Behavior of Vortex Rings

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1. Vorticity and Circulation

The fluid motion associated with circulatory motion around a common centerline, or a flow with closed streamlines is generally called a "vortex", an eddy, a whirlpool, etc. Tornados and hurricanes are also vortices in larger scale. It has such a characteristic form that it is widely used as a symbolic design in arts and literatures describing never-ending motion of fluid, spatiotemporal circulatory motion in social life, etc. (Lugt, 1983).

When a small object is placed in the "vortex", however, two types of fluid motion associated with the circulatory motion are observed, i.e., (i) one type shows a translational motion of that object along a circular orbit without spinning and (ii) the other is the motion accompanied by spinning motion of that object. The fluid motion associated with spinning motion is described by "vorticity" $\boldsymbol{\omega}$, which is mathematically defined by rot$\mathbf{v}$ or curl$\mathbf{v}$ ($\mathbf{v}$ being the velocity field). In the Cartesian coordinate system (x, y, z) with the velocity component $\mathbf{v} = (u, v, w)$, the vorticity $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ is defined as

$$\boldsymbol{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right).$$

The latter type (ii) has vorticity and the flow is called *rotational*, whereas the former type (i) has no vorticity and is called *irrotational*. In terms of the vorticity, a flow with straight streamlines of spatially nonuniform speed (a shear flow) has vorticity and is rotational, even though the stream is not circulating.

A line drawn in a fluid whose element is everywhere tangent to the vorticity vector is called a "vortex line". A "vortex tube" is a tubular surface consisting of the collection of vortex lines which pass through a small closed curve. The limiting case of a vortex tube, in which the cross section of the closed curve is infinitesimal, is called a "vortex filament". In contrast to the mathematical definition of the vortex line, a vortex filament is an idealization of the physical property of spinning fluid region with infinitesimal cross section $d\mathbf{S}$, in which the magnitude of the vortex $\Gamma = \boldsymbol{\omega} \cdot d\mathbf{S}$ characterizes its strength. When a bundle of vortex filaments are distributed over the region S in the fluid, the integral of the vorticity $\boldsymbol{\omega}$ over that region characterizes the total vortex strength:

$$\Gamma = \int_S \boldsymbol{\omega} \cdot d\mathbf{S} = \int_S (\mathrm{rot}\, \mathbf{v}) \cdot d\mathbf{S} = \int_C \mathbf{v} \cdot d\mathbf{l}. \quad (1)$$

Here, C is the closed curve that surrounds the area S in anti-clockwise direction with respect to $\boldsymbol{\omega}$ ($d\mathbf{l}$ being the line element on that curve), and the Stokes theorem is used. The last expression of Eq. (1) is known as a "circulation" (Lamb, 1932; Batchelor, 1967).

2. Vortex Ring

In an inviscid fluid, the circulation Γ along a curve C is conserved (Kelvin's circulation theorem). This theorem does not strictly apply in real fluids because of the viscosity, e.g., vorticity is generated near the solid wall where a strong shear flow is present, while a concentrated vorticity decays its magnitude with time and diffuses into ambient fluid. It is, however, approximately satisfied, once the localized vorticity region is convected into the fluid region. Thus the vortex tube of cross section S with uniform vorticity ω (magnitude of $\boldsymbol{\omega}$ perpendicular to the cross section) has a constant circulation $\Gamma = \omega S$, so that the thinner part of the vortex tube has larger vorticity, and *vice versa*. This explains the strong swirl of the tornados near the ground. Due to the conservation of circulation, a vortex filament cannot disappear in the fluid (a consequence known as Helmholtz's vortex theorem). It must either extend to the boundary of the fluid region or self-connected to form a closed curve. The latter type of fluid motion is called a "vortex ring".

3. Formation of a Vortex Ring

Vortices are formed by frictional or viscous effects in a fluid near solid surfaces. If the fluid is ejected from the tube at an appropriate way (Maxworthy, 1972; Shariff and Leonard, 1992), the vorticity generated on the inner surface of the tube wall separates at the opening, rolls up and convected into the fluid region to form a vortex ring. In the air, a vortex ring can be created by tapping a closed box with a circular orifice in the other side. Some living creatures, like dolphins and squids, are reported to make vortex rings. In any case, the size of the ring, the volume of fluid and the time needed to eject it determine the strength of the vortex ring.

