# Growing Cell Patterns

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(Received July 8, 2014; Accepted January 11, 2015)

**Key words:** Soap Froth, Neumann-Mullins Equation, Self-Similar Growth, Vertex Dynamics Cell Model

## 1. Introduction

Cell patterns are ubiquitous in nature. A typical example of cell pattern is the soap froth, shown in Fig. 1. In this context, a cell is the minimum bounded area, or bubble. Many of these cells expand in space and their boundaries form a network structure. Natural cell patterns exist over a wide range of spatial scales; for example, biological cell tissues ($10^{-3}$ cm), magnetic domains ($10^{-3}$ cm), polycrystal grain structures ($10^{-2}$ cm), soap froths (1 cm), columnar basalt (10 cm), and so on, and the bubble structure of the cosmos ($10^{25}$ cm). Here, the numbers in parentheses are the approximate linear sizes of the cells. The structures, growth mechanisms and dimensions of cell patterns also vary widely. However, if viewed solely as cell patterns, these diverse structures exhibit common features. (Weaire and Rivier, 1984; Weaire and Hutzler, 1999)

## 2. Geometric Constraints

Cell patterns are fundamentally described by Euler's geometric theorem, expressed as $V - E + F = \chi$ in two dimensions and $V - E + F - C = \chi$ in three dimensions. Here $V$, $E$, $F$ and $C$ denote the number of vertices, edges, faces and cells, respectively, and $\chi$ is Euler's characteristic number. Note that in two dimensions, the terms *cell* and *face* are equivalent. Although $\chi$ depends on the boundary condition of the system, it is an integer of order 1; hence we can set $\chi = 0$ for large cellular systems.

In soap froth, each vertex is the joining point of three edges in two dimensions and four edges in three dimensions. These simplest structures are most commonly found in nature because they are topologically stable. According to Euler's theorem, a cell is bounded by six edges (on the average) in two dimensions, and by $12/(6 - \langle n \rangle)$ faces in three dimensions, where $\langle n \rangle$ denotes the average number of edges per face. Any cell pattern must satisfy these geometrical requirements.

## 3. Self-Similar Growth

Under suitable conditions, cell patterns will coarsen or grow. This section introduces some universal properties of growing cell patterns, using the two-dimensional soap froth shown in Fig. 1 as an example. The internal pressure difference between neighboring bubbles induces airflow from the higher pressure side to the lower pressure side through the soap film. Consequently, the bubble at the higher pressure side shrinks while that at the lower pressure side expands. Assuming that the airflow rate is proportional to the curvature of the interfacial membrane, the area of an $n$-sided bubble, $A(t)$, is given by the Neumann-Mullins equation

$$\frac{dA(t)}{dt} = k(n - 6),$$

where $k$ is a positive constant that depends on the properties of interfacial membrane. This equation states that the rate of change of bubble area depends only on the edge number, i.e. the bubble shrinks if enclosed by less than 6 edges and expands otherwise. If a bubble edge becomes short, it triggers varying connectivity with other edges, leading to changes in the edge numbers of the bubble and its neighbors. Bubbles with the minimum edge number ($n = 3$) further shrink and eventually vanish. In this way, individual bubbles repeatedly shrink and expand by changing their areas and edge-numbers. These dynamics lead to a gradual coarsening of the whole system.

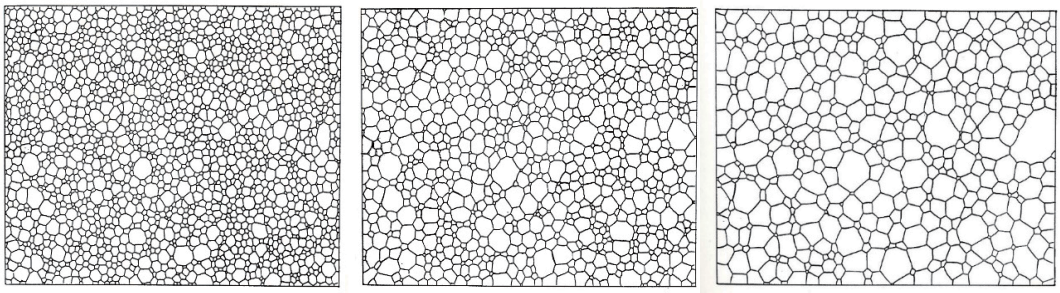
As smaller bubbles are removed from the system, the average area per remaining bubble increases as a power law in time, $t^{\alpha}$. The exponent $\alpha$, which characterizes the growth of the system, is a crucial parameter in research of growing cell patterns. Theoretically, it is expected that $\alpha = 1$, although many experimental studies have reported that $\alpha < 1$. Despite the gradual coarsening of the system, the entire cell pattern does not statistically change (see Fig. 1). In other words, the statistical properties of the pattern are growth-invariant. If we consider a new length unit $R(t)$, defined by the square root of the average area per cell $\sqrt{A(t)}$, then the cell size distribution scales by $R(t)$. The long-term distributions of cell size and edge-number are described by definite functions. Such a state, whose universal properties are independent of the initial states, is called a scaling state. Within the scaling regime, the cell pattern grows in a statistically similar manner.

## 4. Shape Correlation

Many two-dimensional cell patterns obey the well-known Aboav-Weaire law, which specifies the average edge num-



doi:10.5047/forma.2015.s007



(a) 0 h (b) 9 h (c) 27 h

Fig. 1. Growing soap froth in two dimensions. Experimental results obtained by Ogawa and Harada (2004, private communication).

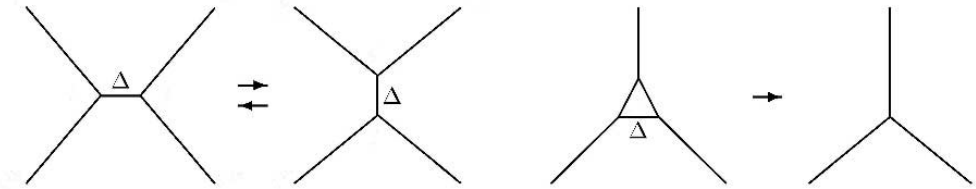
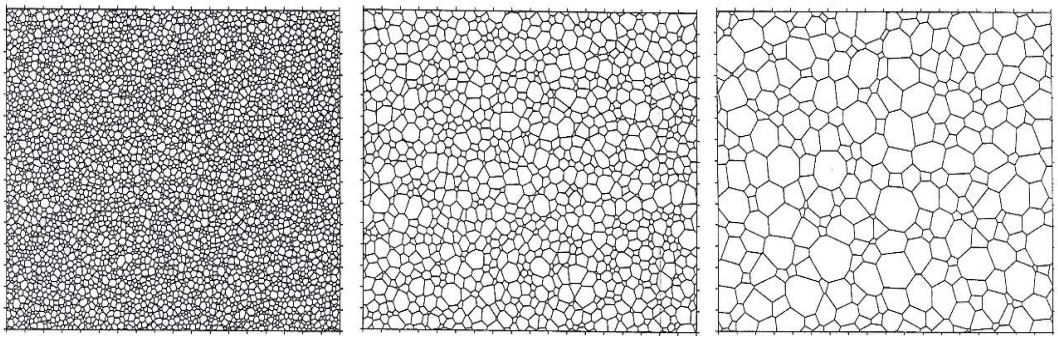



Fig. 2. Elementary processes in a vertex network.



