# Branching Structures in Nature and Human Societies

Ryuji Takaki

*Tokyo University of Agriculture and Technology (emeritus professor), 2-23-12 Yuigahama, Kamakura, Kanagawa 248-0014, Japan*
*E-mail address: jr.takaki@iris.ocn.ne.jp*

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Several examples of branching structures in the nature, social structures and the human body are introduced, and results of their analyses are given. It is shown that the Horton's law, which is confirmed for river branching structures, is satisfied also in variety of branching structures in the nature and human societies. It is suggested that these structures are constructed owing to mechanisms similar to that for river structure. As for the branching structures in human body some trials are introduced to construct them numerically by the use of mathematical models. The object of this review article is to show that the analyses of branching forms are interesting topics as the science of forms.
**Key words:** Branching Structure, Tree, Network, River, Horton's Law, Natural Structures, Social Structures

## 1. Introduction

The branching structures are seen everywhere in the nature and also in human societies. It is easy to nominate their examples in physical, biological, geological and social fields. The purpose of this review article is to point out that many of these branching structures have certain properties in common concerning to their geometrical properties, and that it is possible to understand properties of various branching systems in a certain unified way. Through this way it will be expected that we can make further investigation of branching structures from more many research fields both of natural and social sciences.

To begin with let us define two terms concerned to geometrical property of branching structures, i.e. "tree" and "network". The tree indicates a line shape which includes no loop in it (see Fig. 1(a)). In other words, the tree is defined as follows; choose arbitrary two points in a branching shape (A and B, for example, in Fig. 1(a)) and draw a path within the structure connecting these points, where it is forbidden to trace the same part more than one times. Then, the tree structure has only one such path (dotted line in Fig. 1(a)). On the other hand, a network structure allows to choose a pair of two such points, i.e. one can draw more than one paths connecting these points without tracing the same part more than one times (two dotted lines in Fig. 1(b)).

Familiar examples of tree structure are the most of real trees and rivers, the air ducts in the human lung, and graphic expressions of social structures such as schools and governments. Those of network structures are capillary blood vessels, leaf veins, road systems and graphic expressions of human relations and communication networks. Some of these examples appear in this review article.

As a mathematical framework developed since many years we have the method developed by a geologist Horton (1945), who found so-called Horton's law in the tree-type river branching structures, which is introduced briefly in Sec. 2. This method has a great advantage in a sense that the geometrical properties of tree-type structures can be expressed in terms a single parameter. On the other hand, the network-type branching systems have been often treated successfully by scientists from various fields, but they were not based on a simple method similar to that of Horton. The present author has once proposed a method to treat network-type system for leaf veins and road systems, which was included in a monograph by the present author (Takaki, 1978) and not published as a scientific paper. It is introduced in Sec. 4.

Of course, the framework of analysis of branching systems is not limited to the Horton's method, and some remarkable examples are introduced in the following sections. In particular, an application of the topology (one of mathematical fields) is made by a pathological scientist Shimizu (1992) for analysis of 3-dimensional (3D) network-type structures of blood vessels in human liver along with a topological concept called "Betti number".

Here, it is expected that introductions of various method and concepts would give a larger scope of branching systems, which would stimulate further development of studies in future.

## 2. Horton's Law for River Structures and Its Derivations

### 2.1 Horton's law

A river made of branching streams has a shape belonging to the category of tree-type. The tree structure of riv-



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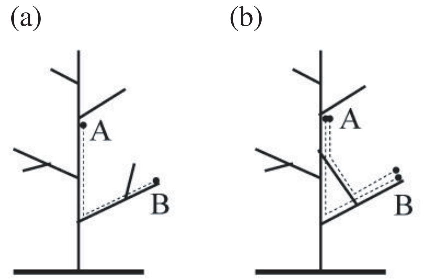


Fig. 1. Definitions of (a) tree and (b) network.

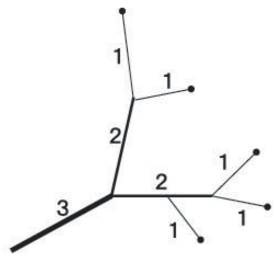


Fig. 2. Definition of the orders of streams. Dots indicate origins. Number of streams with orders 1, 2 and 3 are 5, 2 and 1, respectively.

ers begins its formation with a lot of streams from origins, which merge with each other and form new larger streams. Through these processes the number of streams continues to decrease, and finally they gather to a main stream. It is found by Horton (1945) that if each stream at every stage of merger, including origins and the main stream, is given a number index, called an "order of stream", the numbers of streams of successive orders decrease by a constant ratio, which is called "Horton's law". He found similar laws for other quantities concerned to river structure, as mentioned later. After this paper was published, an improvement was proposed by Strahler (1952) in defining the order of streams so that the Horton's law is established better. In the following the ordering method by Strahler and the Horton's law based on this method are explained.

Figure 2 shows how the orders of streams are given. First, the streams starting from origins (indicated by dots in this figure) have order 1, and new streams produced by merger of two order-1 streams have order 2. If streams of order 1 and 2 merge, the order-2 stream continues to keep its identity. In general, two streams of order $n$ merge to produce an order-($n$+1) stream, and two streams of orders $n$ and $n + m$ ($m > 0$) merge to keep the identity of the order-($n$+$m$) stream. Next, count the number of streams of each order, and plot these data in a coordinate system with abscissa and longitudinal (logarithmic scaling) for the order and the number of streams, respectively. As an example the Amazon is chosen (Fig. 3(a)), and the data obtained by the present author is shown in Fig. 3(b). The four data points are arranged nearly on one line, and the numbers of streams decrease by a ratio 3.2 as the order $n$ increases.

This ratio of decrease of the stream numbers is called a "bifurcation ratio". In many rivers in Japan the branching ratios have values between 4 and 4.5. Horton (1945) found four laws for streams of order $n$ including that for the stream numbers mentioned above, which are listed below.

average number of streams:

$$N(n) = R_b^{\,n_{\max} - n} \quad (n_{\max} \text{ is the order of the main stream}),$$

average length of streams:

$$L(n) = L(1) R_L^{\,n-1},$$

average inclination of streams:

$$S(n) = S(1) R_S^{\,-(n-1)},$$

average area of drainages:

$$A(n) = A(1) R_A^{\,n-1}, \tag{1}$$

where $R_b$, etc., are constant ratios for respective quantities.

### 2.2 Theoretical prediction of Horton's law

A theoretical prediction of the Horton's laws is made by Tokunaga (1966, 1984). He introduced average numbers ${}_\kappa\varepsilon_\lambda$ of streams of order $\lambda$, which meat streams of order $\kappa$, and assumed the following two kinds of relations based on an assumption that tree shapes of river have a certain kind of self-similarity:

$$\begin{aligned} {}_m\varepsilon_{m-1} &= \cdots = {}_2\varepsilon_1 = \varepsilon_1 = \text{constant}, \\ {}_m\varepsilon_{m-2} &= \cdots = {}_3\varepsilon_1 = \varepsilon_2 = \text{constant}, \ \text{etc.}, \end{aligned} \tag{2}$$

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_3}{\varepsilon_2} = \cdots = K = \text{constant}. \tag{3}$$

Then, after some manipulations the ratios in Eq. (1) are derived as

$$\begin{aligned} R_b &= 2 + \varepsilon_1 + K + \left( (2 + \varepsilon_1 + K)^2 - 8K \right)^{1/2}, \\ R_A &= R_b, \ R_L = R_A^{\,1/2}. \end{aligned} \tag{4}$$

These results means that the four laws in Eq. (1) are related each other.

### 2.3 Mechanism of river structure formation

As for the mechanism to produce tree-type structure of rivers, a numerical simulation was made to derive the Horton's law for stream numbers by Kayane (1973). In the simulation the following three rules were set up; (1) the streams change directions randomly, (2) the streams avoid to make a loop, (3) when two streams meet, they merge definitely. The process of simulation is as follows:

1. Prepare a section paper with square lattice.

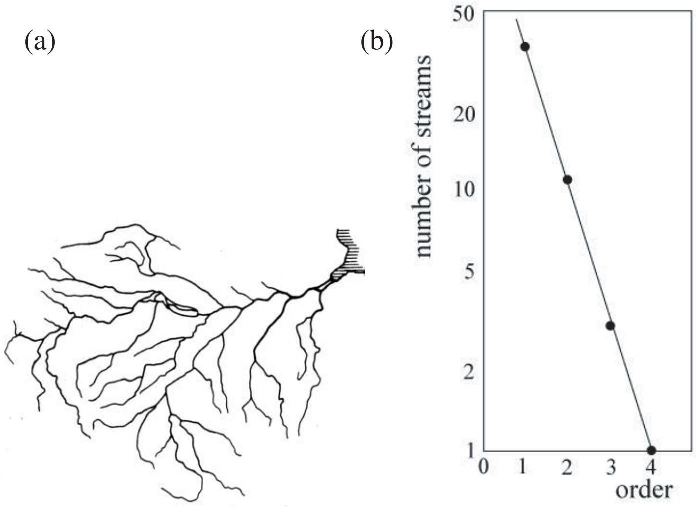


Fig. 3. Examination of the Horton's law for Amazon (reproduced from Takaki (1978)).

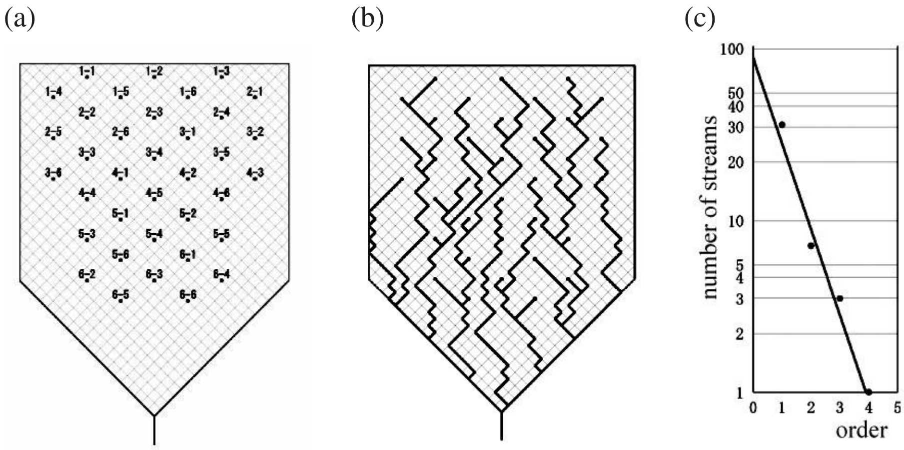


Fig. 4. Simulation of tree shape of river. (a) Distribution of origins, (b) a result of simulation, where stream are directed downwards, (c) confirmation of Horton's law with the bifurcation ratio 3.1 (simulation by students, T. Hashimoto *et al*. (2005, not published)).

2. Choose a point of origin randomly in the lattice.

3. Draw a zigzag stream line along the lattice edges while changing its direction randomly.

4. When the line comes to the periphery of the section paper, i.e. the sea coast, stop drawing.

5. Choose the next origin randomly and draw a zigzag stream line.

6. When the line meets another line, let these lines merge and go to the process 5.

7. When all lattice points are occupied by lines, the simulation is complete.

Then, the inside of the section paper is divided into several areas with one tree shape. It was found that the Horton's law for stream numbers is satisfied for each of these trees. It means that the Horton's law for stream numbers is related to the properties of water flows, which are stated above as three assumptions (1)~(3).

The present author tried to follow this method of simulation with students (T. Hashimoto and others) of Kobe Design University, where they had a course with title "Introduction to the theory of design" for several years. In the simulation the area to draw lines has a pentagonal shape as shown in Fig. 4(a) and is inclined so that the streams have a tendency to go downwards. The 36 points in the area are origins which are chosen randomly by throwing a die. The direction of stream is confined either left-downward or right-downward. On the side edges of the test area the stream should go oblique or go inside. Other rules are the same as given above. After all origins are chosen, the simulation is complete and one tree shape is produced, as shown in Fig. 4(b). The relation between the orders and the stream numbers shows the Horton's first law (Fig. 4(c)).

The reason why this simulation was recommended to students of design is that the present author wanted to let them find a certain kind of beauty in the outcome of natural processes with an exact algorithm, which could not be seen in freely drawn patterns.

In this section the river is assumed to have a tree shape, but natural rivers sometimes includes a part with network

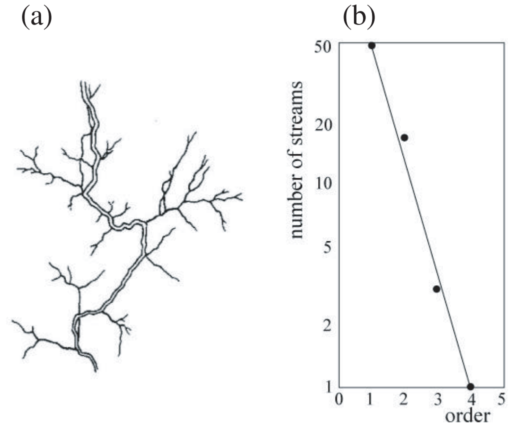


Fig. 5. (a) Lightening of thunder and (b) Horton's law with bifurcation ratio 3.7 ((a) sketch by R. Takaki from Strache (1973)).

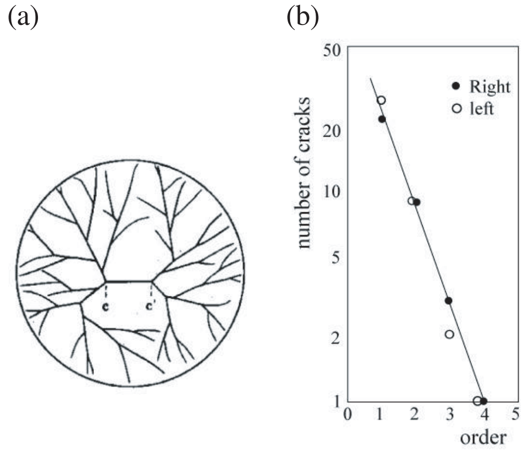


Fig. 6. (a) Cracks in glass plate composed of two tree patterns, (b) Horton's law with bifurcation ratio 2.9 ((a) sketch by R. Takaki from an essay by Hyodo (1974)).

bed to form meandering and braiding is discussed by Parker (1976).

## 3. Horton's Law in Various Natural Systems

The Horton's law has been examined until recently only for river branching structures. In a monograph by the present author (Takaki, 1978) it is shown that this law is satisfied also for other branching structures in the nature. Some of them are shown in this section.

Figure 5 shows a result for a lightening pattern of thunder, where (a) is a 2D projection of a 3D tree-type shape. Figure 6(a) shows cracks in a grass plate whose center was heated suddenly, which are composed of two tree patterns beginning from points C and C′ in the figure. Both of these trees satisfy the Horton's law, as are shown in Figs. 5(b) and 6(b).

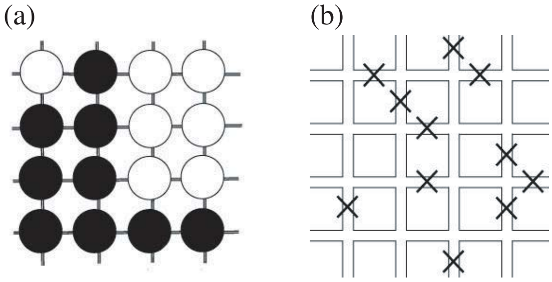


Fig. 7. Two types of percolation. (a) Site percolation, where black circles show lattice points where occupancies of material are allowed, (b) bond percolation, where bonds with × marks are closed.

A physical process called "percolation" often shows branching structures. It is a transport of material through a filter or a porous media. Suppose that the 2D or 3D space is filled with regular arrangement of uniform elements which are connected with neighbors for transportation of material with certain probability. The major problem in the percolation theory is to predict the extent of spreading of material when the probability of connections is given, i.e. to obtain a probability (called a *percolation probability*) for elements to form a network with finite or infinite size for a given value of probability of one site (site percolation) or bond (bond percolation) to allow flow of material (see Fig. 7). An example of percolation phenomena is the spreading of epidemic or secret information. It is known that the size of infected group becomes infinite, if a patient makes more than 4.5 persons infected on the average.

The theory of percolation was introduced by Broadbent and Hammersley (1957) and Hammersley (1957) by a statistical method. A compact review of percolation theory is given by Hori (1972), and an easy monograph by Odagaki (2000) is recommended for general people.

Here, precise description of the percolation theory is avoided and some typical results are given. A theoretical result of 2D site percolation with finite square arrangement is given in Fig. 8, where (a) and (b) show examples of percolation patterns with probabilities below and above the critical value of the percolation probability ($p_c$ = 0.5927) and (c) shows a probability of connection between upper and lower edges of the square region. Note that the pattern tends to form a tree-type or a network type structure below or above the critical condition, respectively. In the case of infinite size the probability is either 0 or 1. In the 3D case with square arrangement of sites a similar analysis is made and the critical value is $p_c = 0.31$.

Of course the values of critical condition depends on the type of arrangement of sites (square one, hexagonal one, etc.), and critical values for various arrangements are given in Odagaki (2000).

There is an interesting case in the percolation theory, which treats a percolation in the *Bethe lattice*, a tree type lattice introduced by the nuclear physicist Bethe (1935), as shown in Fig. 9. It is produced through dichotomous branching at each end, as shown in Fig. 9(a). If it grows infinitely, it attains a pattern without a particular central point, i.e. by choosing any point as a center the whole pattern can be deformed to have a form similar to that in Fig. 9(a). For both site and bond percolations of the Bethe lattice the critical probability for percolation is 0.5. The present author made recently a trial to produce bond per-

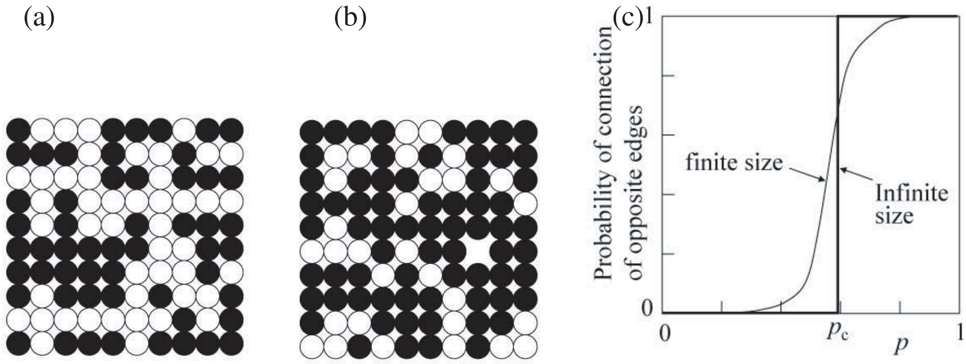


Fig. 8. Examples of 2D site percolation in a 10 × 10 system, where white circles indicate absence of element. Probabilities of site elements are (a) 0.55 and (b) 0.65. (c) Results of theoretical analysis for system with infinite size ($p_c$ = 0.5927) and numerical simulation for system with finite size (19 × 19) ((c) sketch by R. Takaki from Odagaki (2000)).

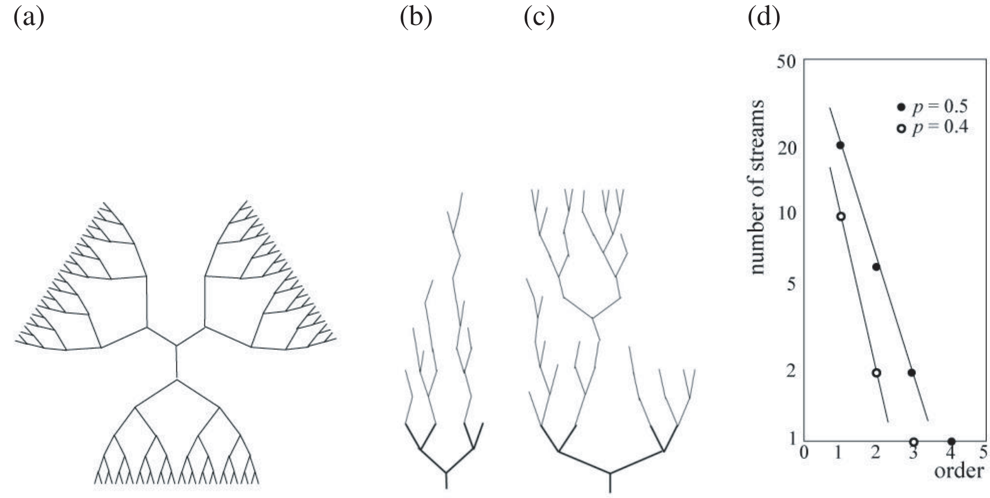


Fig. 9. (a) Bethe lattice. (b), (c) Bond percolations in a 1/3 part of Bethe lattice with $p$ = 0.4 and 0.5, respectively. (d) The numbers of streams for obtained percolation patterns.

colation in the Bethe lattice for probabilities 0.4 and 0.5, whose results are shown in Figs. 9(b) and (c), respectively, where the lattice was cut off at the highest positions in Fig. 9(c).

The Bethe lattice seems to play a role to bridge between the river formation and the percolation, and the present author applied the Horton's method to these percolation patterns. Its result is shown in Fig. 9(d), where the lowest data were neglected in estimating the bifurcation ratios because a special treatment was made for bonds at the central point (corresponding to the main stream of river). Nevertheless, the percolation pattern, especially that in (c) seems to follow the Horton's law.

## 4. Horton's Law in Biological Systems

### 4.1 Leaf veins

A typical example of 2D branching structures in plants is the leaf vein. Figure 10 shows results of trials to examine the Horton's law in leaf veins. The fern shown in (a) is one leaf, which is divided into many parts containing one vein and forms a tree shape. The other two leaves have one main vein at the center (order 4) and more than 10 branches coming out of them (order 3), while smaller veins form network structures. Therefore, a different method is necessary to estimate orders for these smaller veins. These veins can be classified into two groups, one (thicker, order 2) connecting neighboring order-3 veins to form quadrilateral regions with number $N(2)$, while the other (thinner, order 1) forming fine network patches with number $N(1)$ within each quadrilateral region. Figure 10(b) is obtained by counting numbers of elements of these orders.

It should be remarked here that the Horton's law is confirmed for the three plants with the same value of bifurcation ratio, in spite of the fact that they are chosen from quiet different classification groups. Furthermore, biological systems satisfy a similar law to that in non-biological cases. Although reasons of these facts are not clear, it will be interesting to suggest that the structures treated in this review article have a function to distribute materials (mainly liquids) into wide regions, where branching forms assuring the least energy consumption would be chosen.

### 4.2 Branching rules of blood vessels

Geometrical rule at branching points of blood vessels

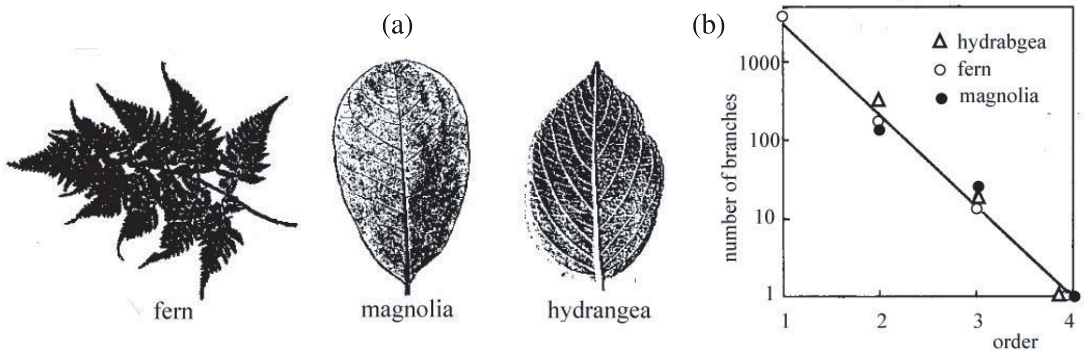


Fig. 10. Examinations of Horton's law for three leaf veins. (a) Single leaves of fern, magnolia and hydrangea. The fern leaf is divided into many parts and forms a tree shape, while the other two leaves include network structures. (b) Horton's law with a common bifurcation ratio of about 14 (reproduced from Takaki (1978)).

has been investigated since many years. In the following a short review is given based on the article by Kamiya and Togawa (1973). As for the branching of blood vessels, Thompson (1917) gave some qualitative rules as follows, which are seen in most blood vessels:

(1) When a mother vessel branches to two daughter ones, cross-sectional area of the mother is smaller than the sum of the daughters' areas.

(2) When the cross-sectional areas of daughters are equal, their angles of deviations from the direction of the mother are equal.

(3) When one daughter has smaller cross-sectional area than the other, it has a larger deviation angle than the other daughter.

More precise study of the branching rule was made by Murray (1926a, b). He considered two kinds of cost for blood vessels to maintain its roles; one is the power to transport the blood and the other is a metabolic cost to refresh the blood. If these costs are considered for a single duct with radius $r$ and length $l$ filled with blood, the former is a product of flow rate $f$ and the pressure difference $\Delta p$, while the latter is proportional to the volume $V$ of the duct, hence the cost function $CF$ is expressed as

$$CF = f\Delta p + kV, \text{ where } \Delta p = \frac{8\mu}{\pi}\frac{fl}{r^4}, \quad V = \pi r^2 l, \tag{5}$$

and $\mu$ and $k$ are the blood viscosity and an unknown constant, respectively. The formula for the pressure difference is derived by minimizing $CF$, i.e. from $\partial CF/\partial r = 0$, and we have the following expression for the flow rate:

$$f = \sqrt{\frac{\pi^2 k}{16\mu}} \cdot r^3 \propto r^3. \tag{6}$$

If we consider a branching of vessels as shown in Fig. 11 with radii and flow rates of mother and daughters $r_0$, $r_1$, $r_2$ and $f_0, f_1, f_2$, respectively, we have $f_0 = f_1 + f_2$, hence we have

$$r_0^3 = r_1^3 + r_2^3. \tag{7}$$

From this result we can derive $r_0^2 = r_1^2 \cdot r_1/r_0 + r_2^2 \cdot r_2/r_0 < r_1^2 + r_2^2$, which agrees with the assertion (1) of D. Thompson. An experimental value 2.7 of the index in Eq. (7) was obtained (instead of 3) by Suwa and Takahashi (1971). A comment is given here on the index 3 in Eq. (7). If it is 2, the sum of cross sectional areas of daughters after branching is equal to that before branching. However, owing to the viscosity of blood the flows in narrow daughters the blood receives strong resistance, which results in loosing much energy. Therefore, the value 3 of the index assures an effective flow distribution.

Furthermore, by minimizing the $CF$ for the combination of three ducts as shown in Fig. 11 through varying the coordinates $(x, y)$ of the connection point B, Murray (1926a, b) obtained the relations between $\theta_1$, $\theta_2$ and $r_0$, $r_1$, $r_2$, as follows:

$$\cos\theta_1 = \frac{r_0^4 + r_1^4 - r_2^4}{2r_0^2 r_1^2}, \quad \cos\theta_2 = \frac{r_0^4 + r_2^4 - r_1^4}{2r_0^2 r_2^2},$$
$$\cos(\theta_1 + \theta_2) = \frac{r_0^4 - r_1^4 - r_2^4}{2r_1^2 r_2^2}. \tag{8}$$

These results agree with the assertions (2) and (3) of Thompson (1917).

Kamiya and Togawa (1972, 1973) proposed another theory for the condition at the branching for the following reason. Blood vessels are connected at the end to tissues through the capillary system, where the blood pressure must balance with the pressure of tissue or osmotic pressure there. On the other hand, the pressure at the beginning, i.e. the heart, is also fixed. In addition, the flow rate of blood must be adjusted to the needs from tissues. Therefore, in order to consider the optimal design of blood vessels, it is not meaningful to include the transportation cost of blood. Hence, they considered only the volume of blood. They chose three quantities $x$, $y$ and $r_0$ in Fig. 11 as variables for optimization. Reason of the choice of $r_0$ for optimization is not mentioned in their paper. According to the guess of the present author, they fixed the sizes of narrow vessels at tissues and tried to construct thicker blood vessels.

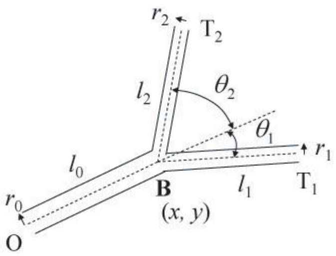


Fig. 11. Parameters at branching of blood vessel.

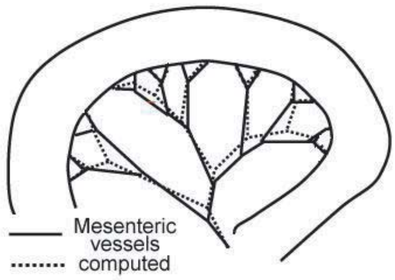


Fig. 12. Mesenteric vessels in a dog, where the starting points on the intestine and the end point are fixed (sketch by R. Takaki from Kamiya and Togawa (1972)).

Then, instead of Eq. (7), they give the following results:

$$\frac{r_0^{\,6}}{f_0} = \frac{r_1^{\,6}}{f_1} + \frac{r_2^{\,6}}{f_2}, \tag{9}$$

and conditions for branching angles, which are the same as Eq. (8) but have different expressions. Equation (9) includes the condition (7) as a special case if $f$ is proportional to $r^3$, hence Eq. (9) is more general. They applied these results to reconstruction of blood vessels in the mesentery (film spanned among parts of intestine) of dog, and obtains a good agreement (see Fig. 12).

### 4.3 Reconstruction of human airway system

The mathematical method for blood vessels can be applied also to the branching system of airway in the human lung. In the following the past studies of airway system and a new trial by those including the present author is introduced based on review articles by Kitaoka and Takaki (1998) and Takaki and Kitaoka (1999). The first mathematical study of airway was made by Weibel (1963), where a branching to equal size ducts is assumed. Extensions of this study to include cases with unequal ducts was given by Horsfield *et al*. (1971). In both of these studies 3D reconstruction is not made. The first 3D reconstruction of airway system is made by Parker *et al*. (1997), who applied a fractal pattern, the Koch curve of tree type, to reconstruct airway tree. But, in this model a branch is assumed to be divided into two equal size branches, and it is not realistic in this sense. The first trial to reconstruct a realistic 3D airway was made by Kitaoka *et al*. (1999).

For creating an algorithm to reconstruct airway, we pose the following four prepositions.

(1) The shape of space for reconstruction is given according to the real shape of lung, including the main trunk and excluding the space of the heart.

(2) The role of lung, i.e. to distribute the air, is considered so that the end points must be distributed uniformly in the given space.

(3) The properties of fluid are considered, such as the minimum energy consumption for transportation.

(4) The timing of change from the convective transport to the diffusion transport within the alveolus is considered.

As for the branching angles, Eqs. (7) and (8) are used,

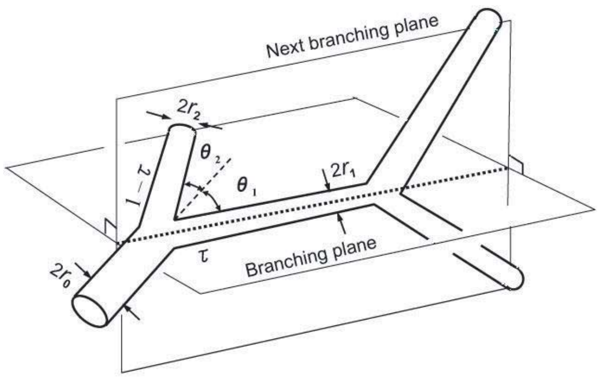


Fig. 13. Geometry at the successive two branchings (sketch by R. Takaki from Takaki and Kitaoka (1999)).

but the index 3 is replaced by $n$ in Eq. (7) and transformed to a different expression including the flow-dividing ratio $\tau$, where the low rate is divided to two branches with a ratio $\tau : (1 - \tau)$, respectively, as is indicated in Fig. 13.

The following rules are set up based on the prepositions stated above (see Fig. 13), where some notes on these rules are given in Appendix:

1: Branching is always limited to dichotomous one.

2: The mother and daughter branches lie on the same plane (called branching plane).

3: The flow rate is conserved after branching.

4: The 3D region governed by a mother branch is divided into two daughter regions by a plane (called space-dividing plane). This plane includes the mother branch, and is perpendicular to the branching plane and extends to the border of the mother region.

5: Flow dividing ratio $\tau$ is equal to the ratio of volumes of daughter regions.

6: Radii and directions of daughters are determined according to the flow-dividing ratio.

7: The lengths of daughters are three times as long as their respective diameters.

8: After a branching the daughters become mothers and the space-dividing plane becomes a new branching planes.

9: The branching process stops when the flow rate becomes smaller than a certain threshold, or when the branch extends out of its own region.

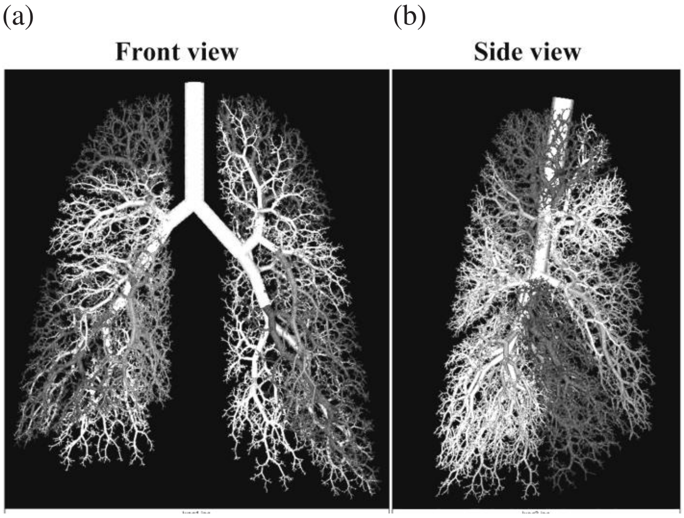


Fig. 14. Reconstructed lung airway, (a) front view and (b) side view. The right-left asymmetry of the front view comes from the effects of the trachea (white ducts) and the heart having asymmetric shapes. The grey levels of ducts indicate different lobes of lung (reproduced from Takaki and Kitaoka (1999); note that Kitaoka *et al*. (1999) contains similar results with slightly different conditions).

Reconstruction of airway system was made by the use of personal computer (Gateway 2000, EV700) with a software C++. An example of reconstructed lung airway is shown in Fig. 14.

Some applications of this reconstruction method are proposed. One is a diagnosis of lung cancer. It is difficult to judge whether a suspicious part is a cancer or a simple inflammation. In order to find a hint for diagnosis, Kitaoka *et al*. (1999) made a computer simulation to deform a part of airway, where either a cancer or an inflammation has appeared. Both cancer and inflammation attract nearby airway branches towards themselves, but in different ways; a cancer strongly attracts only those branches quite close to it, while an inflammation weakly attracts more number of branches. This difference of deformations would be detected by CT.

The end points in the airway are connected to another tissue made of many tiny sacks, called pulmonary acinus, which fill the space out of the airway. Morphology of this tissue has been misunderstood since many years, so that the pulmonary acini are arranged like a bunch of grapes. Recently, Kitaoka *et al*. (2000) began to claim that the pulmonary acini are not like the bunch of grapes, where many vacant spaces are left out of the grapes, but like a 3D labyrinth made of branching paths, which fills the space completely and whose exit is connected to an end point of airway. At first, this claim met strong objections among medical scientists, but it is now getting more supports.

We have reviewed the studies of branching systems in organs, which are aimed at how to reconstruct these systems. On the other hand, there is another problem of how a branching system grows in embryo of real animal. Here, an interesting paper by Honda and Yoshizato (1997) is cited here. Their observation revealed that a branching system was formed from an initial fine and uniform network through a process of selection, i.e. some elements of the network became thick while others shrank. This process is confirmed by computer simulation.

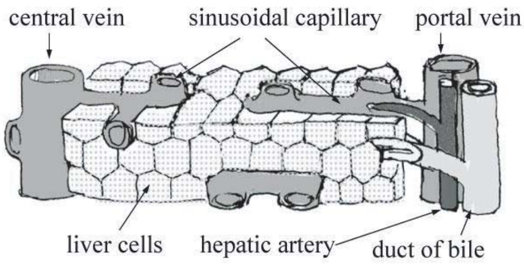


Fig. 15. Structure of a part of lobule for explanation of sinusoidal capillaries and cells (a rough sketch by R. Takaki from a textbook of histology (Fujita and Fujita, 1976)).

### 4.4 Reconstruction of blood vessels in liver

The liver is made of a lot of units called a lobule, which has also a complicated structure, so that it is connected to three kinds of ducts; first, the hepatic artery to supply energy and necessary material, secondly the bile capillaries to carry a liquid called bile and thirdly the portal vein to carry the blood from various parts of body in order to make the liver to detoxify it.

Figure 15 is a rough sketch of a part of lobule. The blood coming into the lobule is collected at the central vein (located at the center of the lobule) and is carried out. The hepatic artery and the portal vein are connected to the central vein through capillaries called “sinusoidal capillaries”. Therefore, the most part of lobule is occupied by liver cells and a network of sinusoidal capillaries. The blood coming to the central vein is carried to the hepatic vein and goes out of the liver.

There are two mathematical problems concerned to the structures of blood vessels in the liver; one is to simulate the branching systems of portal veins and hepatic veins, the other is to simulate the network of sinusoidal capillaries. The present author made some works on these prob-

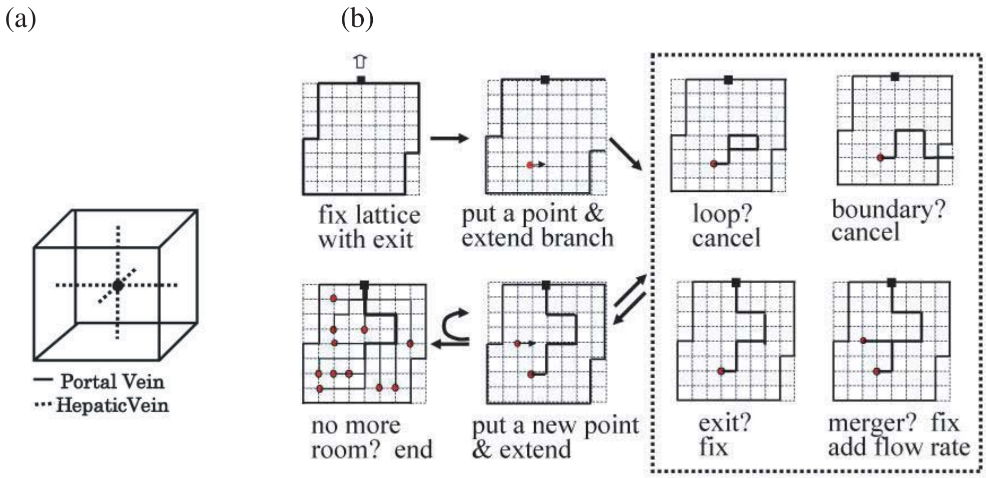


Fig. 16. (a) Lattice structure of simulation of liver veins, (b) algorithm for the vein branching systems (deformed from that in Takaki *et al*. (2003)).

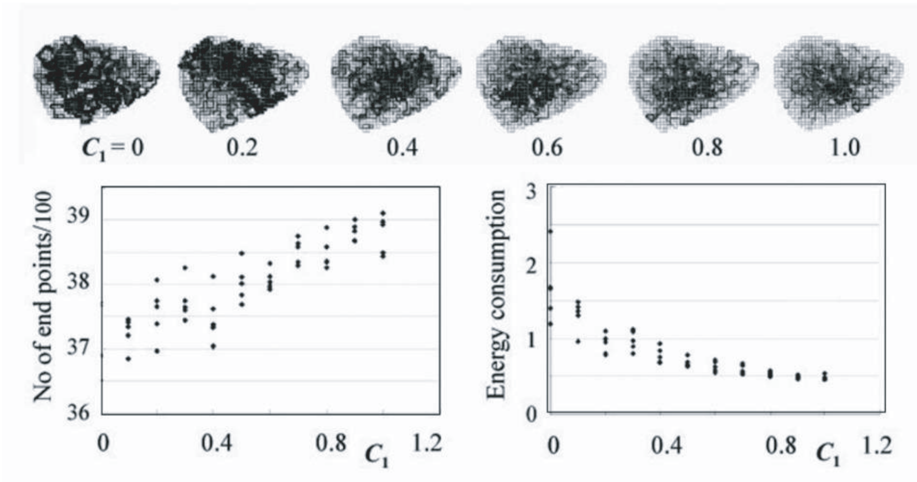


Fig. 17. Results of simulation of veins. Upper: reconstructed vein systems for some values of direction parameter $C_1$, which is larger for larger tendency to extend to the exit (or the entrance) (reproduced from Takaki *et al*. (2003), originally from Nishikawa (2002)).

lems with his collaborators (Takaki *et al*., 2003; Takaki, 2005). They are introduced here briefly. The object of simulation is to examine how the shapes of network of blood vessels are determined.

In the first simulation of portal and hepatic veins the outer boundary of liver and the positions of entrance of artery and the exit of vein are given. The boundary of a liver is given as a realistic but simplified shape within a $30 \times 30 \times 30$ cubic lattice, as shown in Fig. 16(a), where the portal and hepatic veins constitute separated lattices. Algorithm to construct vein networks is shown in Fig. 16(b). The construction of veins (both portal and hepatic) is made by extending a path from a randomly chosen point to the exit point. The basic idea of the simulation is that the each lobule should touch to both kinds of veins, while the number of ducts constituting the network should be reduced in order to economize energy.

For this purpose extension of hepatic vein from the present point is made by choosing one of six neighbors with probabilities, $Pr(\pm x) = 1 \pm C_1 e_x$, $Pr(\pm y) = 1 \pm C_1 e_y$, $Pr(\pm z) = 1 \pm C_1 e_z$, where $(e_x, e_y, e_z)$ is a unit vector from the present point to the exit of the liver and $C_1$ is a positive parameter indicating a tendency to extend towards the exit. For extension of the portal vein, the unit vector is directed from the present point to the entrance of the liver. For constructed veins the total number of end points and the total consumption of energy due to viscous resistance are computed. The results of simulation are shown in Fig. 17. As is seen from this figure, larger value of $C_1$ (larger tendency to the direction of exit or entrance) results in larger number of end points and smaller energy consumption. These results show that the choice of direction at each point of the path extension produces better results both in numbers of end points and the energy consumptions.

Next, results of simulation to construct sinusoidal capillary system within a lobule is introduced. The space of simulation is limited to a cube within a lobule, whose six edges out of twelve constitute either portal veins or central veins (see Fig. 18(b)). The inside of this cube is oc-

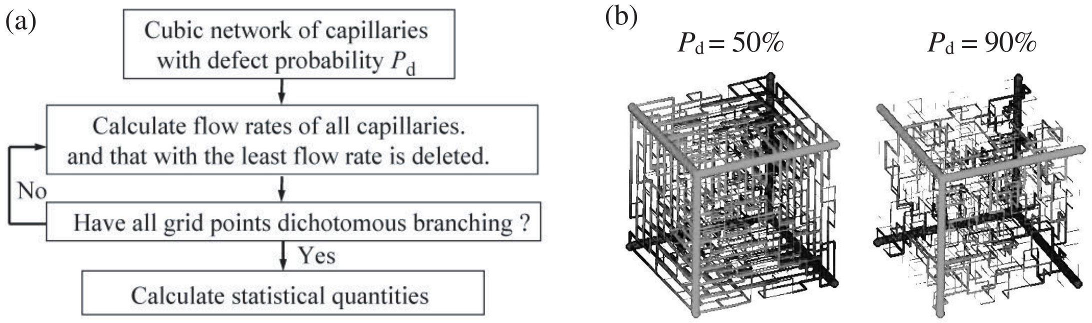


Fig. 18. (a) Algorithm to construct the sinusoidal capillary system. (b) Two examples of simulated sinusoidal capillaries with initial defect fractions given above. Thickness and the color of ducts indicate the flow rate and the pressure (the blood flows from portal vain (grey) to central vein (black)) (reproduced from Takaki *et al.* (2003)).

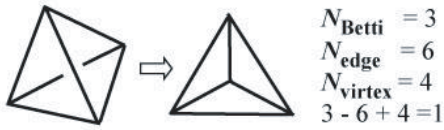


Fig. 19. A simple example of defining Betti number. The frame of tetrahedron shape has three holes, hence the Betti number is 3.

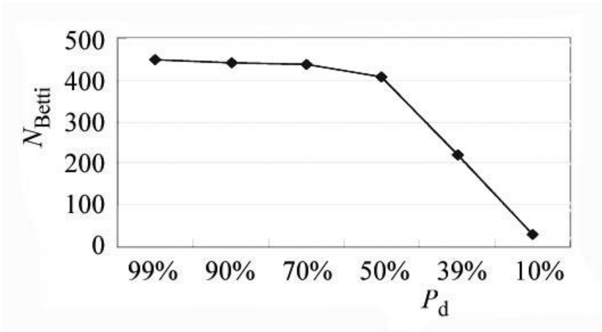


Fig. 20. Betti numbers vs. initial defect fraction obtained from the results of simulation.

cupied by a network of sinusoidal capillaries, which connect these two kinds of veins and touch all liver cells in order to hand over the blood to cells and to receive refreshed blood. Here, a restriction is posed on this network that the branching of capillaries is dichotomous, i.e. a capillary branches only to two new ones. Since the sinusoidal system is not of a tree-type, an algorithm different from that for construction of veins is necessary.

We begin from a cubic network with portal and hepatic veins occupying its six edges, as indicated by thick ducts in Fig. 18(b). These veins are given a pressure difference to give blood flow. Inside of this cube is divided into 10 × 10 × 10 small cubes to span bridges (sinusoidal capillaries). The process to construct a network is shown in Fig. 18(a). Initial condition is a cubic lattice having random defects in the bridge network, i.e. some bridges with fraction $P_d$ of all bridges in number are taken off. Then, flow rates of all bridges are calculated, and the bridge with the smallest flow rate is deleted. This process is repeated until the network includes only dichotomous branching. Two examples of results with $P_d$ = 50% and 90% are shown in Fig. 18(b).

Quantitative treatment of 3D network structures can be made based on the topological evaluation of networks, where a topological parameter $N_{Betti}$, called "Betti number" (a degree of multi-connectedness) plays an important roll. A method to apply this topological concept to analysis of pathological states of human organs, especially the sinusoidal capillaries, is developed by a medical doctor H. Shimizu (Shimizu, 1992, 2012, 2013; Shimizu and Yokoyama, 1994). In the following a brief introduction of the Betti number is given.

The Betti number indicates the number of loops included in a given network structure. For example, the tetrahedron shown in Fig. 19, if it is seen from the top, has three loops, hence it has Betti number 3. For a network with $N_{vertex}$ vertices, $N_{edge}$ edges and Betti number $N_{Betti}$, the following formula is satisfied:

$$N_{Betti} - N_{edge} + N_{vertex} = 1. \tag{10}$$

The Betti numbers for simulated network were counted, as shown in Fig. 20. The Betti number was about 440 for 70% < $P_d$ < 99%, while it decreases rapidly for $P_d$ < 50 %. It is interesting to compare the present result with measurement of Betti number in real liver by Shimizu and Yokoyama (1994). They gave the values of Betti numbers from specimen with size 200 × 200 × 80 $\mu$m, where $N_{Betti}$ = 181 ± 24 for normal examinee and $N_{Betti}$ = 85 ± 19 for examinee with cirrhosis, i.e. pathological hardening of liver.

Now, since the liver sell size is about 20 $\mu$m, their specimen corresponds to the 10 × 10 × 4 lattice, while the number of cells in the present simulation is 10 × 10 × 10. If our results of Betti numbers 440 and 200 for 50 % < $P_d$ < 99% and $P_d$ = 30%, respectively, are multiplied by 4/10, they give 176 and 88, respectively, agreeing well with the data by Shimizu and Yokoyama (1994) as given above. This result suggests that the network of sinusoidal capillaries is formed so that they touch as many cells as possible (choose larger value of $P_d$) while keeping dichotomous branching.

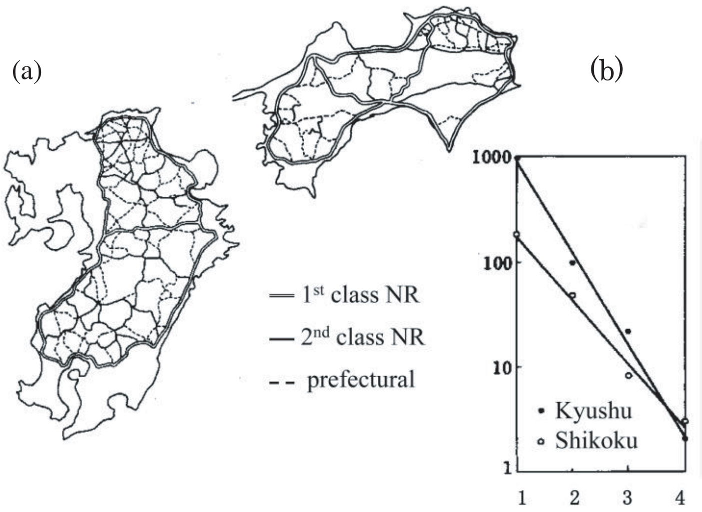


Fig. 21. Analyses of road networks. (a) Objects of analysis in Kyushu (left) and Shikoku (right) islands, where local roads are not shown here. (b) Dependences of numbers of road loops on their orders. Note that the data of Kyushu lie on the steeper line (reproduced from Takaki (1978)).

## 5. Branching Systems in Human Societies and Computers

It is easy to find branching structures in human societies, human cultures and artifacts. Here, two examples are introduced, which are treated by the present author.

### 5.1 Road networks

It was shown in Sec. 3 that an analysis similar to that by Horton is possible for such 2D network structures as leaf veins, where it is possible to classify network elements (loops) into different orders. The road networks also satisfy this requirement, because roads are classified as 1st- and 2nd-class national roads, prefectural road and local road. The present author made an analysis of roads in Kyushu and Shikoku islands in Japan, and derived bifurcation ratios for roads in these islands (Takaki, 1978).

Figure 21(a) shows the road systems in Kyushu and Shikoku islands at the time of 1980, where the 1st and 2nd national roads and the prefectural roads are drawn with different kinds of lines. In the analysis the regions surrounded by 1st class national roads were chosen. Local roads were also treated in the analysis, but are not shown in this figure. Order of a closed loop is defined as follows.

1. A loop made of local roads or of local and higher ones has an order 1.

2. A loop made of prefectural roads or of prefectural and higher ones has an order 2.

3. A loop made of 2nd class national roads or of 1st and 2nd ones has an order 3.

4. A loop made of only 1st class national roads has an order 4.

Note that loops are chosen so that a loop of a certain order does not include a smaller loop of the same order.

