Original Paper — *Forma*, **31**, S29–S40, 2016

European Option Pricing under Fractional Brownian Motion with an Application to Realized Volatility

Takayuki Morimoto

Department of Mathematical Sciences, Kwansei Gakuin University, Japan
E-mail address: morimot@kwansei.ac.jp

(Received February 13, 2015; Accepted October 30, 2015)

This study investigates European option pricing under fractional Brownian motion (fBm) and applies it to realized volatility (RV). The RV measure is selected because it uniquely exhibits simultaneous stationarity and long-range dependency properties in financial time series, as shown in our empirical study. Meanwhile, the Black-Scholes differential equation is not well defined when the underlying assets follow fBm with the Hurst exponent $H \neq \frac{1}{2}$ because fBm is not a semimartingale. Thus, we compute the European option prices using a previously proposed fractional Black-Scholes formula. Our empirical study is conducted on Tokyo Stock Price Index data from January 06, 1997 to December 30, 2013 with a sample size of 4177.

Key words: Fractional Brownian Motion, Realized Volatility, Long-range Dependency, Fractional Black-Scholes Formula, Tokyo Stock Price Index Data

1. Introduction

Fractional Brownian motion (fBm) was mathematically introduced in 1940 by Kolmogorov as a method to generate Gaussian spirals in a Hilbert space as stated in Bardet and Bertrand (2007). The applicability of fBm to real world has been empirically suggested by ancient sages.

> In 1906, a young Englishman named Harold Edwin Hurst arrived in Cairo. It was to have been a short story. But it lasted sixty-two years and ended with his solving one of the great mysteries of the paharaohs—and, inadevertently, providing a clue to the way financial markets work.
>
> *Mandelbrot, Benoit B. "The (Mis)Behavior of Markets," 2004.*

Eventually, Hurst found a long-term relationship between the volatile rain on the Nile River and drought conditions. From these results, he determined the optimum dam sizing and published his seminal paper (Hurst, 1951). Subsequently, long-run dependences in financial time series have been frequently reported. Mandelbrot and Van Ness (1968) first investigated a self-similar process with a long-range dependent incremental process, known as fractional Brownian motion (fBm). Meanwhile Granger (1980), Granger and Joyeux (1980), and Hosking (1981) proposed an autoregressive fractionally integrated moving average (ARFIMA) model for a discrete time series with a long memory. For details of these concepts, see Beran (1994) and Baillie (1996).

Lo (1991) indicated that long-memory components in asset returns are crucial to many paradigms of modern financial economics. For example,

1) If stock returns are long-range dependent, the optimal consumption-savings and portfolio decisions may become extremely sensitive to the investment horizon.
2) The pricing of derivative securities (such as options and futures) by martingale methods is problematic, since the commonly employed class of continuous-time stochastic processes is inconsistent with long-term memory. Traditional tests of the capital asset pricing model and the arbitrage pricing theory are invalidated for persistent time series, because these series cannot be evaluated by typical statistical inferences.
3) The conclusions of more recent tests of "efficient" market hypotheses and stock market rationality also heavily rely on the presence or absence of long-term memory.

This study focuses on implication 2, which indicates that pricing options by martingale methods is inconsistent with long-term memory.

Therefore, we first study the definitions of long-memory processes in discrete time and the autoregressive integrated moving average (ARIMA) model with a fractional differencing parameter. Second, we introduce the R/S statistic proposed by Hurst (1951) and apply it to long-range dependency testing. Third, we construct a realized volatility (RV) measure for our empirical study and unit root tests for testing stationarity. Finally, we investigate European option pricing subjected to fBm and apply it to RV. Our empirical study uses Tokyo Stock Price Index (TOPIX) data from January 06, 1997 to December 20, 2013. The sample size is 4177.

Since the seminal paper of Black and Scholes (1973), numerous papers have described option pricing by a partial differential equation. However, the Black-Scholes dif-

Copyright © Society for Science on Form, Japan.

doi:10.5047/forma.2016.s005

ferential equation is poorly defined when the underlying assets follow fBm with the Hurst exponent $H \neq \frac{1}{2}$ because fBm is not a semimartingale, and the usual Itô integral cannot be integrated with respect to fBm (Biagini *et al.*, 2008). Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) defined a new stochastic integral based on Wick products and Skorohod integration, and showed that it prevents arbitrage opportunities. However, although these models guarantee no arbitrage opportunities, they have no natural economic interpretation (Björk and Hult, 2005). For this reason, we compute the European option prices by the fractional Black-Scholes formula proposed by Norros *et al.* (1999).

The remainder of this study is as follows. Section 2 briefly explains long-range dependency in discrete time. Sections 3 and 4 introduce the concept of fBm and option pricing under fBm, respectively. Section 5 simulates option pricing under fBm. Section 6 presents our empirical result and explains the RV measure and unit root tests for testing stationarity. Section 7 concludes the study.

2. Long-range Dependency

In this section, we briefly survey the stochastic nature of long-range dependence (LRD) in a time series, following Minotani (2001).

The autocorrelations in stochastic processes with LRD, or long memory, decay slowly to zero. A process is called a *short memory* process if its autocorrelation function ρ at lag k takes the form

$$|\rho_k| \leq ca^k, \quad c, a \text{ are constants},$$

and a *long memory* process if ρ_k takes the form

$$\rho_k \approx c_k^{2d-1}, \quad 0 < d < \frac{1}{2},$$

where $d \in \mathbb{R}$ (Brockwell and Davis, 1991). For instance, X_t is expected to follow the first-order autoregressive model AR(1)

$$(1 - \phi_1 L)X_t = \epsilon_t, \quad |\phi_1| < 1, \quad \epsilon_t \sim \text{iid}(0, \sigma^2),$$

(where $\phi_1 \in \mathbb{R}$ and L denotes the lag operator), the autocovariance function of process γ_k becomes

$$\gamma_k = \frac{\phi_1^k \sigma^2}{1 - \phi_1^2} \to 0, \quad k \to \infty,$$

and the autocorrelation function is

$$\rho_k = \phi_1^k \to 0, \quad k \to \infty.$$

Hence, AR(1) is found to be a short memory process. As another example, if X_t is expected to follow the autoregressive moving average (ARMA) model of order (p, q), denoted by ARMA(p, q), and the process is presumed stationary and invertible, the autocovariance function becomes

$$\gamma_k \approx c\lambda_1^k \to 0, \quad k \to \infty, \quad |\lambda_1| < 1,$$

where $\lambda_1, \ldots, \lambda_p$ are the roots of the characteristic equation of the AR(p) and λ_1 is closest to 1. Hence ARMA(p,q) is also a short memory process.

On the other hand, X_t has a long memory property if its autocovariance function is expressed as Lo (1991)

$$\gamma_k \sim \begin{cases} k^{2H-2}L(k), & \frac{1}{2} < H < 1, \\ -k^{2H-2}L(k), & 0 < H < \frac{1}{2}, \end{cases}$$

where H denotes the Hurst exponent and $L(k)$ is an arbitrary function such as $\lim_{t\to\infty} L(tk)/L(t) = 1$.