(a) $t = 4.0$ (b) $t = 25.0$ (c) $t = 100.0$

Fig. 3. Snapshots of vertex dynamics cell model simulation. Relaxation to stationary state (scaling state) from a random Voronoi state with 30,000 cells at $t = 0$. Time $t$ is dimensionless and periodic boundary conditions are assumed.

ber $m(n)$ of neighboring cells of a $n$-sided cell by the relationship

$$m(n) = k_1 + \frac{k_2}{n}.$$

Here, $k_1$ and $k_2$ are positive constants that depend on the force range of inter-cellular interactions. In soap froths and grain aggregates, $k_1 \sim 5$ and $k_2 \sim 7.5$. The Aboav-Weaire Law expresses the shape correlation between cells. Cells with many edges are surrounded by cells with few edges, and vice versa. This law holds for a system in statistical equilibrium after sufficient relaxation of local strains, which describes the above-mentioned scaling state.

## 5. Computer Simulations

The self-similar growth of cell patterns has been investigated in numerous computer simulations. The scaling state of self-similar growth is a stationary state, which can be elucidated only by simulating systems with sufficiently large number of cells for sufficiently long time. The scaling state is regarded as an intrinsic state of the system, independent of both initial state and boundary conditions. This section introduces a model that effectively simulates the self-similar behavior of cell systems; namely, the vertex dynamics cell model (Nagai *et al.*, 1988; Kawasaki *et al.*, 1989). The model illustrates the essences extracted by coarse-graining of real cell systems. The two-dimensional version is discussed here; the three-dimensional version is detailed in Nagai *et al.* (1990) for soap froths and grain aggregates and in Honda *et al.* (2004) for biological cell systems.

Growing cell patterns are characterized by the dynamics of their cell boundary networks. To express these dynam-

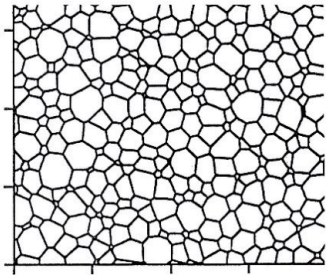


Fig. 4. Magnification of the left-bottom part of Fig. 3(a). The average cell size in this figure is identical to that of Fig. 3(c).

ics, we describe the system as an assembly of intersection points of cell boundaries (i.e. vertices). In this model, each vertex is bound to three neighboring vertices with straight cell boundaries, forming a vertex network. The vertices are assumed to move such that the total energy $U$ of the cell boundaries always decreases. In two dimensions, the spatiotemporal dynamics satisfy

$$\eta_i \frac{d\boldsymbol{r}_i}{dt} = -\nabla_i U(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots),$$

where $\boldsymbol{r}_i$ denotes the position vector of the $i$-th vertex, $\eta_i$ is the friction coefficient and $t$ denotes time. The above equation relates the frictional force (left-hand side) to the potential force that reduces the total potential energy $U$ (right-hand side). Furthermore, the topological dynamics of the vertex network are described by two elementary processes, recombination T1 and triple condensation T2. In T1, two vertices approaching within a small distance $\triangle$ recombine as shown in Fig. 2(a). In T2, three vertices approaching within a small distance $\triangle$ coalesce into a single new vertex as shown in Fig. 2(b). Here, $\triangle$ is the smallest length in the coarse-grained model. The vertex dynamics cell model thus describes growing cell patterns by an equation of motion for the vertices and a pair of elementary processes. The model is quite general and applicable to many systems. Below, we apply the model to soap froth or grain aggregates.

In soap froths and grain aggregates, the total energy of cell boundaries can be expressed as $U = \sum_{\langle ij \rangle} \sigma_{ij} |\boldsymbol{r}_i - \boldsymbol{r}_j|$, where the sum is taken over cell boundary $\langle ij \rangle$ and $\sigma_{ij}$ denotes the interfacial linear energy density of $\langle ij \rangle$. This approach is valid because the potential energy induced by inhomogeneity is proportional to the length of the cell boundary. Thus, the potential force acting on a vertex depends on the directions of the three cell boundaries raying from it, but not their lengths. In soap foam and isotropic grain systems, the interfacial linear energy density $\sigma_{ij}$ is independent of the boundary $\langle ij \rangle$ and can be simply expressed as $\sigma$. The friction coefficient $\eta_i$ is considered to be proportional to the total length of the three cell boundaries meeting at vertex $i$ and is thereby written as $\eta_i = \eta_0 \sum_j^{(i)} |\boldsymbol{r}_i - \boldsymbol{r}_j|/2$, where $\eta_0$ is a positive constant and the sum is taken over the three neighbors $j$ of $i$ connected by their cell boundaries. We performed computer simulations of such isotropic cell systems, and present the results below.

The initial state was a random pattern of 30,000 Voronoi cells. As the time elapsed, this system continued to coarsen by repeating the two elementary processes T1 and T2.

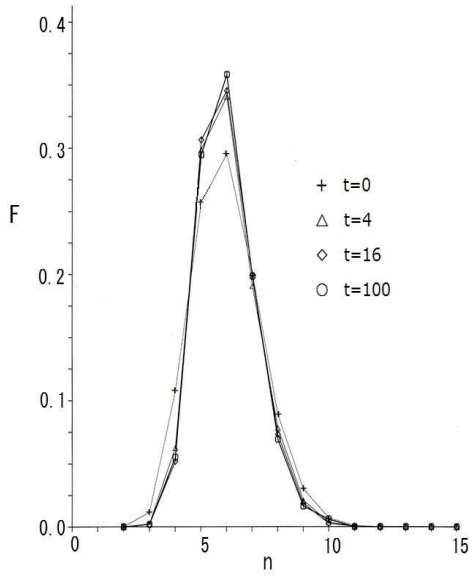



Fig. 5. Edge number distribution of cells. Relaxation to an asymptotic function $F^*(n)$ from an initial distribution of 50,000 random voronoi cells.

Through a series of intermediate states, the initial Voronoi pattern faded from memory and the system gradually approached a stationary state. Snapshots of the simulation at various times are presented in Fig. 3. The system and its evolution to steady state closely resemble Fig. 1. Paralleling the pattern dynamics, the average cell area $A(t)$ gradually became power-law in time (i.e. tended to $t^\alpha$ with $\alpha = 1$). Figure 4 shows the pattern obtained by magnifying Fig. 3(a) by a factor of 4.57. This factor is the ratio of the average linear cell size at $t = 100$, $R(100)$, to $R(4)$ at $t = 4$. That is, the average linear cell size in Fig. 4 is identical to that in Fig. 3(c). Note that no recognizable difference exists between these two figures. Similarly, magnifying Fig. 3(b) yields a pattern that is hardly distinguishable from Fig. 3(c).

The self-similarity of the above-mentioned cell system can be quantified by the area distribution function of cells scaled by the average cell area, or by the edge-number distribution function of cells. Figure 5 plots the distribution function $F(n, t)$ of $n$-sided cells at time $t$. This figure confirms that the initial random Voronoi pattern approaches a definite state described by a corresponding function $F^*(n)$. This trend is exactly the self-similar growth of a cell system.

## References

Honda, H., Tanemura, M. and Nagai, T. (2004) A three-dimensional vertex dynamics cell model of space-filling polyhedra simulating cell behavior in a cell aggregate, *J. Theor. Biol.*, **226**, 439–453.

Kawasaki, K., Nagai, T. and Nakashima, K. (1989) Vertex models for two-dimensional grain growth, *Philos. Mag. B*, **60**, 399–421.

Nagai, T., Kawasaki, K. and Nakamura, K. (1988) Vertex dynamics of two-dimensional cellular patterns, *J. Phys. Soc. Jpn.*, **57**, 2221–2224.

Nagai, T., Ohta S., Kawasaki, K. and Okuzono, T. (1990) Computer simulation of cellular pattern growth in two and three dimensions, *Phase Transitions*, **28**, 177–211.

Weaire, D. and Hutzler, S. (1999) *The Physics of Foams*, Oxford University Press, Oxford, U.K.

Weaire, D. and Rivier, N. (1984) Soap, cells and statistics-random patterns in two dimensions, *Contemp. Phys.*, **25**, 59–99.