Analysis of the Motion of the Pop-up Spinner

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The pop-up spinner is a pop-up card. As the completed card opens and closes, its nested frames spin. We analyze the motion of frames of the pop-up spinner, and represent by a nested epitrochoid mechanism.
Key words: Pop-up spinner, Epitrochoid

1. Introduction

The pop-up spinner is a pop-up card (see Fig. 1). Figure 2 shows a template for the construction of pop-up spinner, having cuts, mountain folds, and valley folds lines. As their names imply, a mountain fold is a crease that bumps outward, while a valley fold is a crease that dents inward. Try cutting template and crease as indicated. As the completed card opens and closes, its nested frames spin. The pop-up spinner has the central frame (like a beak) with one slit and other frames (like wings) with two slits. We can see the animation by A. Nishihara in the Nishihara web page [2]. The name of the pop-up spinner was given by A. Nishihara ([2]). It is not well-known when and who designed this first; Due to [3], the pop-up spinner was invented in Japan by an unknown student at Musashino Art University in 1988. Due to the support web page of “Origami no suuri” [3], it seemed to have already existed in 1970’s.

In [3], J. O’Rourke pointed out that the central zig-zag path of mountain and valley creases is a chain with the fixed angle by the construction as in Fig. 3. When the card is closed, this chain is curled up into a spiral configuration. When the card is opened, the chain heads toward the planar staircase configuration which achieves the maximum possible end-to-end distance of the chain, known as the *maxspan*.

In this article, we study the motion of frames of the pop-up spinner when the card is closing. We set x-axis and the origin O in the horizontal line of the opened template as in Fig. 4, y-axis vertical to the opened template. The template closes in the positive direction of y-axis as in Fig. 4. Let n be the number of frames. For $1 \le k < n$, we denote by O_k and A_k the vertices of the edge of the k-th frame (like a wing) from the outside as in Fig. 4. And, we denote by O_{n-1} and A_n the vertices of the edge of the n-th frame (the central frame like a beak) as in Fig. 4. For convenience, we assume that the length of OA_1 is n, the length of O_kA_k for $1 \le k < n$ is 1 and the length of $O_{n-1}A_n$ is 1.

We treat the ideal pop-up spinner. That is, we assume that the angle of the crease in the edge of each frame is equal to

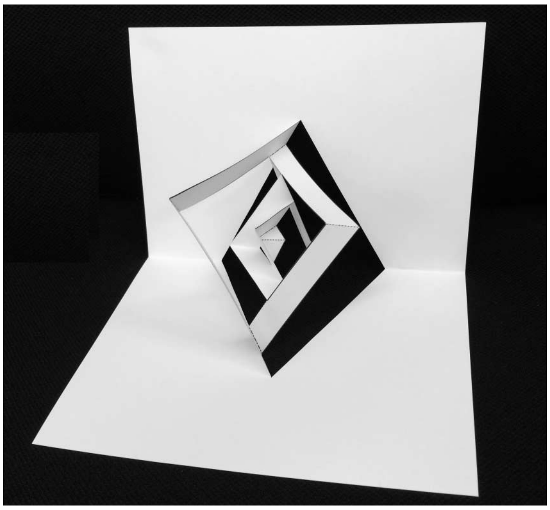

Fig. 1. Pop-up spinner.

the fold angle of the card and that the ideal pop-up spinner is symmetric with respect to the xy plane on the way of the folding. Then note that the vertices O_k and A_k are on xy plane.