2.1 ARFIMA model

The ARIMA process takes the form

$$\phi(L)(1 - L)^d X_t = \theta(L)\epsilon_t,$$

where $\phi, \theta \in \mathbb{R}$, and d is the difference parameter, which must be integer valued. The ARFIMA model extends the ARIMA model to allow fractional order of differencing d (Granger, 1980; Granger and Joyeux, 1980; Hosking, 1981). An ARFIMA(p, d, q) has the following d-dependent properties (Chen *et al.*, 2008; Das and Pan, 2011).

1) If $d = 1/2$, the ARFIMA$(p, -1/2, q)$ process is stationary but non-invertible.
2) If $-1/2 < d < 0$, the ARFIMA(p, d, q) process has short memory and monotonically and hyperbolically decays to zero.
3) If $d = 0$, the ARFIMA$(p, 0, q)$ process can be white noise.
4) If $0 < d < 1/2$, the ARFIMA(p, d, q) process is a long-memory stationary process. This model appropriately describes LRD. The autocorrelation of a LRD time series slowly decays as a power law function.
5) If $d = 1/2$, the ARFIMA$(p, 1/2, q)$ process is a discrete time $1/f$ noise.

2.2 R/S statistic

Hurst (1951) and Mandelbrot (1972) developed the R/S statistic for testing whether LRD exists in a time series. First, we define the range R of X at time T as

$$R_T = \max_{1\leq k\leq T} \sum_{t=1}^{k} (X_t - \bar{X}) - \min_{1\leq k\leq T} \sum_{t=1}^{k} (X_t - \bar{X}),$$

where $\bar{X}$ denotes the sample mean of X_t. Second, we estimate the sample standard deviation of X_t as

$$S_T = \left[\frac{1}{T} \sum_{t=1}^{T} (X_t - \bar{X})^2 \right]^{\frac{1}{2}}.$$

The quantity $Q = R_T/S_T$ is called the R/S statistic.

Hurst (1951), Mandelbrot (1972), and Lo (1991) showed that

$$\text{p}\lim_{T\to\infty} \{T^{-H}(R_T/S_T)\} = \text{constant}.$$

They also approximated the above equation as

$$\log[\text{E}(R_T/S_T)] \approx \text{constant} + H \log(T).$$

Rearranging this formula, the Hurst exponent H is approximated by

$$H \coloneqq \frac{\log(R_T/S_T)}{\log(T)}.$$

The properties of the Hurst exponent H depend on its value Peters (1994). In the original theory, $H = 1/2$ implies an independent process, whereas $1/2 < H \leq 1$ implies a persistent time series, which is characterized by long memory effects. Theoretically, what happens today forever influences the future. $0 \leq H < 1/2$ signifies anti-persistence. An anti-persistent system covers less distance than a random one, because it self-reverses more frequently than a random process. In addition, the Hurst exponent H is directly related to the fractional differencing operator d of the ARFIMA model as

$$d = H - \frac{1}{2}. \tag{1}$$

In fact, when testing for LRD in X_t, we require the sample distribution of the R/S statistic. Again, we introduce Q_T as

$$Q_T = \frac{R_T}{S_T} = \frac{1}{S_T}\left[\max_{1\leq k\leq T}\sum_{t=1}^{k}(X_t - \bar{X}) - \min_{1\leq k\leq T}\sum_{t=1}^{k}(X_t - \bar{X})\right].$$

Assuming that $X_t \sim$ iid is true, we have

$$\frac{1}{\sqrt{T}}Q \Rightarrow V,$$

where $\Rightarrow$ denotes convergence in law, V is the range of Brownian bridge in $[0, 1]$, $\mathrm{E}(V) = \sqrt{\frac{\pi}{2}}$ and $\mathrm{Var}(V) = \frac{1}{6}\pi(\pi - 3)$; see Lo (1991).

2.3 Lo's modified R/S statistic

The type I error probability is known to exceed the nominal size of the R/S statistic. Consequently, a short memory process may be incorrectly deemed as a long memory one. To correct this problem, Lo (1991) proposed the following modified R/S statistic:

$$Q^* = \frac{R_T}{\hat{\sigma}_T(q)},$$

where $\hat{\sigma}_T^2(q) = S_T^2 + 2\sum_{j=1}^{q}\omega_j(q)c_j$, $c_j = \hat{\gamma}_j$ and $\omega_j(q) = 1 - \frac{j}{q+1}$ for $q < T$. The Q^* and Q differ in one respect only; the divisor of Q^* is $\hat{\sigma}_T(q)$, whereas that of Q is S_T. For instance, when X_t follows an AR(1) process, $q = 1$ and

$$\hat{\sigma}_T^2(1) = S_T^2 + c_1.$$

Moreover, since $Q^* < Q$ by virtue of $c_1 > 0$, we have $\hat{\sigma}_T^2(1) > S_T^2$. Q^* also has the following asymptotic property:

$$\frac{1}{\sqrt{T}}Q^* \Rightarrow V,$$

where V is the range of the Brownian bridge in $[0, 1]$ as stated before. However, we notice that a large lag q may lower the power of the test, indicating that if q is oversized, Q^* cannot properly detect the LRD in a time series. Hence, the right choice of q is essential in Lo's method (Teverovsky *et al.*, 1999).

3. Fractional Browninan Motion

A stochastic process Z on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is called fBm $\{Z_t\}_{t\geq 0}$ with Hurst exponent $H \in (0, 1)$ if the following conditions are satisfied:

1) Z_t has strictly stationary increments, that is, $Z_{t+s} - Z_t$, $s \geq 0$ is strictly stationary.
2) $Z_0 = 0$ and $\mathrm{E}[Z_t] = 0$ for all t almost surely.
3) $\mathrm{E}[Z_t^2] = |t|^{2H}$ for all t and $H \in (0, 1)$.
4) Z_t follows a Gaussian distribution.
5) Z_t is almost surely continuous.

The stochastic process is defined by

$$Z_t - Z_s = c_H\left[\int_s^t (t-u)^{H-1/2}dw_u + \int_{-\infty}^{s}\left((t-u)^{H-1/2} - (s-u)^{H-1/2}\right)dw_u\right], \tag{2}$$

where w_t is a standard Brownian motion (Bm) and c_H is given by

$$c_H = \left[\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(H + \frac{1}{2})\Gamma(2 - 2H)}\right]^{1/2},$$

with Γ being the gamma function. Setting $s = 0$ in Eq. (2) we can write

$$Z_t = c_H\left[\int_0^t (t-u)^{H-\frac{1}{2}}dw_u + \int_{-\infty}^{0}\left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)dw_u\right].$$

If $H = \frac{1}{2}$, fBm reduces to standard Brownian motion.

$$Z_t = c_{\frac{1}{2}}\left[\int_0^t (t-u)^{\frac{1}{2}-\frac{1}{2}}dw_u + \int_{-\infty}^{0}\left((t-u)^{\frac{1}{2}-\frac{1}{2}} - (-u)^{\frac{1}{2}-\frac{1}{2}}\right)dw_u\right], = \int_0^t dw_u = w_t,$$

where

$$c_{\frac{1}{2}} = \left[\frac{2\cdot\frac{1}{2}\Gamma(\frac{3}{2} - \frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2})\Gamma(2 - 2\cdot\frac{1}{2})}\right]^{1/2} = 1.$$

As described in the previous section, fBm with $1/2 < H \leq 1$ implies a persistent time series and $0 \leq H < 1/2$ signifies antipersistence.