Numbers of loops in Kyushu and Shikoku islands were counted according to this rule, and the results are shown in Fig. 21(b). It is remarkable that the data for both islands follow the Horton's law, i.e. the number of loops decreases exponentially with the order. This situation might have been realized through many years owing to the human's desire to construct a convenient road system. In addition the difference of steepness of lines in Fig. 21(b) could be understood by assuming that the social system of Kyushu is more developed than that of Shikoku, so that society in Kyushu needed more number of roads of lower orders.

It is noted here that the above results may contribute well in a planning of road system in large scale regions.

### 5.2 Hierarchy of organization

A representative branching system in human society would be the graphic expression of organizations, such as schools, companies and governments. Most of them have tree-type structures, otherwise they must have confusions in information transmission and requests of jobs. It will be easy to imagine that a system of network-type suffers from great confusion through receiving various requests contradicting each other from many sections.

Figure 22 shows a construction of a Japanese university (Musashino Art University in 1970s), which was made of sections with four levels. An analysis similar to that for rivers was made for this construction by the present author (Takaki, 1978), where orders of sections are determined in the following way. The sections at the right ends have order 1, and the upper sections composed of lower sections acquire higher orders, where the rules in the Horton's analysis are followed.

It is remarkable that the graph in Fig. 22 shows an exact linearity. It should be noted here that this kind of social structures are constructed so that they function in the best way through continuous improvement, where managers of the structures are not conscious of the Horton's law. However, the resulting structure satisfies this law. This situation is similar to that for construction of road networks.

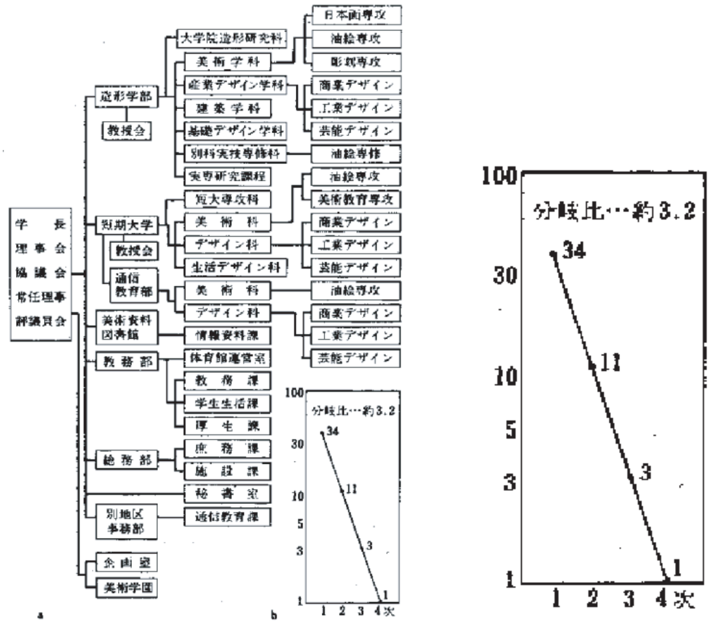


Fig. 22. An example of analysis of social structures by applying the Horton's method. In the graph on the right the abscissa is the order of sections and the ordinate is the number of sections.

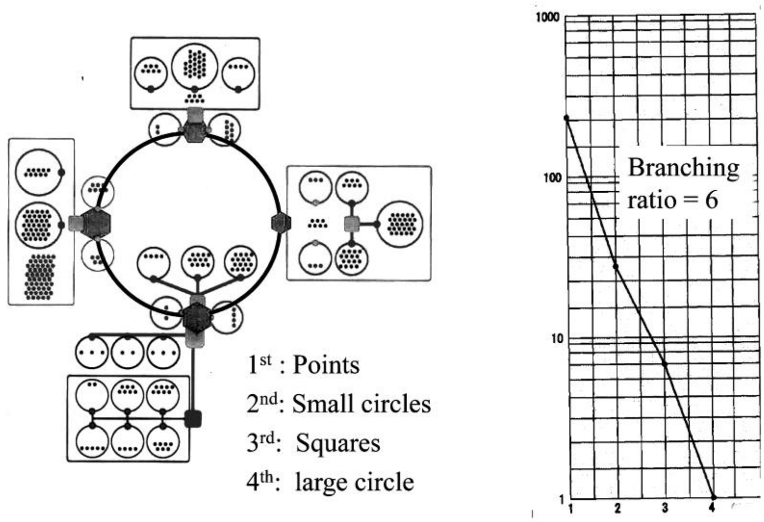


Fig. 23. Analysis of file structure in a personal computer by a design student (N. Sugimura, 2004, private communication). The small dots indicate individual files (order 1), the small circles are folders (order 2) including order 1 files, the squares are folders of order 3 and the large thick circle is the main folder (order 4). The left figure indicates the Horton's law in this file structure.

The present author does not know well whether some theoretical works are made as to the relation between structures of social systems and their functions. Analysis of this relation would be a difficult problem, because it should include both geometry of systems and human behavior in societies. However, it must be an important problem.

It is noted here that data files stored in personal computers have a certain kind of branching structure. In computers data files are stored within a main folder, which includes individual files and also some folders. These folders also include individual files and folders. Therefore, the total files can be expressed as a branching structure of tree type. A student, who took a course of the present author at Kobe Design University in 2004, examined his file structure and found that it satisfies the Horton's law approximately (see Fig. 23). It is hoped to confirm that this tendency is universally found in many personal computers.

## 6. Concluding Remarks

In this review article branching and network structures are discussed based on the Horton's method for analysis of river structure, and it is shown that the Horton's law is found universally among both natural and social phenomena. It might suggest an existence of a more fundamental law governing phenomena from various fields. Although the present author has at present no idea on how this kind of law looks like, but it might be meaningful to investigate it in an interdisciplinary activities, such as those by the Society for Science on Form.

Here, some notes are given on concepts concerned to complicated systems in general, i.e. the complex systems and the chaos. The term "complex systems" is given to a group of systems made of many elements which interact each other with nonlinear way, while the term "chaos" is concerned to systems with small number of elements with nonlinear interaction. The former attracts scientists because it seems to bridge natural sciences and social ones. The latter also attracts scientists because the chaos systems show complicated behavior in spite of the fact that they are made of small numbers of elements.

On the other hand, the topics treated in this review article seem to be somewhat different from the above two concepts. Both in complex and chaos systems we are interested in their dynamical behavior, while the topics of branching structures are concerned to their geometrical shapes, i.e. they are rather static. Although in appearances of branching systems certain kinds of dynamical processes must have worked, our interest in branching structures is mainly their geometrical natures. Here, we mainly investigate relations between "form" and "functions", which is considered to be one of important problems in the Science of Forms.

## Appendix

Several notes are given here to explain the rules given in Subsec. 4.3, which are proposed for constructing lung structure numerically.

*Note on rule 4*: a supplementary rule (4a) is posed so that the end points are distributed uniformly within the whole space (for precise, refer Kitaoka *et al*. (1999) and Kitaoka and Takaki (1998)).

*Note on rule 6*: a supplementary rule (6a) is posed for correcting the branching angles so that the daughters are directed to the center of their regions.

*Note on rule 7*: a supplementary rule (7a) is posed so that the daughter branches do not come out of the mother region or are not too short for supplying air.

*Note on rule 8*: the angle between the successive two branching planes (called rotation angle) is, in principle, the right angle, but a supplementary rule (8a) is posed to correct the rotation angle so that the volume ratio between two divided regions is not too small.

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