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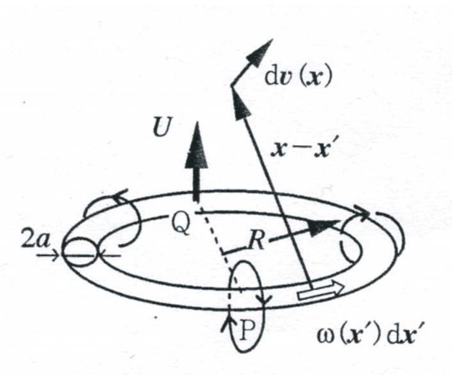

Fig. 1. Velocity due to a vortex ring.

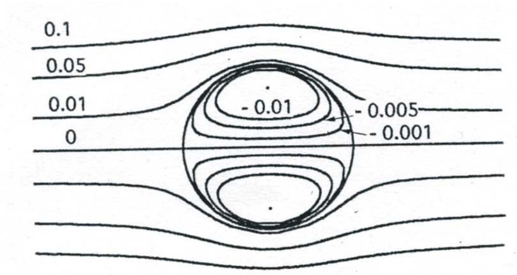

Fig. 2. Hill's vortex ring.

4. Velocity Induced by a Vortex Ring

Velocity field $d\mathbf{v}$ induced by a line element $d\mathbf{x}'$ on the vortex ring with vorticity $\omega(\mathbf{x}')$ is given by

$$d\mathbf{v} = \frac{\omega(\mathbf{x}')d\mathbf{x}' \times (\mathbf{x} - \mathbf{x}')}{4\pi|\mathbf{x} - \mathbf{x}'|^3}, \tag{2}$$

which is the same form as the Biot-Savart' law known in the electromagnetism. The velocity at any fluid point $\mathbf{x}$ is given by integrating the contribution from the velocity induced by $\omega(\mathbf{x}')$.

If a vortex ring of radius R has uniform vorticity ω over the core region of radius a (see Fig. 1), the velocity at any point on the vortex ring Q induced by the other part of the vortex ring is the same due to the axisymmetry of the ring, so that the vortex ring itself translates at a certain speed U without deformation. In the case of a thin vortex ring (Lord Keivin, 1867; Dyson, 1893),

$$U \approx \frac{\Gamma}{4\pi R}\left[\log\left(\frac{8R}{a}\right) - \frac{1}{4}\right]. \tag{3}$$

On the other hand, a thick vortex ring of a spherical shape, i.e., thick enough to allow no fluid through the ring, creates a velocity field described in spherical coordinate system (r, θ, ϕ) by

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r},$$

where ψ is the stream function given by

$$\psi = \frac{3U}{4a^2}r^2(r^2 - a^2)\sin^2\theta, \quad r < a$$

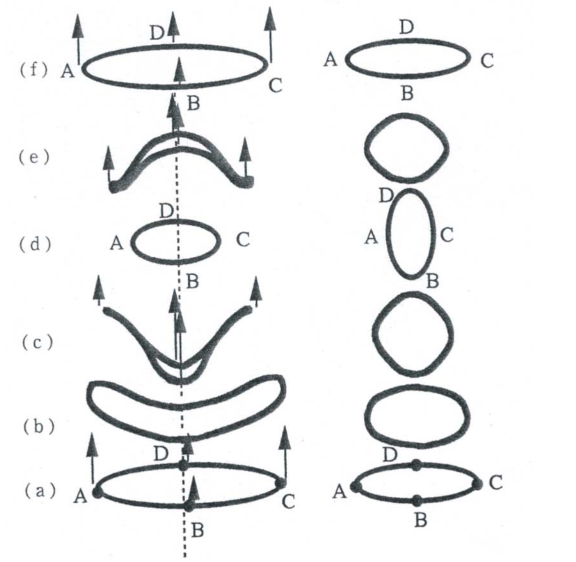

Fig. 3. Deformation of a vortex ring of elliptic shape; bird's-eye view (left column) and plan view (right column).

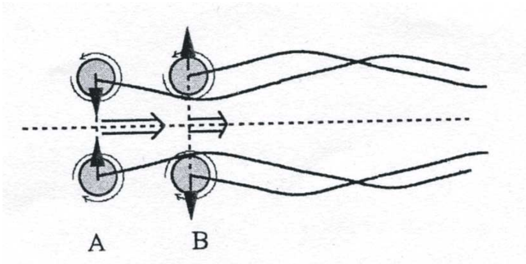

Fig. 4. Interaction of two vortex rings in tandem position.

$$\psi = \frac{U}{2}\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta. \quad r > a \tag{4}$$

Evidently, the fluid volume inside the sphere of radius a moves with the vortex ring (Hill's spherical vortex, see Fig. 2) (Hill, 1894), In general, a vortex ring behaves like a particle with characteristic velocity, momentum, and energy kept constant.

5. Non-circular Vortex Ring

According to Eq. (3), a vortex ring of smaller radius moves faster, and *vice versa* under the same Γ. This implies that if the vortex ring is elliptic with AC a longer diameter (see Fig. 3(a)), the part with the smaller radius (near the points A and C in Fig. 3(a)) moves faster than the other part, so that the vortex ring ceases to lie on a plane and deforms like Figs. 3(b) and (c). Then, the local radius near the points B and D becomes smaller, so that the parts near the latter catch up, and that the vortex ring recovers to be in a plane with BD a longer diameter (Fig. 3(d)). Further deformation is repeated from Figs. 3(d) to (f), similarly to that from Figs. 3(a) to (d) with longer diameter 90° rotated (Kambe and Takao, 1971). A vortex ring with much more waves along the ring behaves similarly. It proceeds with a speed in accordance with the average ring radius, in which the deformation of alternating concave and convex parts is

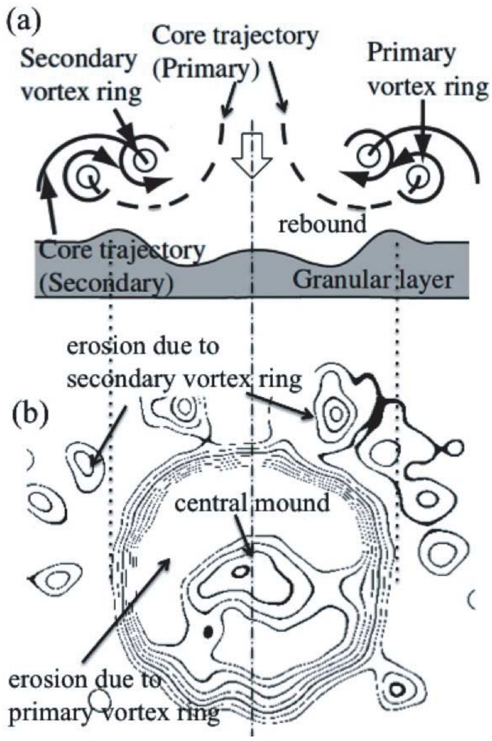

Fig. 5. Collision of a vortex ring with a granular plane; (a) cross section of the surface (schematic), and (b) contour lines observed at the collision of a stronger vortex ring.

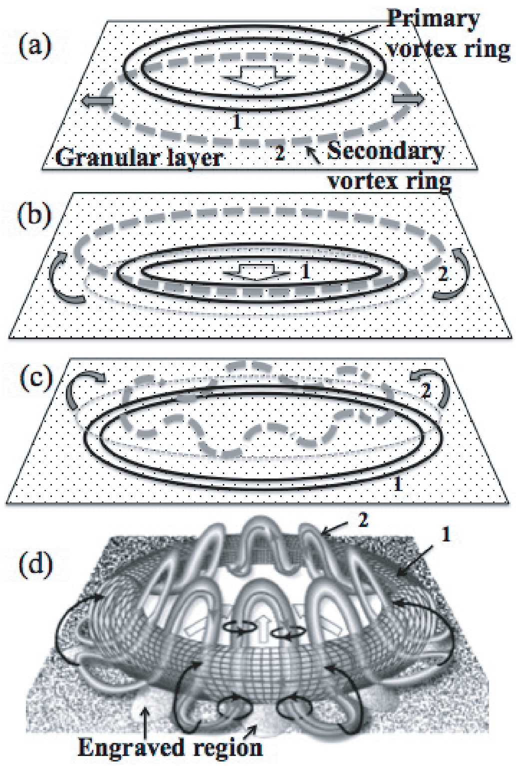

Fig. 6. Schematic pictures showing the creation of "dimple pattern".

accompanied.

An initially circular vortex ring in a viscous fluid can be deformed as a result of instabilities inherent to the vortex ring (Lugt, 1983; Shariff and Leonard, 1992), such as Widnall instability (Widnall and Sullivan, 1973; Widnall *et al.*, 1974; Saffman, 1978), curvature instability (Fukumoto and Hattori, 2005), etc. Wavy motions along the ring develop, and the latter finally decays due to viscous dissipation.

6. Interaction of Two Vortex Rings in Tandem Position

When two vortex rings of equal circulation are placed along a common axis of symmetry, a mutual slip-through motion is observed (see Fig. 4). The vortex ring A placed behind the vortex ring B induces a velocity field which increases the latter radius, and *vice versa*. The translational speed of the ring B decreases, whereas that of A increases, so that the ring A passes through ring B. Once this happens, the role of ring A and ring B alternates, so that the latter passes through the former. This mutual slip-through motion will be repeated indefinitely in an inviscid fluid (Dyson, 1893). In the viscous fluid, however, only a few times slip-through motion is observed because of the dissipation of the vorticity (Oshima *et al.*, 1975; Yamada and Matsui, 1978).

7. Collision of a Vortex Ring with an Obstacle

As mentioned in Section 4, a vortex ring has its own mass, momentum, and energy, so that it shows a self-propelled motion in a fluid similarly to an isolated particle. A vortex ring can blow out the candles at a distant place to which it is directed.

Collision of a vortex ring with a fluid plane (an imaginary boundary keeping a planar form but moving freely in the tangential direction) is almost the same as a head-on collision of two vortex rings of equal and opposite magnitude (Oshima, 1978; Lim and Nickels, 1992). Continuity of the tangential velocity and stresses is satisfied, so that the fluid plane remains undeformed. In the absence of viscosity, the radius of the vortex ring increases indefinitely with the approach to the plane, because of the flow induced by the other (mirror) vortex ring. On the other hand, the collision of a vortex ring with a solid plane is rather complicated (Boldes and Ferreri, 1973; Yamada *et al.*, 1982; Walker *et al.*, 1987; Orlandi and Verzicco, 1993; Chu *et al.*, 1993).

When the vortex ring approaches the solid plane from a distant position, its radius increases similarly to the collision with the fluid plane. Near the solid boundary, however, a shear layer with vorticity opposite to that of the ring is induced owing to the no-slip condition. If the magnitude of the primary vortex ring is large enough, the shear layer near the solid boundary rolls up to form a secondary vortex ring, which interacts with the primary one. Here, the secondary vortex ring moves inward (toward the axis of the ring) around the primary one, whereas the primary one moves outward (away from the axis of the ring to increase its radius) accompanied by a slight departure from the solid plane ("rebound"), so that the entangled motion of the two rings is recognized (similar behavior is illustrated in the upper figure of Fig. 5). For much stronger primary vortex ring, secondary vortex ring, tertiary vortex ring, $\cdots$ will be generated.

Collision of a vortex ring with a granular surface, which has both fluid and solid properties, leaves rather complicated erosion patterns. Depending on the strength of the vortex ring, "hardness" of the granular layer and the distance to the granular plane, several types of erosion patterns are reported (Munro *et al.*, 2009; Bethke and Dalziel, 2012; Masuda *et al.*, 2012; Yoshida *et al.*, 2012; Yoshida and Sano, 2015). A weaker vortex ring, but above a certain critical strength, will create a shallow circular depression on the granular surface, which reflects the shape of the axisymmetric vortex ring. The rim of the crater may be wavy if the vortex ring travels enough distance to develop instabilities described in Section 5. When a vortex ring is strong enough and the distance to the granular layer is shorter, the shape of the ring is kept circular immediately before the impact, and creates a circular secondary vortex ring. These vortices are coupled and erode the surface of the granular layer, so that characteristic surface patterns like radial grooves in the main circular crater or discrete dimples outside the ridge of circular crater are observed. Figure 5 is an example of "dimple pattern", where a nearly circular depression with a central mound is created by the primary vortex ring. The process of dimple formation is illustrated in Fig. 6. The secondary vortex ring moves around the primary vortex ring (Figs. 5(a) and 6(b)), becomes wavy as it is convected to the interior region of the primary vortex ring (Fig. 6(c)), develops to a row of hairpin-like vortices (Fig. 6(d)), which creeps into the space between the primary vortex and the granular boundary. These hairpin-like vortices engrave the granular layer and create discrete smaller depression, called dimples, outside of ridge of the nearly circular crater due to the primary vortex ring (Fig. 5(b)).

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