3.1 A fractional Brownian market model

We introduce a riskless asset A_t and a risky asset S_t by means of a geometric fBm driven by $dA_t = rA_t dt$ and $dS_t = \mu S_t dt + \sigma S_t dB_t^H$, respectively. The parameters for the riskless interest rate r as well as the drift μ and the volatility σ are constant. The stochastic differential equation can be interpreted in multiple ways depending on the chosen stochastic integration calculus, which are pathwise integration, and in contrast, Wick-based integration (see Rostek and Schöbel, 2013).

Numerous articles have been published choosing fractional Brownian motion as an underlying diffusive process. There has been an ongoing discussion about the usage of fBm within financial models since Rogers (1997). Some publications also discuss market microstructure foundations of fBm while most of the literature focuses on arbitrage and its exclusion. Shiryayev (1998) constructed an explicit arbitrage strategy within the fractional market setting based on pathwise integrals. Duncan *et al.* (2000) provided a stochastic integration calculus with respect to fBm based on the Wick product. For more detailed information, see Rostek and Schöbel (2013).

3.2 Wick-Itô integral

Let w be a Brownian motion and F be a left-continuous, adapted and locally bounded deterministic process. The mean squares of the Riemann-Stieltjes sums converge to a random variable called the Itô integral of F with respect to w, which is defined up to time T:

$$\int_0^T F(s)dw_s = \lim \sum F(t_i)(w_{t_{i+1}} - w_{t_i}).$$

However, if the integrand is no longer deterministic, convergence in the mean square sense is not necessarily given, see Rostek (2009). Therefore, following Duncan *et al.* (2000), we replace the ordinary multiplication within the Riemann-Stieltjes sums with a different multiplicative concept called the Wick product (denoted by a diamond symbol $\diamond$. The fractional Wick-type integral, called the Wick-Itô integral, is the limit of the according sequence of Riemann-Stieltjes sums:

$$\int_0^T F(s)dw_s^H = \lim \sum F(t_i) \diamond (w_{t_{i+1}}^H - w_{t_i}^H).$$

The Wick-Itô integral cannot exhibit martingale behavior since the integrator of fBm is not a semimartingale. Meanwhile, the Wick-Itô approach is formally compatible with classical Brownian theory when pricing options (Rostek, 2009).

Following Biagini *et al.* (2008), we now explain the Wick product in the above equation. First, we define the nth-order Hermite polynomial:

$$h_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}), \quad n \geq 0,$$

and the nth order Hermite function:

$$\tilde{h}_n := \pi^{-1/4}(n!)^{-1/2}2^{-n/2}h_n(x)e^{-x^2/2}, \quad n \geq 0,$$

also we have

$$\mathcal{H}_\alpha(\omega) = \Pi_{i=1}^n h_{\alpha_i}(< \tilde{h}_i, \omega >),$$

where $\omega \in \Omega$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\alpha_i \in \mathbb{N} \cup \{0\}$. Let two random variables $F(\omega), G(\omega) \in L_2(\Omega)$ be chaotically expanded as

$$F(\omega) = \sum_\alpha c_\alpha \mathcal{H}_\alpha(\omega) \quad \text{and} \quad G(\omega) = \sum_\beta d_\beta \mathcal{H}_\beta(\omega),$$

respectively, where $c_\alpha, d_\alpha \in \mathbb{R}$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\beta_i \in \mathbb{N} \cup \{0\}$. The Wick product is then defined as

$$(F \diamond G)(\omega) = \sum_{\alpha,\beta} c_\alpha d_\beta \mathcal{H}_{\alpha+\beta}(\omega).$$

The Radon-Nikodym derivative for fBm is provided in Appendix A and for the discrete approximation of fBm, see Appendix B.

Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) implemented the Wick product into the definitions of the portfolio value and/or self-financing property. The seemingly encouraging result of a fractional Black-Scholes market excluding arbitrage provided by Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) entailed further models. Athough this pricing approach grew in popularity, some serious concerns questioned the usage of Wick products beyond pure integration theory. The suitability of Wick-based definitions of fundamental economic concepts was first doubted by Sottinen and Esko (2003).

3.3 Option pricing with fBm

When the underlying assets follow fBm with $H \neq \frac{1}{2}$, fBm is not a semimartingale, and the usual Itô integral cannot be integrated with respect to fBm (Biagini *et al.*, 2008). Under this circumstance, the Black-Scholes differential equation is ill-defined. To circumvent this problem, Hu and Øksendal (2003) and Elliott and Van Der Hoek (2003) defined a new stochastic integral based on Wick products and Skorohod integration, which ensures the absence of arbitrage opportunities. However, as already mentioned, these models lack a natural economic interpretation (Björk and Hult, 2005). We introduce self- and Wick-financing portfolios following Björk and Hult (2005) in Appendix C.

3.4 Fractional Black-Scholes formula

In the study, we compute European option prices using the fractional Black-Scholes formula proposed by Norros *et al.* (1999).

Suppose that a stock price follows geometric Brownian motion. Then, the price of a European call option C_{bs} is given by the celebrated Black-Scholes formula

$$C_{\text{bs}} = s\Phi(d_1) - k\exp(-rt)\Phi(d_2),$$

where s is the value of the underlying asset, k is the strike price of the option, t is the expiry time of the option, r is the risk-free interest rate, and σ is the volatility of the underlying asset. $\Phi(\cdot)$ represents the cumulative distribution function of a standard normal variable, and d_1 and d_2 are given by $d_1 = \frac{(\log(\frac{s}{k})+(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$ and $d_2 = d_1 - \sigma\sqrt{t}$. As shown by Norros *et al.* (1999), one can define a centered Gaussian martingale (the fundamental martingale) that generates the same filtration as fBm (Kozlowski, 2012). Since the filtration (rather than the stochastic process itself) represents the information provided by the market, this martingale may reasonably be used for option pricing. By this approach, formulas analogous to the classical Black-Scholes formulas are easily obtained, and coincidence is achieved by setting the Hurst index to $1/2$. The fractional European call option price C_{fbs} is given by

$$C_{\text{fbs}} = s\Phi(d_3) - ke^{-rt}\Phi(d_4),$$

where $d_3 = \frac{\log(\frac{s}{k})+(r+\frac{1}{2}c_2^2\sigma^2t^{1-2H})t}{c_2\sigma t^{1-H}}$ and $d_4 = d_3 - c_2\sigma t^{(1-H)}$. If $H = \frac{1}{2}$, we find that $C_{\text{bs}} = C_{\text{fbs}}$. That is, the differences between the fractional and the standard Black-Scholes prices are solely determined by the value of the

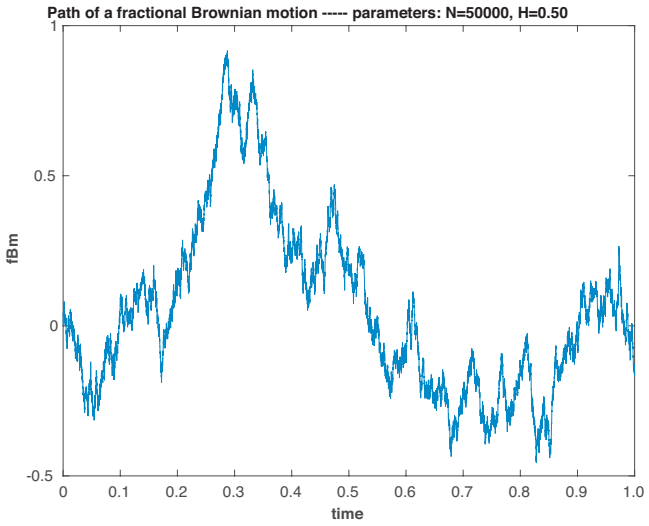

Fig. 1. Simulated fBm with $H = 0.50$.