We show the following theorem:

THEOREM. *Let n be the number of frames and θ be the fixed angle of the central chain.*

(1) *The angle $A(\theta)$ of the spin of the n-th frame (the central frame like a beak) of the pop-up spinner is given by $A(\theta) = n\theta$.*
(2) *The edge $O_{n-1}A_n$ of the n-th frame of the pop-up spinner has the orbit represented by the following coordinate:*

$$O_{n-1} = \left(\sum_{\ell=1}^{n-1}(-1)^{\ell-1}(n-\ell)\cos(\ell\alpha),\ \sum_{\ell=1}^{n-1}(-1)^{\ell-1}(n-\ell)\sin(\ell\alpha)\right),$$

$$A_n = O_{n-1} + \left((-1)^{n-1}\cos(n\alpha),\ (-1)^{n-1}\sin(n\alpha)\right),$$

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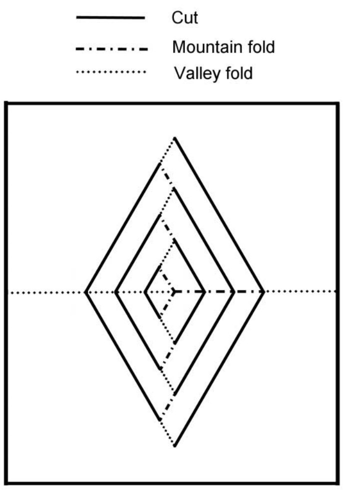

Fig. 2. A template for the construction of the pop-up spinner.

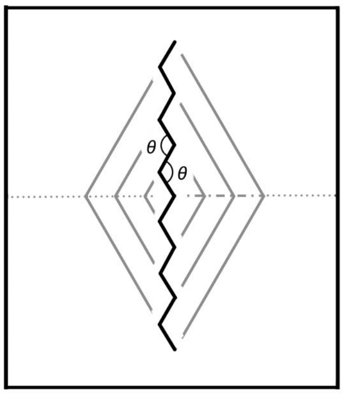

Fig. 3. A central chain with the fixed angle θ.

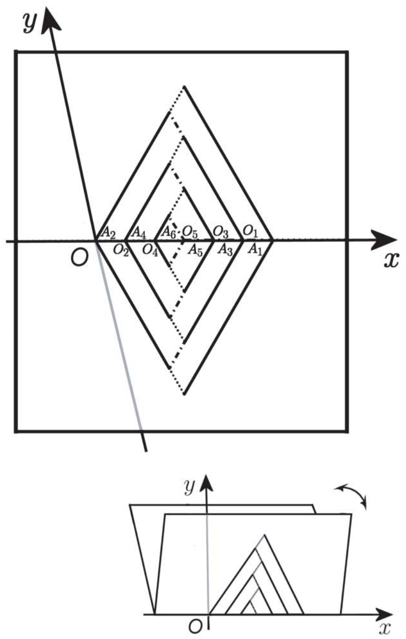

Fig. 4. The x-axis and the y-axis.

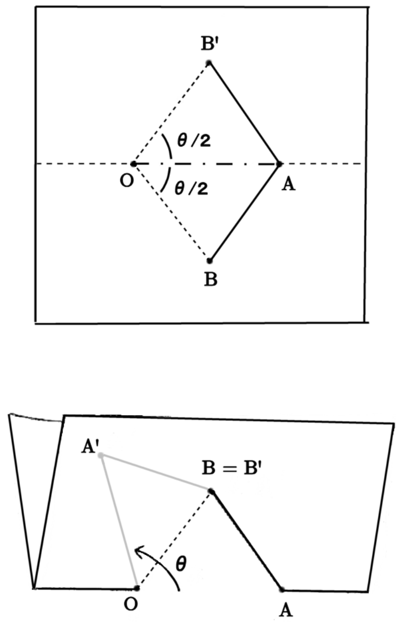

Fig. 5. A basic unit of the spin of the frame.

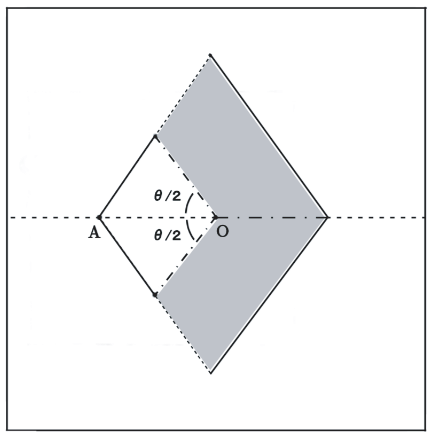

Fig. 6. An outside frame (the gray part) and an inside frame.

where $0 \leq \alpha \leq \theta$ and when the card is opened (resp. closed), $\alpha = 0$ (resp. $\alpha = \theta$).

The third author and his students conjectured the formula of Theorem (1) by observing the pop-up spinner actually made, and tried designing the template for the construction of a pop-up spinner having bi-directional spin ([1]).

An epitrochoid is defined to be a roulette traced by a point attached to a disc of radius r rolling around the outside of a fixed disc of radius R, where the point has a distance d from the center of the exterior disc.

We consider the following mechanism of epitrochoids with a nested structure: A rolling disc of one epitrochoid mechanism is concentrically attached on a fixed disc of another epitrochoid mechanism as in Figs. 11–14. These two

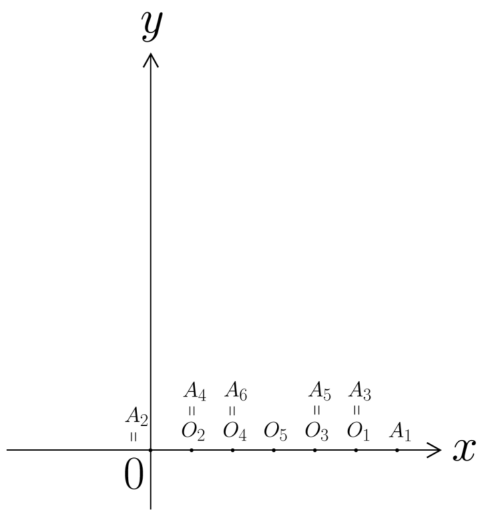

Fig. 7. The configuration of edges of frames in the opened card.

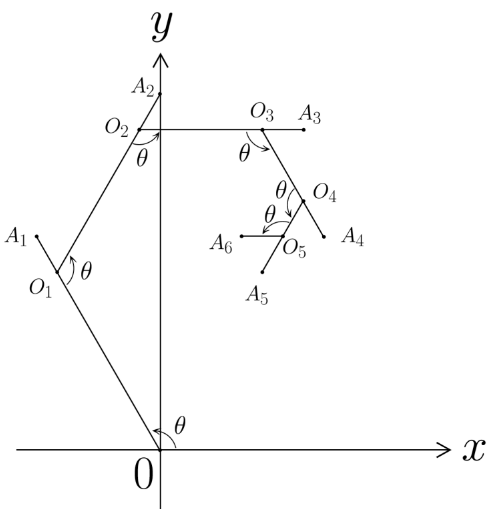

Fig. 8. The configuration of edges of frames in the closed card.

discs rotate together. A mechanism of multiple epitrochoids with a nested structure is obtained by repeating this procedure. We call such a mechanism a nested epitrochoid mechanism.

Then, Theorem (2) implies the following corollary:

COROLLARY. *The motion of frames of the pop-up spinner can be represented by a nested epitrochoid mechanism.*

2. The Proof of Theorem and Corollary

PROOF OF THEOREM. (1) We see that A_1 rotates around O and that A_k rotates around O_{k-1} for $2 \leq k \leq n$. These rotations drive the spin of frames. So, the basic unit of the spin of the frame is a pop-up structure as in Fig. 5. The frames move while spinning. As in Fig. 6, the inside frame works as the basic unit for the outside frame (the gray part). An inside frame is rotated by the spin of an outside frame. An inside frame rotates itself. Therefore, the spin of an inside frame is obtained by adding them. So, we have only to obtain the angle of the spin of the frame in the basic unit

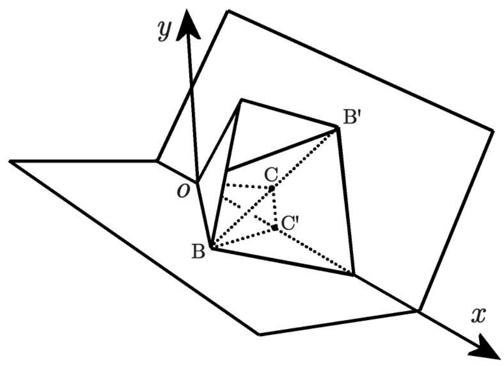

Fig. 9. The basic unit with the fold angle β.

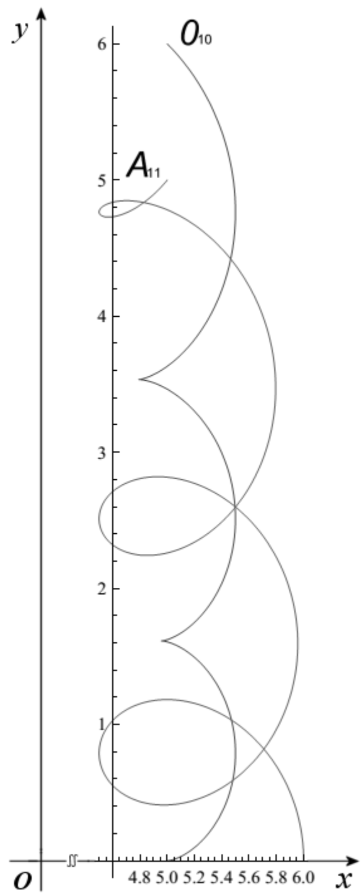

Fig. 10. The orbit of O_{10}, A_{11}.

of Fig. 5. As the card closes, OA rotates to OA' around O in the positive direction as in Fig. 5. Then, we can see that the angle $\angle AOA'$ is θ as in Fig. 5. Hence, the angle $A(\theta)$ of the spin of the n-th frame is given by $A(\theta) = n\theta$ for the pop-up spinner with n number of frames as in Fig. 8. The proof of Theorem (1) is completed.

(2) We can draw the configuration of edges of frames as in Fig. 7 (resp. Fig. 8) when the card is opened (resp. closed). Recall the assumption that the angle of the crease in the edge of each frame is equal to the fold angle of the card. On the way of the folding, we put $\alpha_1 = \angle OO_1O_2$, $\alpha_k = \angle O_{k-1}O_kO_{k+1}$ $(2 \leq k \leq n-2)$ and $\alpha_{n-1} =$

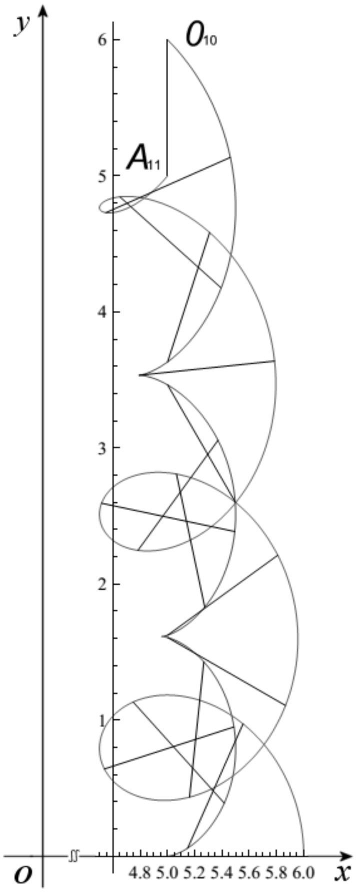

Fig. 11. The orbit of the edge of 11-th frame.

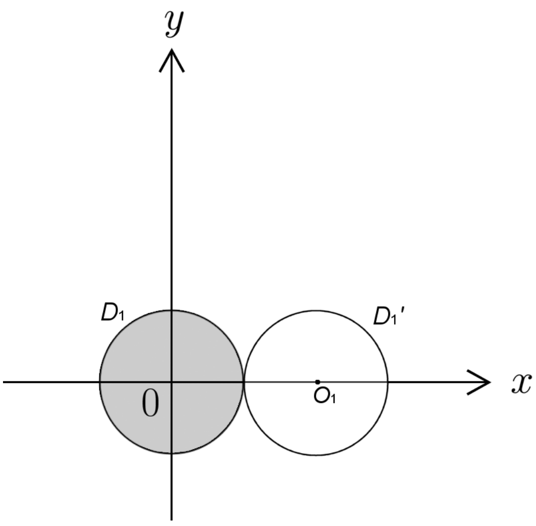

Fig. 12. The fixed disc D_1 and the rolling disc D_1'.

$\angle O_{n-2}O_{n-1}A_n$.

In order to obtain the relation between α_k and the fold angle β of the card, it suffices to observe the basic unit with the fold angle β (Fig. 9). In Fig. 9, C denotes the midpoint between B and B', and C' denotes the orthogonal projection of C to the x-axis. Then, we have three right triangles: $\triangle OCC'$ with $\angle OC'C = \frac{\pi}{2}$ and $\angle COC' =$

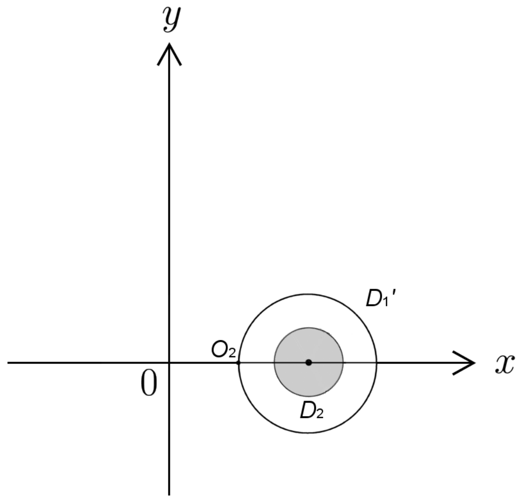

Fig. 13. The attached disc D_2.

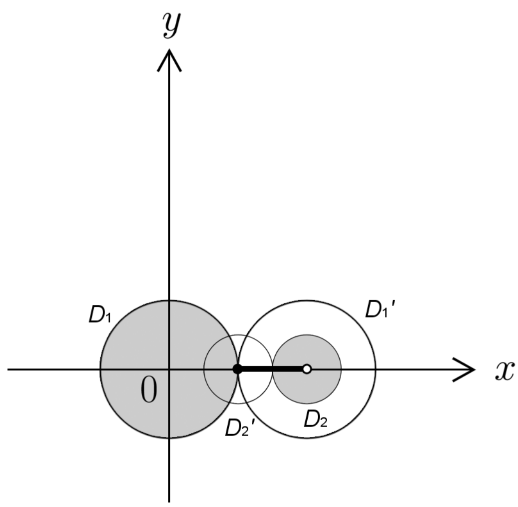

Fig. 14. A nested epitrochoid mechanism ($n = 3$).

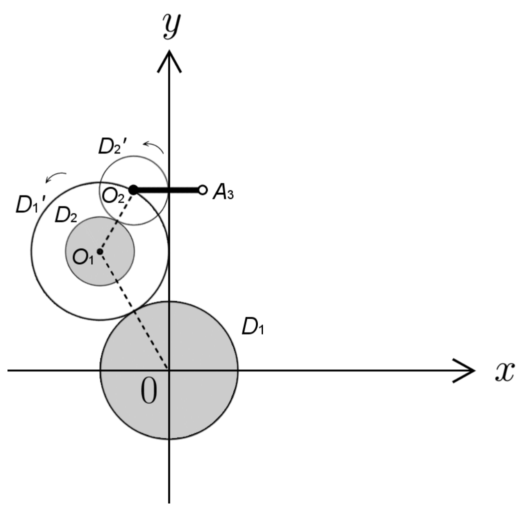

Fig. 15. A nested epitrochoid mechanism ($n = 3$).

$\frac{\alpha_k}{2}$, $\triangle OBC'$ with $\angle OC'B = \frac{\pi}{2}$ and $\angle BOC' = \frac{\theta}{2}$, and $\triangle BCC'$ with $\angle BCC' = \frac{\pi}{2}$ and $\angle BC'C = \frac{\beta}{2}$. By calculating the length of CC' in two ways: using $\triangle OCC'$ and using $\triangle OBC'$ and $\triangle BC'C$, we can find the relation $\tan(\alpha_k/2) = \tan(\theta/2)\cos(\beta/2)$ $(1 \le k \le n-1)$. Thus, we have $\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1}$. Here, we put $\alpha = \alpha_k$.

The coordinates of O_k is given by components of the position vector $\overrightarrow{OO_k}$. We can see that $\overrightarrow{OO_k} = \overrightarrow{OO_{k-1}} + \overrightarrow{O_{k-1}O_k}$ and that $\overrightarrow{O_{k-1}O_k} = ((n-k)\cos(k\alpha - (k-1)\pi), (n-k)\sin(k\alpha - (k-1)\pi)) = \big((-1)^{k-1}(n-k)\cos(k\alpha), (-1)^{k-1}(n-k)\sin(k\alpha)\big)$ and $\overrightarrow{O_kA_k} = \big((-1)^{k-1}\cos(k\alpha), (-1)^{k-1}\sin(k\alpha)\big)$ (cf. Fig. 8 in the case that $\alpha = \theta$). Hence, we obtain the following coordinates of O_k and A_k for $k = 1, 2, \ldots, n-1$:

$$\begin{aligned} O_k &= \left(\sum_{\ell=1}^{k}(-1)^{\ell-1}(n-\ell)\cos(\ell\alpha),\right. \\ &\qquad \left.\sum_{\ell=1}^{k}(-1)^{\ell-1}(n-\ell)\sin(\ell\alpha)\right), \\ A_k &= O_k + \big((-1)^{k-1}\cos(k\alpha), (-1)^{k-1}\sin(k\alpha)\big), \end{aligned}$$

Moreover, we obtain the coordinate $A_n = O_{n-1} + \big((-1)^{n-1}\cos(n\alpha), (-1)^{n-1}\sin(n\alpha)\big)$. The proof of Theorem (2) is completed.

REMARK. (1) By the proof of Theorem (2), the orbit of the edge O_kA_k of the k-th frame can be represented by the above coordinates for $k = 1, 2, \ldots, n-1$.
(2) In fact, we draw the orbit of the edge of 11-th frame by Mathematica as in Figs. 10 and 11 when the number of frames is 11 and $\theta = \pi/2$. In Fig. 11, we add 15 line segments of edges of 11-th frame to Fig. 10. We can verify Theorem (1) by pursuing the rotation of the edge in Fig. 11.

PROOF OF COROLLARY. We demonstrate the way of adding epitrochoid discs with the nested structure by Figs. 12–15 in the case of $n = 3$. Recall the assumption that the angle of the crease in the edge of each frame is equal to the fold angle of the card. First, we fix the disc D_1 with the center O and radius $|OO_1|/2 = (n-1)/2$, where $|OO_1|$ denotes the length of the line segment OO_1 (Fig. 12). And, we set the rolling disc D'_1 with the center O_1 and radius $(n-1)/2$ around D_1 (Fig. 12). Moreover, we attach a concentric disc D_{k+1} with radius $|O_kO_{k+1}|/2 = (n-k-1)/2$ to the disc D'_k for $1 \le k \le n-2$ (Fig. 13). Note that concentric discs D'_k and D_{k+1} rotate together. And, we set a rolling disc D'_{k+1} with the center O_{k+1} and radius $(n-k-1)/2$ around D_{k+1} (Fig. 14). Finally, we attach a bar $O_{n-1}A_n$ with the length 1 to D'_{n-1} (Fig. 14). Thus, we can construct the desired nested epitrochoid mechanism (Fig. 15).

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