Large Deviation Statistics Derived from First Passage Times —Inspired by *Shishi-odoshi*—

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Shishi-odoshi is a traditional device found in Japanese gardens composed of a bamboo tube that when filled with water revolves to empty and makes a clanking sound. It consists of a water-filled bamboo tube which clacks against a stone when emptied, and the sound scares beasts and birds from gardens. For a fluctuating flow rate, intervals between the clacks are distributed. The flow rate per unit time and distribution function of the clack interval can be respectively identified as a velocity of a random walker and first passage time distribution. The rate function of the flow rate per unit time is derived not according to its definition, but by use of the distribution function of a first passage time. Difficulties in applying this novel idea to various actual water flows are disscussed. Experimental verification might be easier for a diffusion system than for water flow.
Key words: *Shishi-odoshi*, Large Deviation, Rate Function, First Passage Time

1. Introduction

As mentioned in a preceding study of one of the authors [1], one of the most remarkable points about deterministic chaos is the duality consisting of irregular dynamics and fractal structure of the attractor in the phase space. Amplifying this relationship between dynamics and geometry, we can characterize an interesting geometry by considering its dynamics. For example, water drops from a dripping faucet take on various irregular shapes, as shown in Fig. 1. Instead of a direct characterization of the irregular shapes of the water drops, we can indirectly characterize those by considering fluctuations in the flow rate of the faucet. Large deviation statistics facilitate this.

Shishi-odoshi consists of a segmented tube, usually of bamboo, pivoted to one side of its balance point. At rest, its heavier end is down and resting against a rock. A trickle of water into the upper end of the tube accumulates and eventually shifts the tube's center of gravity past the pivot, causing the tube to rotate and dump out the water. The heavier end then falls back against the rock, making a sharp sound, and the cycle repeats. This noise is intended to startle any herbivores such as deer or boar which may be grazing on the plants in the garden. Examples are illustrated in [2]. In Japan, its sounds are enjoyed in traditional gardens.

In this paper, we assume that the flow rate or the amount of water pouring into *Shishi-odoshi* per unit time fluctuates. Furthermore, we discuss the relationship of the distribution of intervals between the clacks to large deviations of the flow rate per unit time.

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2. Formalism

Let V be the volume of *Shishi-odoshi*'s water container. The time-dependent flow rate per unit time is denoted as $f(t)$. At $t = t_0$, we start to pour water into *Shishi-odoshi*, and it is filled at $t = t_0 + n$. In this case, the relation $\int_{t_0}^{t_0+n} f(t)\mathrm{d}x = V$ is satisfied, in which n is the interval between the clacks of *Shishi-odoshi*. In the following, an ideal *Shishi-odoshi* is considered, which instantaneously discharges the total amount of water when full. One may regard f, V, and n respectively as the velocity of a random walker starting from the origin, a distant goal, and the first passage time to the goal. Thus, measuring the intervals between the clacks of *Shishi-odoshi*, we can construct a distribution of the first passage time.

The local average z of the flow rate per unit time is given by:

$$z = \frac{\int_{t_0}^{t_0+n} f(t)\mathrm{d}x}{n} = \frac{V}{n}.$$

The first passage times n distribute, as well as the local averages z due to the above relation. The distribution of z depending on n is denoted as $P(n, z)$, from which we can obtain large deviation statistics of the flow rate per unit time. If n is much larger than its average auto-correlation time of $f(t)$, $P(n, z)$ is scaled as:

$$P(n, z) = P(n, \overline{z}) \exp[-n\psi(z)], \qquad (1)$$

in which $P(n, \overline{z})$ is an algebraic factor depending on n and $\psi(z)$ is called the rate function of the flow rate per unit time [3]. Let $\overline{z}$ be the long-time average as z. The rate function is concave up, which satisfies $\psi(z)|_{z=\overline{z}} = \left.\frac{d\psi(z)}{dz}\right|_{z=\overline{z}} = 0.$

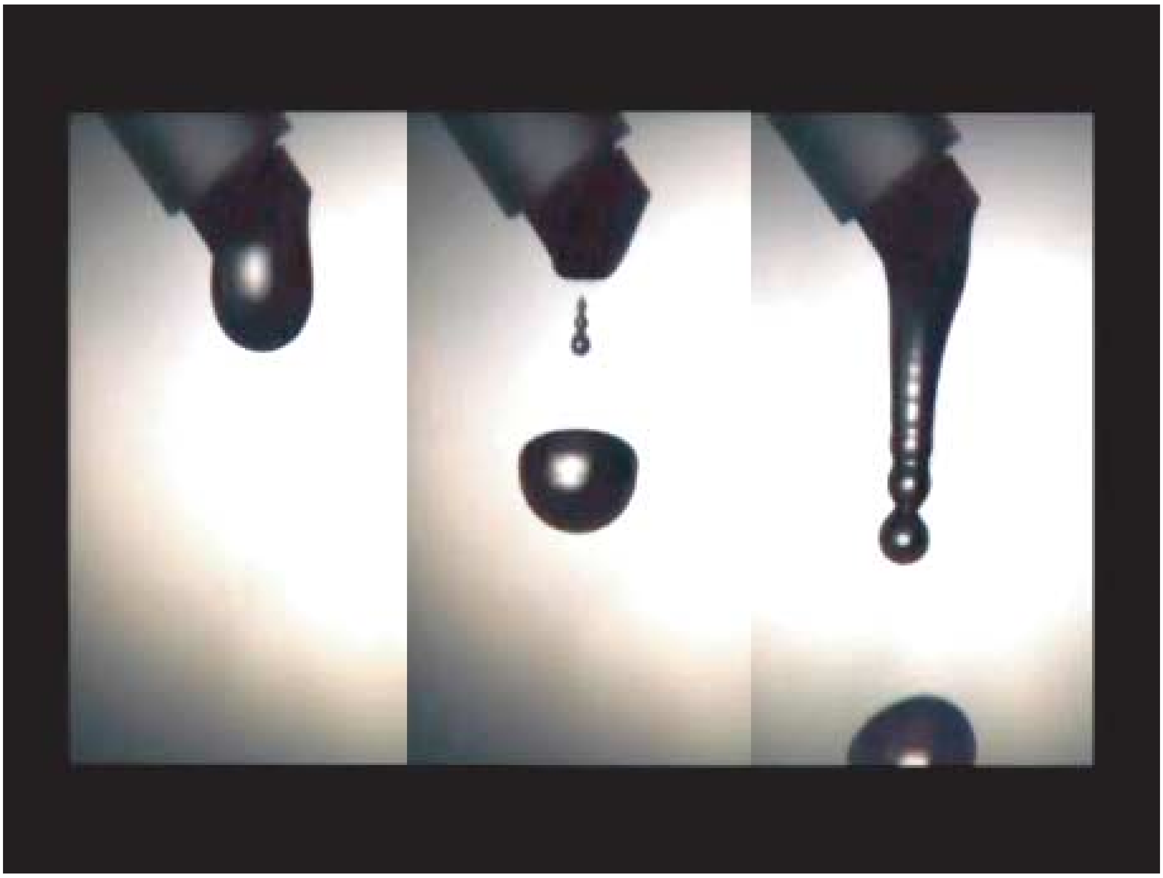

Fig. 1. High-speed photography of waterdrops courtesy of Hiroki Hata (Kagoshima University).

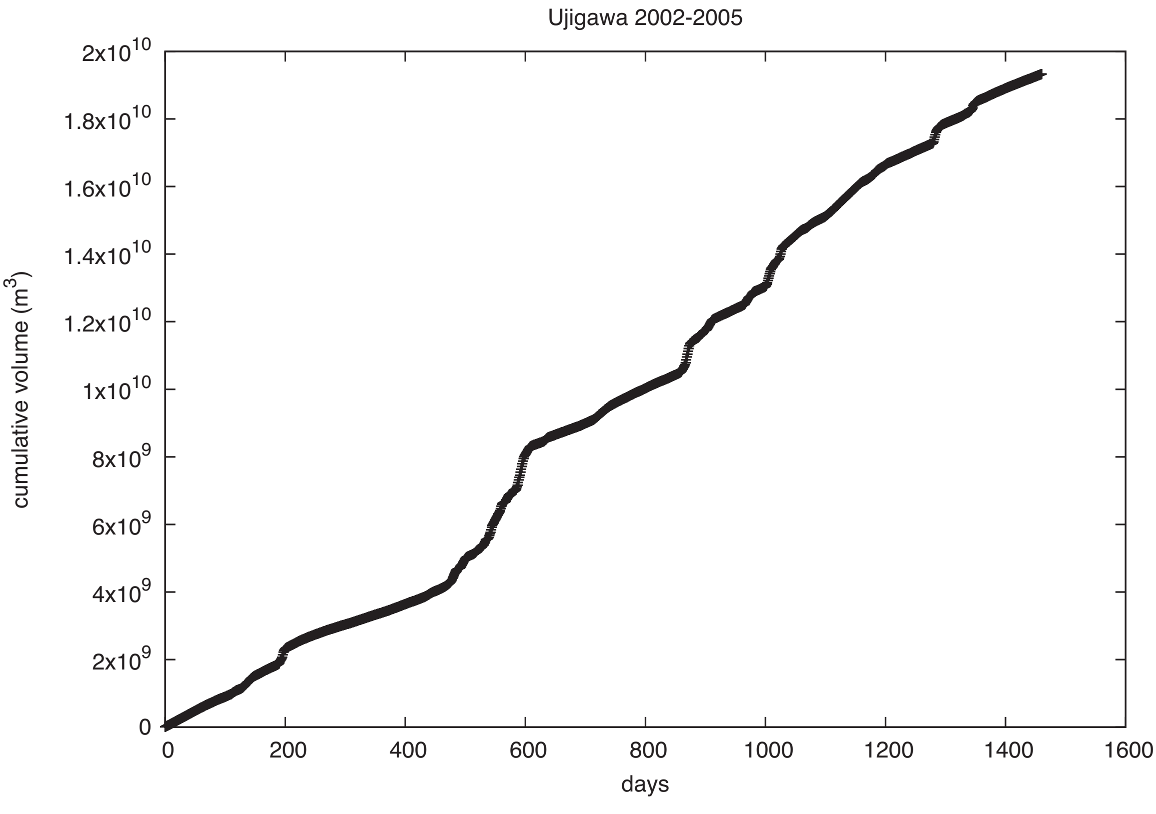

Fig. 2. Cumulative flow volume of Ujigawa (Uji-river) for four years from 2002 to 2005.

As a consequence of the central limit theorem, the rate function is quadratic around $z = \overline{z}$.

From our novel viewpoint inspired by *Shishi-odoshi*, we do not directly observe the local average z or its instantaneous value of the flow rate per unit time, but the first passage time n corresponding to the interval between the clacks in the case of *Shishi-odoshi*. The distribution $P(n, z)$ of z can be regarded as a distribution $Q(V, n)$ of n via the relation $z = V/n$.

Transformation of the variable from z to $V = nz$ satisfies the conservation of probability $P(n, z)dz = Q(V, n)dV$, so that we have:

$$P(n, z) = Q(V, n)\frac{dV}{dz} = nQ(V, n),$$

$$P(n, \overline{z}) = \overline{n}Q(V, \overline{n}),$$

and the rate function $\psi(z)$ is indirectly obtained from:

$$-\frac{1}{n}\log\frac{nQ(V, n)}{\overline{n}Q(V, \overline{n})}$$

plotted against $z = V/n$, where $\overline{n} = V/\overline{z}$ is the long-time average of the first passage time.

3. An Analytical Model and Real Water Flows

A dripping faucet is a real example of chaotic dynamics [4]. We obtained analytical results for coin-tossing-like dynamics [5]. The interval between successive waterdrops from a real faucet is related to the diameter of the waterdrop. A longer interval yields a larger waterdrop, so that the flow rate of the dripping faucet is nearly constant and fluctuates little [4].

Let us consider a long rigid pipe full of water. Such a water flow is subject to Bernoulli's principle, so that its fluctuation is strongly limited in this case.

River flow rates in Japan are recorded on an open website [6] controlled by the Ministry of Land, Infrastructure, Transport and Tourism. For example, the cumulative flow volume of Ujigawa (Uji-river) for four years from 2002 to 2005 can be obtained from the above website, and is shown in Fig. 2. A constant flow rate is given by a straight line. Thus, we expect large deviations of these data. However, river flow rates generally obey a log-normal distribution and do not follow the scaling (1), so that the rate function is not suitable to describe river flow rates with marked intermittency.

In summary, it is difficult to observe fluctuating flow rates in a realistic system, in which the rate function in the form of (1) is well-defined. In the second section, we consider an ideal *Shishi-odoshi*, in which it instantaneously discharges the total amount of water when full. A promising realization of fluctuating flow rates is a realistic *Shishi-odoshi*, which can be regarded as a chaotic oscillator yielding a fluctuating output flow rate even for a constant input flow rate. An example of such a study was performed by Japanese pupils belonging to Shizuoka Prefectural Hamamatsu Kita High School, awarded the *Yamazaki-prize* in 2013. The related information written in Japanese is available on the websites [7].

4. Concluding Remarks

Although we found inspiration in *Shishi-odoshi* and gave an idea of derivation of the rate function from virtual first passage times, it will be easier for a random walk, a Brownian motion, a chaotic diffusion, and those *genuine* first passage times to examine our idea than using water flows and virtual first passage times with respect to the flow rate. The analytical model of waterdrops [5] can also be regarded as a one-directional random walk, where the random walker either stops or jumps in a positive direction.

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