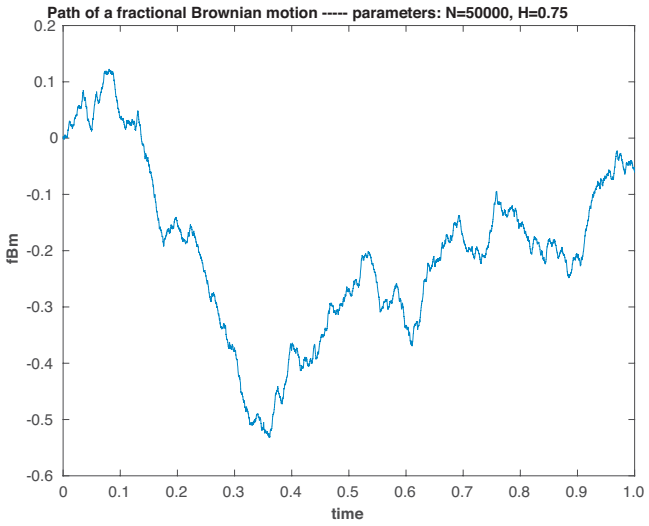

Fig. 2. Simulated fBm with $H = 0.75$.

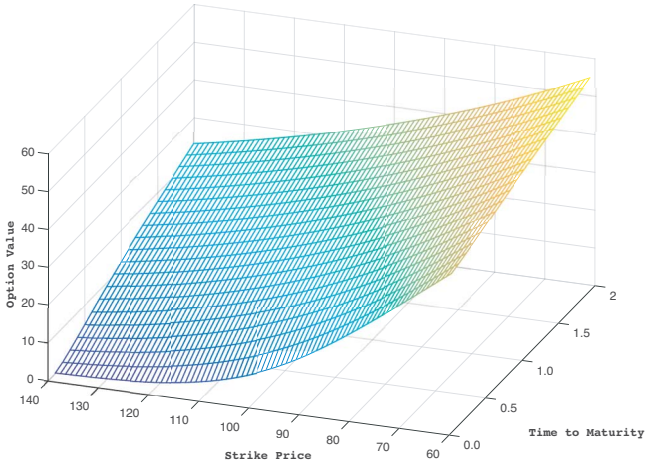

Fig. 3. Call option prices with $H = 0.50$ (current stock price $s = 100$; volatility $\sigma = 0.5$; risk-free interest rate $r = 0.1$).

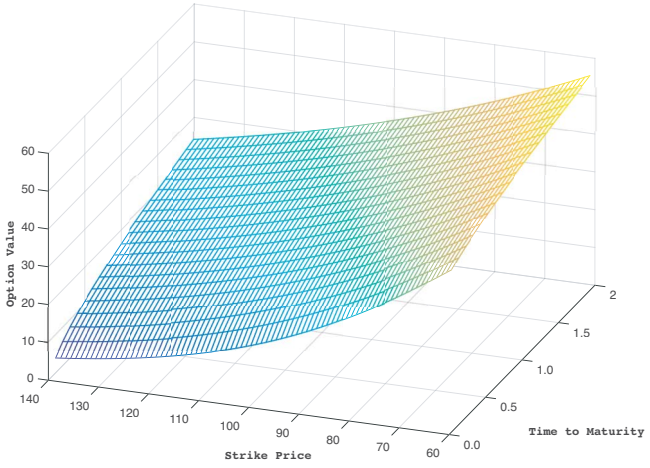

Fig. 4. Call option prices with $H = 0.75$ (other parameters fixed as in Fig. 3).

long memory parameter H (i.e., the Hurst exponent). The fractional European put option price P_{fbs} is also determined by the standard put-call parity relation

$$P_{\text{fbs}} = C_{\text{fbs}} + ke^{-rt} - s.$$

4. Simulation Study

In the section, we examine how the fractional European call option prices are evaluated in simulation studies.

Figures 1 and 2 are examples of fBm paths with $H = 0.50$ and $H = 0.75$, respectively. Simulations were performed using the dvfBm[*1] package function circFBSM() in the R language. As noted above, $H = 0.50$ (Fig. 1) describes standard Brownian motion, while $H = 0.75$ (Fig. 2) describes a process with a long memory property. Generation of the fractional Brownian paths is detailed in Coeurjolly (2000).

[*1] http://cran.r-project.org/web/packages/dvfBm/

Figures 3 and 4 show the European call option prices calculated by the fractional Black-Scholes formula introduced above. The Hurst parameters are set to $H = 0.50$ and $H = 0.75$ respectively, the current stock price $s = 100$, the volatility $\sigma = 0.5$, and the risk-free interest rate $r = 0.1$.

Since Figs. 3 and 4 are visually very similar, we clarify their differences in Figs. 5 and 6.

Figure 5 shows how the European call option prices differ between $H = 0.50$ and $H = 0.75$ with other parameters fixed at $s = 100$, $\sigma = 0.5$, and $r = 0.1$. According to this figure, the price difference is enhanced around the at-the-money (here denoted by the strike price $k = 100$), and widens as the time to maturity reduces. Figure 6 shows how the European call option prices as the Hurst exponent H varies from 0 to 1. Other parameters are fixed at $s = 100$, $\sigma = 0.5$, $r = 0.1$, and $k = 100$. From this figure, we observe that the difference greatly increases as the time to maturity reduces and the Hurst exponent H increases.

5. Empirical Study

In this section, we examine the long memory property of real financial data such as time series of stock prices, log-

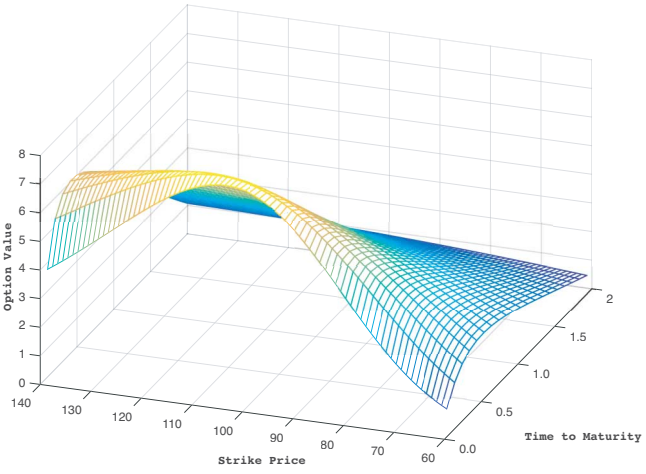

Fig. 5. Differences in call prices with fixed $H = 0.50$ and $H = 0.75$ ($s = 100$; $\sigma = 0.5$; $r = 0.1$).

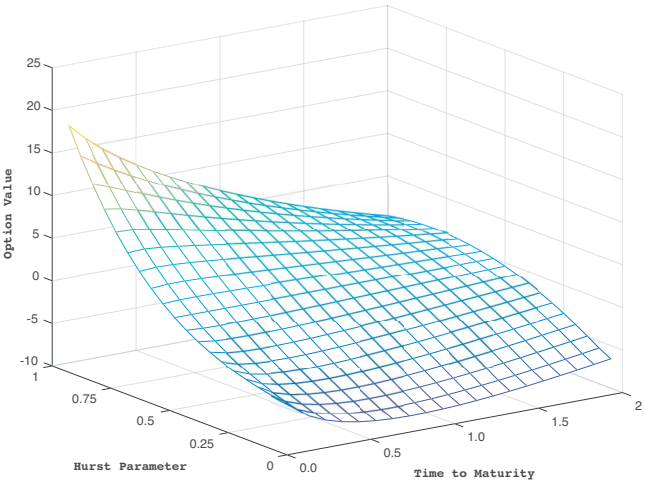

Fig. 6. Differences in call prices as H is varied from 0 to 1 ($s = 100$; $\sigma = 0.5$; $r = 0.1$; $k = 100$).

return, and volatility. We employ the daily closing price data of the Tokyo stock price index (TOPIX) from January 1997 to December 2013, denoting its value at time t by S_t. The daily logarithmic return at time t is defined as $r_t = \log S_t - \log S_{t-1}$. Our analysis also includes the RV defined at time t as the sum of the intraday squared returns (Andersen *et al.*, 2001):

$$RV_t = \sum_{i=1}^{n_t} r_{t,i}^2,$$

where $r_{t,i}^2$ denotes a squared log-return (the ith observation on day t) and n_t is the number of data points in t. Regarding the underlying log-price process as the continuous martingale part in a semimartingale model setup, the RV can be viewed as a proxy variable of the integrated variance calculated from the intraday full high-frequency log-returns. Consequently, the RV estimation requires the full high-frequency data over 24 h as a daily volatility measure. However, the Japanese stock market is divided into two sessions by a lunch break, i.e., the morning session lasts from 09:00 to 11:00 and the afternoon session from 12:30

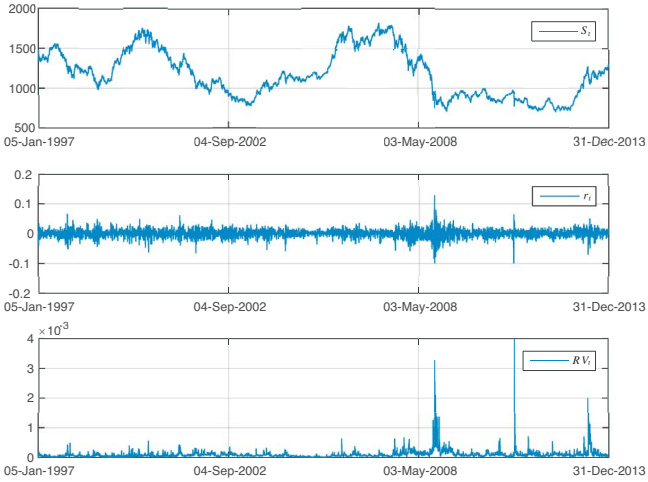

Fig. 7. Paths of S_t, r_t and RV_t time series in the TOPIX data (1997–2013).

Table 1. Summary statistics of TOPIX data (1997–2013).

	S_t	r_t	RV_t
Mean	1173.374	−0.0000	0.01025
Median	1145.760	0.0002	0.00716
Maximum	1816.970	0.1286	0.39879
Minimum	695.510	−0.1001	0.00001
Std. Dev.	293.302	0.0141	0.01489
Skewness	0.317	−0.2925	11.76425
Kurtosis	2.018	8.402	223.907
Obs.	4177	4177	4177
LB(10)	41133.82∗	22.47	8548.13∗

Note that LB(10) denotes the Ljung-Box test statistics at lag 10 and ∗ indicates the rejection of the null hypothesis that the process is not autocorrelated.

to 15:00. Thus, we adopt the weighted RV proposed by Masuda and Morimoto (2012), which is a modified version adjusted to the Japanese market (Hansen and Lunde, 2005). The weighted RV with estimated optimal weights λ_1, λ_2, λ_3 and λ_4 is defined by

$$wRV_t = \lambda_1 Y_{t,1}^2 + \lambda_2 RV_{t,2} + \lambda_3 Y_{t,3}^2 + \lambda_4 RV_{t,4},$$

where $Y_{t,1}^2$, $RV_{t,2}$, $Y_{t,3}^2$, and $RV_{t,4}$ denote the square of the close-to-open return, the RV in the morning session, the square of the lunch break return, and the RV in the afternoon session, respectively, on the tth day. Hereafter, we replace the weighted RV wRV by RV for notational simplicity. In addition, we set the sampling frequency to 1 min, the minimum observation interval of the Japanese stock market. The resulting sample sizes of the morning and afternoon sessions are 120 and 150, respectively.

5.1 Data description

In the empirical analysis, we first describe the three time series data S_t, r_t and RV_t discussed above. Figure 7 depicts the paths of S_t, r_t and RV_t over the sample period, and Table 1 presents the descriptive statistics of these data. The null hypothesis is that the data are independently distributed. According to the Ljung-Box (10) statistics for serial correlation in Table 1, we cannot reject the null hypothesis for

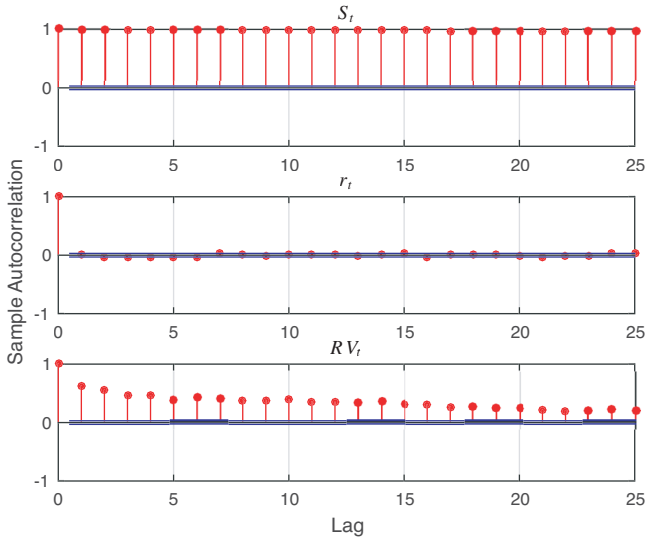

Fig. 8. Sample autocorrelation functions of S_t, r_t and RV_t time series in the TOPIX data (1997–2013).

Table 2. Results of t-ratio $\hat{\tau}_\mu$ estimated by the DF test.

Assumed models	S_t	r_t	RV_t
AR(1)	−0.632	−63.28	−24.68
	(0.421)	(0.001)	(0.001)
AR(1) with drift	−1.929	−63.27	−31.05
	(0.328)	(0.001)	(0.001)
TS	−1.796	−63.28	−31.85
	(0.694)	(0.001)	(0.001)

Note that TS represents trend-stationary and p-values for the null hypothesis are reported in parentheses.

Table 3. Results of the R/S analysis.

Used methods	S_t	r_t	RV_t
Hurst-Mandelbrot			
$\widehat{V}$	1245.19	66.35	561.00
$\widehat{H}$	0.855	0.503	0.759
Lo			
$\widehat{V}$	880.86	65.68	440.13
$\widehat{H}$	0.813	0.502	0.730

r_t at the 0.01 significance level, but the null hypothesis for S_t and RV_t is rejected at this level. Thus, the S_t and RV_t series show apparent serial correlations. As a graphical verification, we present sample autocorrelation functions for S_t, r_t and RV_t in Fig. 8. The correlogram impressively shows that the sample autocorrelation functions of both S_t and RV_t slowly decay, whereas that of S_t is one at all lags.

5.2 Stationarity and long-range dependency

In this subsection, we examine the stationarity and long-range dependency of the series before setting the pricing options under fBm.

First, we perform a unit root test on the series, namely the Dickey-Fuller (DF) test proposed by Dickey and Fuller (1979). The unit root problem in a time series arises when either the autoregressive or moving average polynomial of an ARMA model has a root on or near the unit circle (Brockwell and Davis, 2002). We provide a brief theoretical explanation of the unit root test in Appendix D since a unit root in either of these polynomials has important implications for modeling. Table 2 shows the t-ratio $\hat{\tau}_\mu$ estimated by the DF test for each time series analyzed by the two models. While the S_t exhibits no obvious stationarity property, the r_t and RV_t series are probably stationary processes.

Second, to examine the LRD of the data, we conducted an R/S analysis using the Hurst-Mandelbrot and the Lo methods introduced in the previous section. The results of this analysis are presented in Table 3. The Hurst exponent H of the r_t by each method is approximately 0.5, implying that the process follows a standard Brownian motion. In contrast, the S_t and RV_t series are likely to have long-range dependency since their Hurst exponents lie within $(1/2, 1)$.

In addition, we estimated the memory parameter d in the ARIMA model, adopting the Sperio estimator proposed by Reisen (1994). Table 4 shows the estimated d and H for each time series, calculated by Equation (1). All the estimated Hurst exponents in Table 4 are slightly higher than those in Table 3.

Table 4. Estimated results of the ARFIMA model.

Estimates	S_t	r_t	RV_t
$\hat{d}$	0.388	0.119	0.341
$\widehat{H}$	0.888	0.619	0.841

Table 5. Summary of the stationarity and long-range dependency result.

	S_t	r_t	RV_t
Stationarity	No	Yes	Yes
Long-range dependency	Yes	No	Yes

5.3 Option pricing under fBm

Finally, we examined option pricing under fBm by the method of Norros *et al.* (1999). We confined this analysis to the RV_t time series since the RV_t data exhibit simultaneous stationarity and long-range dependency properties, as shown in Table 5.

Figure 9 shows how the European call option prices differ between $H = 0.50$ and $H = 0.7592$, estimated by the Hurst-Mandelbrot method. For comparison, the price differences between $H = 0.50$ and $H = 0.7301$ estimated by Lo's method are presented in Fig. 10. In both figures, the differences are enhanced around the at-the-money (here denoting the strike price $k = 100$), and increase as the time to maturity decreases. These figures are plotted identically to Fig. 5 in the simulation study, but they exhibit a distinctly different shape. These shape differences might be explained by the different values of the volatility parameters in the simulation and the empirical study ($\sigma = 0.5$ and $\sigma = 0.01489$, respectively). The volatility is well known as the most sensitive input parameter in pricing options.

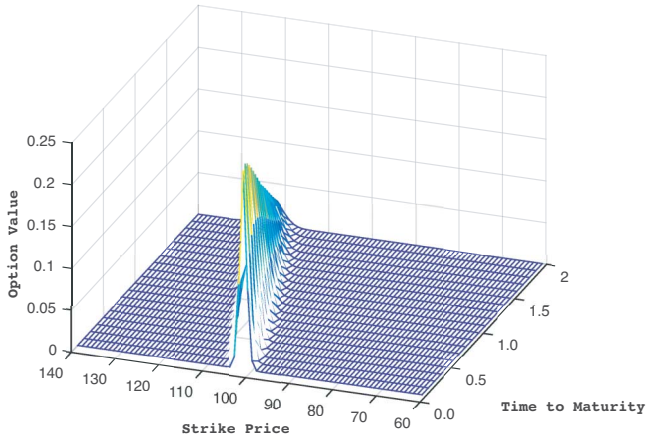

Fig. 9. Differences in call prices between fixed $H = 0.50$ and $H = 0.7592$.

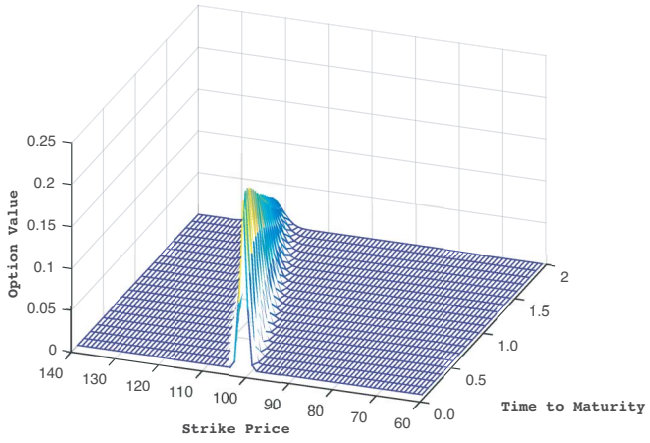

Fig. 10. Differences in call prices between fixed $H = 0.50$ and $H = 0.7301$.

5.4 Discussion

The classical R/S statistic intrinsically depends on ranges chosen arbitrarily by researchers. According to Teverovsky *et al.* (1999), the Lo's test, which is based on the modified R/S statistic V, tends to be very conservative in rejecting the null hypothesis of non-long-range dependence series. The test performs very well and gives the correct results when a series exhibits only short-range dependence. However, the test still tends to incorrectly accept the null hypothesis when the series is long-range dependent.

Hence we introduce alternative semi-parametric methods of estimating the Hurst exponent $H = d + \frac{1}{2}$ which are known as a class of local Whittle estimators mainly analyzed by Robinson (1995), Shimotsu and Phillips (2005) and Shimotsu (2010). We will now give a brief explanation of these methods following Beran *et al.* (2013) and Kumar (2014).

The first alternative is the local Whittle (LW) method proposed by Künsch (1987) and Robinson (1995), which assumes the behavior of the spectral density $f(\lambda)$, when $\lambda = 0$, Suppose that X_t is a stationary process with spectral density

$$f_X(\lambda) \sim c_f |\lambda|^{1-2H} \text{ as } \lambda \to 0$$

Table 6. Estimated results of the LW and 2ELW tests.

Used methods	S_t	r_t	RV_t
LW			
$\hat{d}$	1.058	0.053	0.465
$\widehat{H}$	1.558	0.553	0.965
2ELW			
$\hat{d}$	1.059	0.057	0.455
$\widehat{H}$	1.559	0.557	0.955

Note that $\widehat{H}$ is naïvely estimated from $\widehat{H} = \hat{d} + 0.5$.

$$\text{where } -\frac{1}{2} < d < \frac{1}{2} \text{ and } c_f \neq 0.$$

The empirical analog to the spectral density is the periodgram

$$I_{n,X}(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} X_t e^{-it\lambda} \right|^2 \text{ for } \lambda_j = \frac{2\pi j}{n}.$$

Then, given $d \in \Theta \subseteq (-\frac{1}{2}, \frac{1}{2})$, the LW estimator of d is defined by

$$\hat{d}_{LW} = \arg\min_{d \in \Theta} K_m(d),$$

where

$$K_m(d) = \log G_m(d) - d \left(\frac{2}{m} \sum_{j=1}^{m} \log \lambda_j \right),$$

and for the spectral density satisfying the equation, the Whittle log-likelihood function is given as:

$$\hat{c}_f = G_m(d) = \frac{1}{m} \sum_{j=1}^{m} \frac{I_{n,X}(\lambda_j)}{\lambda_j^{-2d}},$$

where $\lambda_j = 2\pi j/T$ and G is a constant. Robinson (1995) derived the asymptotic distribution of $\hat{d}_{LW}$ under some assumptions that mimic those for the GPH estimator suggested by Geweke and Porter-Hudak (1983):

$$\sqrt{m}(\hat{H} - H) \Rightarrow N(0, 1/4) \quad \text{as} \quad N \to \infty.$$

For more details, see Beran *et al.* (2013).

The LW estimator introduced above is consistent for $d \in (-0.5, 1)$ and asymptotically normal for $d \in (-0.5, 0.75)$. The issue of nonstationarity has been addressed more completely by Shimotsu and Phillips (2005). They proposed correcting the discrete Fourier transform by adding a complementing term, thereby ensuring a valid approximation that holds for every value of d. This is a so-called exact LW (ELW) procedure. Furthermore, Shimotsu (2010) proposed the two-step ELW (2ELW) estimator which is consistent and has the same limiting distribution as the LW and ELW estimators under $-0.5 < d < 2$.

Table 6 shows the estimated results of the LW and 2ELW tests using MATLAB codes[*2] provided by Professor Shimotsu. The estimate of $\widehat{H}$ obtained from S_t is abnormally

[*2] http://shimotsu.web.fc2.com/Site/Matlab_Codes.html

large while the values of r_t and RV_t are reasonable in comparison to the previous results. It has been considered that the relational expression $\widehat{H} = \hat{d} + 0.5$ cannot be simply applied to a non-stationary time series such as S_t.

6. Conclusion

The objective of this paper was to study European option pricing under fBm with application to RV. We confined our study to the RV measure because the RV uniquely expressed both stationarity and long-range dependency properties in the financial time series, as shown in our empirical study. The Black-Scholes differential equation is not well defined when the underlying assets follow fBm with $H \neq \frac{1}{2}$ because the fBm is not a semimartingale. Thus, we computed European option prices using the fractional Black-Scholes formula proposed by Norros *et al.* (1999).

From our simulation study, we concluded that the European call option prices are much more sensitive to the Hurst exponent H as the time to maturity reduces (where H ranges from 0 to 1). The price differences also widen with increasing H. Furthermore, the differences in the European call option prices exhibited substantially different behaviors in the simulation and empirical studies. We attributed these differences to the different volatility parameters $\sigma = 0.5$ in the simulation and $\sigma = 0.01489$ in the empirical study. The volatility is well recognized as the most sensitive input parameter in pricing options.

Acknowledgments. This work was supported by JSPS KAKENHI Grant Number 26380279 and 15K03406.

Appendix A. Radon-Nikodym Derivative for fBm

Let X_t be a stochastic process driven by

$$X_t = Z_t + at,$$

where $a \in \mathbb{R}$ and Z_t is an fBm under the probability measure P. We can change the measure of process X_t via the Radon-Nikodym derivative of Q with respect to P:

$$\frac{dQ}{dP} = \exp\left(-aM_t - \frac{1}{2}a^2\langle M, M\rangle_t\right), \tag{A.1}$$

which yields a stochastic process X without drift under the new probability measure Q. M_t in Eq. (A.1) is given by

$$M_t = \int_0^t c_1 u^{\frac{1}{2}-H}(t-u)^{\frac{1}{2}-H} dw_u,$$

where $c_1 = \left[2H\Gamma\left(\frac{3}{2}-H\right)\Gamma H + \frac{1}{2}\right]^{-1}$ and M_t is a martingale with independent increment and zero mean. The variance of M_t is given by

$$\mathrm{E}[M_t^2] = c_2^2 t^{2-2H}, \quad c_2 = \frac{c_H}{2H\sqrt{2-2H}}.$$

In terms of this expression, we can rewrite the Radon-Nikodym derivative (A.1) as

$$\frac{dQ}{dP} = \exp\left(-aM_t - \frac{1}{2}a^2c_2^2t^{2-2H}\right). \tag{A.2}$$

When $H = \frac{1}{2}$, Eq. (A.2) reduces to the change of measure formula in standard Brownian motion.

Appendix B. Discrete Approximation of fBm

Discrete approximation problems for fBm can be categorized into three types depending on their H values (Neuenkirch *et al.*, 2010) as follows:

1) When $H > 1/2$, the Euler approximation converges at the rate $n^{-(2H-1)+\epsilon}$ for arbitrarily small $\epsilon > 0$ (Neuenkirch and Nourdin, 2007; Davie, 2008; Mishura and Shevchenko, 2008).
2) When $H = 1/2$, the approximation reduces to a discrete approximation of standard Brownian motion (Kloeden and Platen, 2010).
3) When $1/3 < H < 1/2$, the solution should be approximated by a Milstein-type scheme or a more sophisticated scheme (Lyons and Qian, 2003; Gubinelli, 2004). Moreover, it is easily seen that when $H < 1/2$, the standard Euler scheme does not converge as the step size reduces to zero, even in the one-dimensional case. In fact, consider the one-dimensional standard differential equation

$$dX_t = X_t dw_t^H, \quad X_0 = 1,$$

whose exact solution is given by $X_t = \exp(w_t^H)$. The Euler approximation of this equation at $t = 1$ is given by

$$X_1^{(n)} = \prod_{k=0}^{n-1}(1 + (w^H_{(k+1)/n} - w^H_{k/n})).$$

For $n \in \mathbb{N}$ sufficiently large, a Taylor's expansion gives

$$\begin{aligned} &X_1 - X_1^{(n)} \\ &= \exp(w_1^H) \\ &\quad - \exp\left(\sum_{k=0}^{n-1}\log(1 + (w^H_{(k+1)/n} - w^H_{k/n}))\right), \\ &= \exp(w_1^H) \\ &\quad - \exp\left(w_1^H - \frac{1}{2}\sum_{k=0}^{n-1}|w^H_{(k+1)/n} - w^H_{k/n}|^2 + \rho_n\right), \end{aligned}$$

where $\rho_n \overset{a.s.}{\to} 0$ for $n \to \infty$ for $H > 1/3$. Now, it is well known that

$$\sum_{k=0}^{n-1}|w^H_{(k+1)/n} - w^H_{k/n}|^2 \overset{a.s.}{\to} \infty,$$

when $H < 1/2$, so we have $X_1^{(n)} \overset{a.s.}{\to} 0$. However, such one-dimensional equations converge under a Milstein-type scheme (Gradinaru and Nourdin, 2009).

Furthermore, Neuenkirch *et al.* (2010) demonstrated three different regimes for the convergence rate of the exact root mean square in the Euler scheme depending on the Hurst parameter $H \in (1/4, 1)$.

1) When $H < 3/4$, the exact convergence rate is $n^{-2H+1/2}$.
2) When $H = 3/4$, the exact convergence rate is $n^{-1}\sqrt{\log(n)}$.
3) When $H > 3/4$, the exact convergence rate is n^{-1}.

where n denotes the number of discretization subintervals.

Based on the above discussion, we now derive theorem 1 in Neuenkirch *et al.* (2010). Let $B = (B^{(1)}, B^{(2)})$ be a two-dimensional fBm with Hurst parameter $H \in (1/4, 1)$ indexed by $\mathbb{R}$. We approximate

$$X_t = \int_0^T B_s^{(1)} dB_s^{(2)}, \tag{B.1}$$

by the Euler and a trapezoidal scheme based on equidistant discretizations. The standard Euler approximation of (B.1) is explicitly given by

$$X_T^n = \sum_{i=0}^{n-1} B_{iT/n}^{(1)} \left(B_{(i+1)T/n}^{(2)} - B_{iT/n}^{(2)} \right). \tag{B.2}$$

From this expression, we can determine the exact L^2-convergence rate of the Euler scheme.

Theorem B.1 (Theorem 1 in Neuenkirch *et al.* (2010)) *Let X_T and its Euler approximation X_T^n be given by expressions (B.1) and (B.2), respectively. In addition, let*

$$\alpha_1(H) = c_0 + 2\sum_{k=1}^{\infty} c_k \quad \text{and} \quad \alpha_2(H) = \frac{H^2(2H-1)}{4(4H-3)},$$

where c_0 and c_k are constants defined in Neuenkirch et al. (2010). Then, $\sum_{k=1}^{\infty} c_k$ is a convergent series if $H \in (1/4, 3/4)$ and

$$\mathbf{E}|X_T - X_T^n|^2 = \begin{cases} \alpha_1(H) \cdot T^{4H} \cdot n^{-4H+1} + o(n^{-4H+1}) & \text{for } H \in (1/4, 3/4), \\ \frac{9}{128} \cdot T^3 \cdot \log(n) n^{-2} + o(\log(n) n^{-2}) & \text{for } H = 3/4, \\ \alpha_2(H) \cdot T^{4H} \cdot n^{-2} + o(n^{-2}) & \text{for } H \in (3/4, 1). \end{cases}$$

At the end of this section, we introduce a limit theorem for the asymptotic error distribution of the Euler scheme proposed by Neuenkirch *et al.* (2010).

Theorem B.2 (Theorem 3 in Neuenkirch *et al.* (2010)) *Define X_T, X_T^n and $\alpha_1(H), \alpha_2(H)$ as above. Moreover, let Z be a standard normal random variable. Then*

1) *Case $1/4 < H \leq 3/4$. The following central limit theorems hold:*

$$\lim_{n\to\infty} n^{2H-1/2}(X_T - X_T^n) \overset{\mathcal{L}}{=} \sqrt{\alpha_1(H)} T^{2H} \cdot Z,$$

for $H \in (1/4, 3/4)$ and

$$\lim_{n\to\infty} n(\log(n))^{-1/2}(X_T - X_T^n) \overset{\mathcal{L}}{=} \frac{3}{4\sqrt{8}} T^{3/2} \cdot Z,$$

for $H = 3/4$ where $\overset{\mathcal{L}}{=}$ denotes convergence in law.

2) *Case $H > 3/4$. Let R_1 and R_2 be two independent Rosenblatt processes (Neuenkirch et al., 2010) for the definition). Then we have*

$$\lim_{n\to\infty} n(X_T - X_T^n) \overset{\mathcal{L}}{=} \sqrt{2\alpha_2(H)} T^{2H} \cdot (R_1 - R_2).$$

Appendix C. Portfolio Strategies

Self-financing portfolio Consider a financial market with $n+1$ asset price processes $S^0, S^1, \ldots, S^n$, and denote the corresponding vector process by S. We consider an adapted portfolio process $h = (h^0, h^1, \ldots, h^n)$ and define the value process V^h associated with h by the standard formula

$$V_t^h = \sum_{i=0}^{n} h_t^i S_t^i = h_t S_t,$$

where the equality between random variables is interpreted as equality P-almost surely. In continuous time, the self-financing concept becomes more complicated. However, we can specify a putative minimum requirement; that a buy-and-hold portfolio, i.e., a portfolio that remains constant over a fixed time interval, be self-financing over that interval. Let us consider the time interval $[t_0, t_1]$ and a portfolio h that is constant over that interval. At any time $t \in [t_0, t_1]$ the portfolio value will be $V_t = h_t S_t$. Since h is constant, the portfolio value changes over the interval by an amount

$$V_{t_1} - V_{t_0} = h_{t_1} S_{t_1} - h_{t_0} S_{t_0} = h_{t_0}(S_{t_1} - S_{t_0}) = \int_{t_0}^{t_1} h_t dS_t, \tag{C.1}$$

where the integral is defined trajectory-wise. Thus, we have the standard Itô value dynamics

$$dV_t = h_t dS_t.$$

Wick-financing portfolio According to Eq. (C.1), the buy-and-hold portfolio will satisfy

$$V_{t_1} - V_{t_0} = h_{t_0}(S_{t_1} - S_{t_0}). \tag{C.2}$$

However, to qualify as Wick-financing, the portfolio should instead satisfy the condition

$$V_{t_1} - V_{t_0} = \int_{t_1}^{t_1} h_{t_0} S_u \diamond dw_u^H, \tag{C.3}$$

which generally differs from (C.2) since (C.3) is not generally consistent with

$$h_{t_0} = \int_{t_0}^{t_1} S_u \diamond dw_u^H,$$

by non-commutativity of the Wick product i.e., $(F \diamond G) \cdot H \neq F \diamond (G \cdot H)$.

As an illustrative example, we construct a portfolio strategy that is self-financing in the standard sense but is not Wick-financing. To this end, we refer to an example in Björk and Hult (2005). We also demonstrate that the use of "risk-neutral" pricing formulas based on the Wick-financing concept, as suggested by Elliott and Van Der Hoek (2003), leads to easily implementable arbitrage possibilities in the standard naive sense.

Example C.1 (Example 1 in Björk and Hult (2005)) *Consider the following portfolio strategy with initial capital $x > 0$.*

1) *At $t = 0$, we deposit all our money into a bank account and wait until $t = 1$.*
2) *Since the short rate equals zero, amount x remains in the account at $t = 1$.*
3) *At $t = 1$, we transfer all our money into a risky asset. