \quad Z = z + \epsilon^2 \pi \tau,$$

$$\text{where } \tau = \pi^2 t (1 + \epsilon^2 \pi^2)^{-3/2}, \quad (14)$$

which indicates a vortex filament with a helical shape moving steadily in the z -direction.

Deformation of a vortex filament which behaves as a soliton is obtained by Hasimoto (1972). A rough sketch of the filament shape of soliton mode is shown in Fig. 6 along with a sketch of tornado observed actually. The deformation of the filament moves vertically, while the fluid element within it nearly conserves its height.

3. Entanglement of Two Vortex Filaments

Two vortex filaments are sometimes arranged parallel to each other both in the nature (for example, two nearby tornados) and in experiments. If they have the same sense of rotation they go around each other owing to the velocity induced by the other. When the vortices are not straight but curved, they make a complicated interaction so that they are deformed to a helical shape and tangle with each other. This phenomenon was observed in an experiment by Chandraseda *et al.* (1978). A theoretical treatment of this phenomenon is made by Takaki and Hussain (1984a, b), who derived a governing equation for dynamics of the vortex filaments and obtained a solution with regular helical shape. In the following the process of deriving the governing equation is explained briefly.

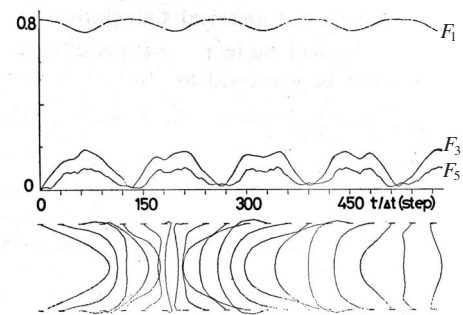


Fig. 5. Computer simulation of deformation of a vortex filament with an initial trigonometric shape. F_1 , F_3 and F_5 indicate the basic, the 3rd and the 5th Fourier components of the filament shapes, which are shown below (reproduced from Takaki (1975)).

In order to make a simple analysis it is assumed that the two vortex filaments are deformed to the same shape and arranged axisymmetrically with respect to the central axis (z -axis), as shown in Fig. 7. These vortices are assumed to have the same sense of rotation with circulation Γ . The object of analysis is to derive an equation governing the velocity $\mathbf{u}(\mathbf{x}, t)$ of the fluid element at the position $\mathbf{x} = (x, y, z)$ on the right filament, which is made of two contributions; one from the local curvature of the right vortex at $\mathbf{x}$ (self-induction), and the other from induced velocity by the part of the left vortex near to the point $\mathbf{x}$ (mutual induction). The variables in the left vortex are shown with hats. Then, the velocity $\mathbf{u}$ is given by the following equation:

$$\mathbf{u} = A \frac{\partial \mathbf{x}}{\partial \ell} \times \frac{\partial^2 \mathbf{x}}{\partial \ell^2} - \frac{\Gamma}{4\pi} \int \frac{\mathbf{s} \times \widehat{d\ell}}{s^3}, \quad (15)$$

where A is given in Eq. (10) and $\mathbf{s} = \mathbf{x} - \hat{\mathbf{x}}$.

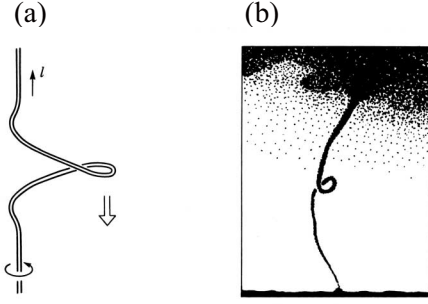


Fig. 6. Soliton-like deformation of a vortex filament. (a) Rough sketch of filament shape obtained by Hasimoto (1972), (b) sketch of the photo of a tornado taken in Oklahoma state, USA in 1978 (reproduced from Takaki (1988)).

Furthermore, it is assumed that the filaments are nearly straight in the z -direction and the curvature c of the filaments is small enough to neglect terms with higher order of ϵ , where

$$\frac{\partial x}{\partial \ell}, \frac{\partial y}{\partial \ell} = O(\epsilon), \quad \frac{\partial z}{\partial \ell} = O(1), \quad \frac{\partial^2 x}{\partial \ell^2}, \frac{\partial^2 y}{\partial \ell^2} = O(\epsilon^2),$$

and that the mutual distance of the vortices $|\mathbf{x} - \hat{\mathbf{x}}|$ is much larger than the core size σ of the vortices and much smaller than the range $(-\hat{L}, \hat{L})$ of integration in Eq. (15), i.e. $\sigma \ll |\mathbf{x} - \hat{\mathbf{x}}| \ll \hat{L}$.

Next, we normalize the length and time by L and $4\pi L^2/\Gamma$, respectively, and introduce new expressions of $\mathbf{x}$ and $\hat{\mathbf{x}}$ in terms of a complex function $W(z, t)$, defined by $W(z, t) = x(z, t) + iy(z, t)$, so that

$$\mathbf{x} = (W(z, t), z), \quad \hat{\mathbf{x}} = (-W(z, t), z), \quad (16)$$

where the variables ℓ and $\hat{\ell}$ are replaced by z , which is allowed because the vortices are almost directed to the z -axis. Then, the equation for $W(z, t)$ is derived from Eq. (15), as follows (see Takaki and Hussain (1984a)):

$$\begin{aligned} \frac{\partial W}{\partial t} = & A^* i W'' \\ & + \frac{i}{W} \left(1 - \frac{1}{2} W' \bar{W}' + \frac{(\text{Re} W \bar{W}')^2}{W \bar{W}} - \text{Re}(W \bar{W}'') \right) \\ & + \frac{\text{Im} W \bar{W}'}{W \bar{W}} W' + O(\epsilon^4), \end{aligned} \quad (17)$$

where $A^* = \ln(Ls_0/\hat{L}\sigma)$ and s_0 is a length with the same order of magnitude as $|s|$. Since $s_0 \gg \sigma$, we can expect that $A^* > 0$.

As a particular solution of Eq. (17) we assume a helical shape $W = R \exp i(kz - \omega t)$, where R , k and ω are real constant. Substituting this into Eq. (17), we have

$$\omega = \left(A^* + \frac{1}{2} \right) k^2 - \frac{1}{R^2}, \quad \text{hence} \quad \frac{\omega}{k} = \left(A^* + \frac{1}{2} \right) k - \frac{1}{kR^2}. \quad (18)$$

The quantity ω/k is the phase velocity of the wave, which the side view of the helical filament shows. Since the length is normalized by L , the amplitude R is much smaller than

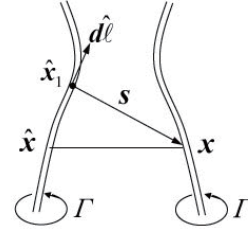


Fig. 7. Pair of two vortex filaments deformed to the same shape and arranged axisymmetrically.

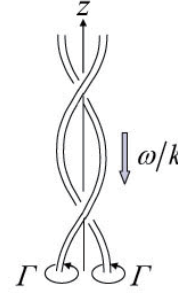


Fig. 8. Helical shape of filaments, which moves downwards.

unity, and the wave progresses downwards, as shown in Fig. 8. Of course, the downward wave motion does not necessarily mean that the material within the filament moves downwards.

Another interesting particular solution of Eq. (17) is a case where the two vortex filaments are straight and parallel to each other in far region ($z \rightarrow \pm\infty$), and they tangle several times within a finite region. A nonlinear wave analysis of this case is made by Ohtsuka *et al.* (2003).

4. Reconnection of Two Vortex Filaments

In the previous sections dynamics of vortex filaments has been discussed for the cases, where the fluid is assumed to be inviscid and the effect of filament thickness is not considered. However, vortices sometimes show a unique behavior of so-called reconnection, where they touch each other and cut themselves at the touching points and make connections to the other filaments.

This phenomenon was first reported by Crow (1970) for the reconnection of trailing vortices behind an airplane, as shown in Fig. 9(a). A similar reconnection process had been observed by Hama (1962) for a vortex filament produced in the boundary layer (see Fig. 9(b)). Since then, the reconnection of vortex rings were observed by many scientists, such as Kambe and Takao (1971), Fohl and Turner (1975) and Oshima and Asaka (1977) (see Fig. 9(c)).

Computer simulations of the reconnection process were made by a lot of scientists, examples in the early stage of studies being those by Chamberlain and Liu (1985) and Melander and Hussain (1988). A precise review of the problem of vortex reconnection is given by Kida (1994). However, the mechanism of reconnection has not been investigated theoretically, except for that by the present author and his collaborator (Takaki and Hussain, 1985, 1988). In the fol-

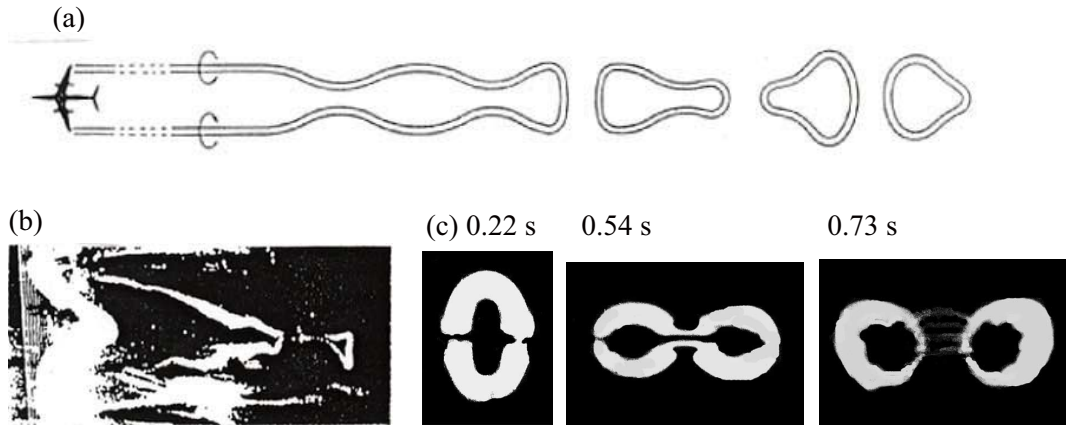


Fig. 9. Observations of vortex reconnection. (a) Trailing vortices behind an airplane by Crow (1970). (b) A small triangular ring vortex produced in a boundary layer by Hama (1962). (c) Two ring vortices produced by Oshima and Asaka (1977), where times after ejection of smoke are shown above ((a), (c): sketches by R. Takaki made from respective papers, (b) is reproduced from Hama (1962)).

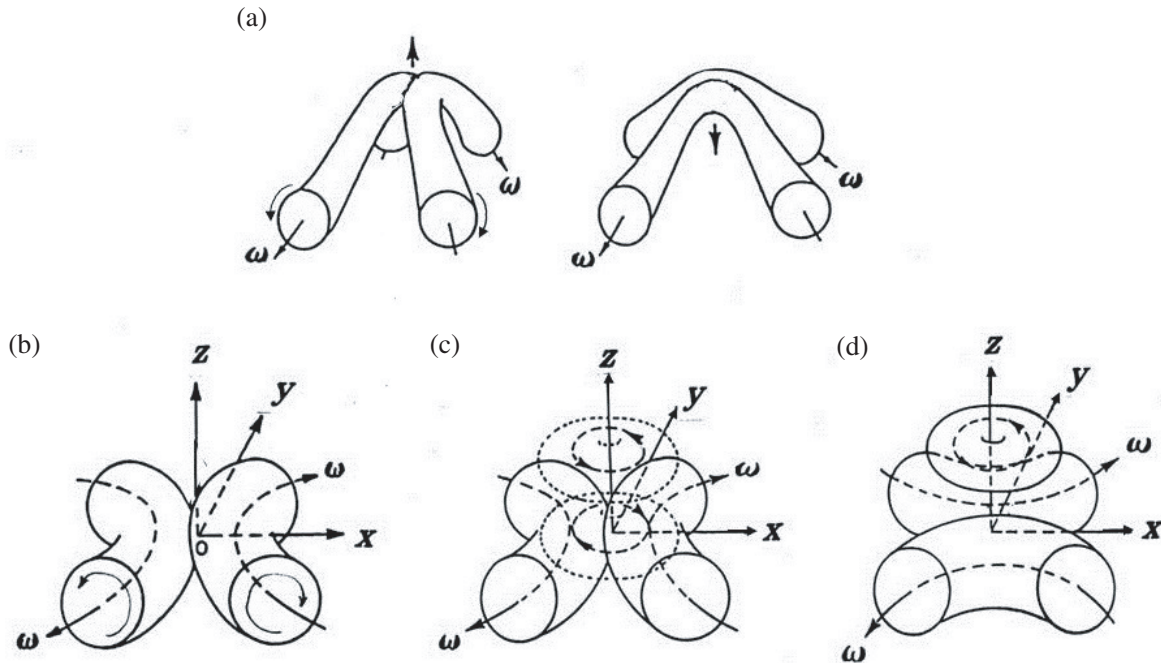


Fig. 10. A simple model of reconnection process. (a) Lift up motion of close parts (hatched regions) and push down motion of reconnected parts. (b) Initial state in a short interval of reconnection. (c) Superposition of two ring vortices. (d) final state of the reconnected filaments and the remaining ring vortex (simplified sketches of those in Takaki (2002)).

lowing an outline of this theory is explained along with an experimental verification of this theory based on a review article by Takaki (2002).

The reconnection begins with an approaching of parts of two vortex filaments with the same magnitude of circulation but with an opposite sense of rotation. As shown in Fig. 10(a), these parts (hatched regions) are lifted up by the mutual induction of velocity, and then they are pushed together by their self-induction motions. Thus, these interacting parts make a strong interaction with each other. After the reconnection they are pushed down also by the mutual induction, and get apart by the self-induction. Therefore, the reconnection occurs when these parts attain the high-

est positions, and it can be assumed that these parts do not change their positions during a short interval of reconnection. The starting state in this interval is shown in Fig. 10(b).

Here, the reconnection is looked upon as a superposition of a ring vortex on the interacting parts of both filaments, so that the vorticities at the superposed regions are cancelled out and new vorticities are born at the space between the two filaments. Since this ring vortex has a momentum in the vertical direction, we must superpose one more ring vortex with opposite direction of vorticity, as shown in Fig. 10(c), so that conservation of momentum is assured. Thus, superposing this pair of ring vortices, we have the final state in the short interval as shown in Fig. 10(d), which is

composed of the reconnected filaments and a ring vortex with upward momentum.

So far is a qualitative understanding of the reconnection process. We must go further to apply the vorticity equation (2) for a viscous incompressible fluid. It can be confirmed easily that we need the viscosity terms in this equation for analysis of reconnection in the following way. Let us consider the distribution of ω_x on a plane P spanned by the y - and z -axes. It satisfies the x -component of the vorticity equation, i.e.

$$\frac{\partial \omega_x}{\partial t} + (\mathbf{u} \cdot \nabla) \omega_x - (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} = \frac{\mu}{\rho} \Delta \omega_x.$$

Note that ω_x and the x -component of the velocity $\mathbf{u}$ vanish on this plane. Then, this equation shows that ω_x on the plane P continues to vanish, if the viscosity term (right-hand side) is neglected. On the other hand, ω_x after reconnection should have finite value on the plane P. This finite value is produced through the viscous effect, i.e. through diffusion from both sides of the plane P with non-zero distribution of ω_x .

Now, since precise analyses of velocity and vorticity fields are difficult, we assume simple polynomial expansions up to the second order of coordinates for components of $\mathbf{u} = (u, v, w)$, $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ and $\Delta \boldsymbol{\omega}$ in the initial and final states (Figs. 10(b) and (d)) based on their spatial symmetries, i.e. whether they are even or odd functions of coordinates. For example, the u is an odd function of x and an even function of y , hence it is expressed as $-mx + kxz$ up to the second order. On the other hand, v in the initial state is very weak and assumed to vanish. As for w , it is even function of x and expressed as $w_0 + mz - lx^2/2 + kz^2/2$. Coefficients in these expressions are adjusted so that they satisfy the continuity equation for incompressible fluid, i.e. $\text{div } \mathbf{u} = 0$. The expressions of velocity components in the final state are assumed in the same way.

Velocity components u and v of the superposed ring vortex are given simple expressions, by assuming that they are composed of even or odd functions of x and y . Since the ring vortex has an axisymmetric distribution of w around z -axis, w in it should depend on x and y through $x^2 + y^2$. Hence, by superposing the two expressions for the filaments and the ring, w is expressed as $-n - 2mz - kz^2/2 + l(x^2 + y^2)/2$. In addition, strength of this ring vortex is assumed to grow from 0 and asymptote to a finite value, hence the expressions for the ring vortex and the filaments have a common factor $T(t)$, which satisfies $T(t) \rightarrow 0$ for $t \rightarrow -\infty$ and $T(t) \rightarrow 1$ for $t \rightarrow \infty$.

After all, the superposition of the initial state and their components are expressed as follows:

$$\begin{aligned} u &= -m(1 - T(t))x + k(1 - T(t))xz, \\ v &= mT(t)y + kT(t)yz, \\ w &= w_0 - \delta T(t) + m(1 - 2T(t))z \\ &\quad - \frac{l(1 - T(t))x^2}{2} + \frac{lT(t)y^2}{2} + \frac{k(1 - 2T(t))z^2}{2}, \end{aligned} \quad (19)$$

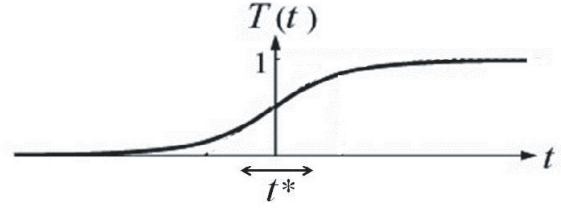


Fig. 11. Solution of $T(t)$, showing the process of reconnection.

$$\begin{aligned} \omega_x &= (k + l)T(t)y - nT(t)yz, \\ \omega_y &= (k + l)(1 - T(t))x - n(1 - T(t))xz, \\ \omega_z &= 0, \Delta \omega_x = g_x T(t)y, \\ \Delta \omega_y &= g_y(1 - T(t))x, \Delta \omega_z = 0, \end{aligned} \quad (20)$$

where expansions of $\Delta \omega_x$ and $\Delta \omega_y$ are made up to the first order by assuming that they have the same forms as ω_x and ω_y .

Substituting the above expressions into the vorticity equation (2) and taking terms up to the first order of coordinates, we have the following equation for $T(t)$:

$$\frac{dT}{dt} = \frac{T}{t^*} - \frac{T^2}{t^{*2}}, \quad \text{where } t^* = \frac{\delta n}{k + l} = \frac{\mu}{\rho}(g_x - g_y) - 2m. \quad (21)$$

Solution of this equation satisfying the boundary conditions for $t \rightarrow \pm\infty$ is given below.

$$T(t) = \frac{e^{t/t^*}}{1 + e^{t/t^*}}, \quad (22)$$

whose behavior is shown in Fig. 11.

This result is convincing since $T(t)$ grows and makes the reconnection smoothly. The time t^* needed for reconnection decreases for smaller viscosity.

Here, it should be noted that the coefficients, l, m, n , etc., are left undetermined; their values should be determined so that the velocity and the vorticity distributions in the reconnecting region match to those in outer flow. For that purpose a polynomial expansion up to higher orders is necessary, which is not yet done because a highly complicated manipulation are required.

The present author and his collaborator made an experiment and a computer simulation to confirm the theoretical result (Takaki and Kakizaki, 1992). Figure 12(a) shows the side view of experimental apparatus, where two ring vortices were ejected by the use of speaker connected to computer and the velocity distribution was measured by LDV (Laser Doppler Velocimeter). Sketches of vortex shapes are shown in Fig. 12(b). The ejected two rings made the first reconnection to produce a distorted ring, which made the second reconnection to produce two rings. They measured vorticity distribution during the reconnection process, which agreed considerably with the visualization (Fig. 12(b)) and the theoretical result (Eqs. (20)–(22)).

After the theoretical work introduced above no progress has been made in theoretical understanding of the reconnection process. It remains to be one of the largest mysteries in fluid dynamics.

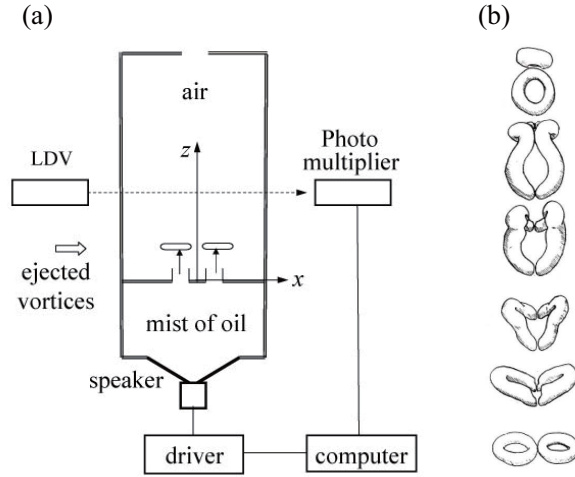


Fig. 12. (a) Experimental apparatus to observe reconnection process. (b) Observed reconnections of two ring vortices moving from lower to upper (sketch from photos) ((b) is reproduced from Takaki (2002)).

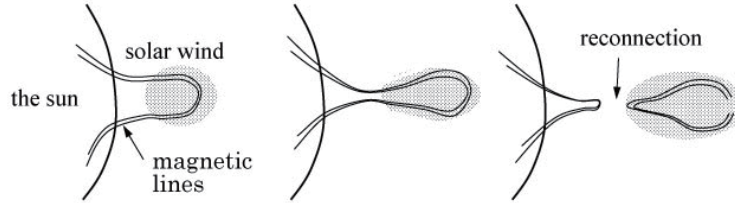


Fig. 13. Reconnection of magnetic lines as the plasma region goes away from the sun (reproduced from Takaki (2002)).

5. Dynamics of Line Singularities in Other Materials

Reconnection processes have been observed in systems of other materials, as is mentioned in a review by Takaki (2002). Here, two cases are introduced briefly; the reconnection of magnetic lines in a plasma and the reconnection of microscopic vortex filaments in superfluid helium. A representative system with reconnection of magnetic lines would be a solar wind, which is a large scale plasma including magnetic lines. Figure 13 shows a process of ejection of a solar wind, where the magnetic lines connected to the sun is reconnected in order that the wind can go away from the sun. A hydromagnetic analysis of this phenomenon is made by Yeh and Axford (1970).

What is interesting here is that the functional expressions of velocity and the magnetic fields in the Yeh and Axford's analysis are similar to those of velocity and the vorticity in our reconnection analysis. It is noted here that there are two ways of analogy between the vorticity field and the magnetic field. One is based on the similarity of formulae $\omega = \text{rot } \mathbf{u}$ and $\mathbf{i} = \text{rot } \mathbf{H}$, where the velocity $\mathbf{u}$ corresponds to the magnetic field $\mathbf{H}$, and the vorticity ω to the current density $\mathbf{i}$. In this analogy, the reconnection of vortex filaments does not directly correspond to that of magnetic lines. Another analogy is based on the similarity of governing equations of both fields, where the vorticity is governed by Eq. (2), while the magnetic field in plasma is governed by the

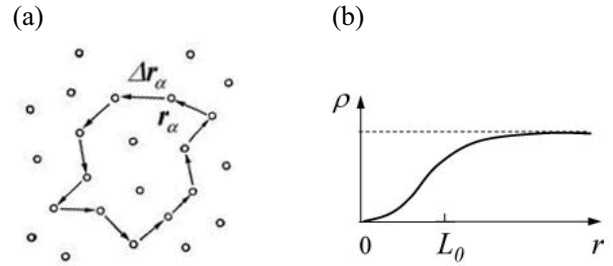


Fig. 14. Quantized vortex in superfluid helium. (a) Displacements of helium atoms along a closed path. (b) Distribution of helium density within a quantum vortex filament as a solution of Schroedinger equation, where $L_0 = \hbar / \sqrt{2} V_0 \rho_0 = O(0.1 \text{ nm})$.

following equation:

$$\frac{\partial \mathbf{H}}{\partial t} + (\mathbf{u} \nabla) \mathbf{H} - (\mathbf{H} \nabla) \mathbf{u} = \eta \Delta \mathbf{H}, \quad (23)$$

where η is the magnetic diffusivity defined by $\eta = 1/(\sigma \mu_0)$ (σ is the electric conductivity of the plasma and μ_0 is the magnetic permeability). This equation is derived by combining the Maxwell's equation and the fluid dynamical equation along with Ohm's law.

As in the vorticity field, the second and the third terms of the left-hand side of Eq. (23) stand for the convection of magnetic lines by fluid motion and the thinning of bundle of

magnetic lines by stretching, respectively. The right-hand side stands for the diffusion of magnetic lines by electric conductivity. Comparing Eqs. (2) and (23), we can convince ourselves that the magnetic lines correspond to vortex filaments so long as their dynamics are concerned.

Existence of vortex filament in the superfluid was suggested by Feynman (1955) based on the following logics. Suppose that positions and velocities of atoms in superfluid helium are denoted by $\mathbf{r}_\alpha$ and $\mathbf{v}_\alpha$, respectively, as shown in Fig. 14, and the wave function of atoms is given with an analogy of the wave function of single particle with momentum p , i.e. $\exp(i\mathbf{p} \cdot \mathbf{x}/\hbar)$, as follows:

$$\Phi = \exp\left(i \sum m \mathbf{v}_\alpha \cdot \mathbf{r}_\alpha / \hbar\right) \Phi_0, \quad (24)$$

where m is the mass of helium atom and Φ_0 is the wave function of the static state.

When atoms have moved along a closed path to the position of the next atom as shown by arrows, the wave function changes from that in Eq. (24) by a factor $\exp(i \sum m \mathbf{v}_\alpha \cdot \Delta \mathbf{r}_\alpha / \hbar)$. Since these movements do not change the state of atoms, the quantity in the parenthesis of this factor must be a multiple of $2\pi i$, hence we have $\sum \mathbf{v}_\alpha \cdot \Delta \mathbf{r}_\alpha = 2\pi \hbar n / m = (h/m) \cdot n$, where n is an integer. The sum $\sum \mathbf{v}_\alpha \cdot \Delta \mathbf{r}_\alpha$ is a circulation along the closed path (see Eq. (4)), and is denoted by Γ . Thus, the vortex filament in superfluid has a quantized circulation, as expressed by $\Gamma = (h/m) \cdot n$. The magnitude of the h/m is about $0.94 \times 10^{-3} \text{ cm}^2/\text{s}$ and has a macroscopic order. It is known that the smallest value of circulation corresponds to $n = 1$, and existence of a vortex filament with this value of circulation is confirmed, which is called a quantum vortex filament. If a circulation along a closed path has a value with $n > 1$, the closed path includes more than one quantum vortex filaments within it.

Since the superfluid has no viscosity, the quantum vortex filament has no mechanism to increase its core size, and keeps its size with order of inter-molecular distance. Or, one should say that the quantum vortex filament does not have a core of usual sense with constant angular velocity or vorticity (see Fig. 1). If the fluid is moving with velocity u along a circular path with radius r , its circulation is $\Gamma = 2\pi r u$, and if the circulation is constant we have the relation $u = \Gamma / 2\pi r$ for r down to the scale of inter-atomic distance.

On the other hand, a circular motion of this high speed ($\sim 10^6 \text{ m/s}$) within the region of inter-molecular scale will produce a very low pressure at the center of vortex filament, hence the density of atoms should be very low there. An analysis based on a hydrodynamical expression of the Schrodinger equation gives the radius $L_0 = \hbar / \sqrt{2} V_0 \rho_0$ of this low density region of about 0.1 nm, where V_0 stands for the repulsive potential between helium molecules and ρ_0 is the density of superfluid far from the vortex filament. Thus, the core of the quantum vortex filament is not a region with vorticity but a hollow region, where the density of atoms is low.

Existence of the vortex reconnection in superfluid is already confirmed by Schwarz (1985) and Koplik and Levine (1993) by numerical simulation based on the Schrodinger equation. Now, the present author would like to suggest that a superposition of ring vortices on nearly parallel line

vortices might give a simple model for theoretical analysis of reconnection of quantized vortex filament, where these vortices do not have vortical cores but low density cores without vorticity.

Finally, the present author would like to suggest that vortex dynamics is related to many fields in physics, and that it provides a common method for theoretical studies of problems in these fields.

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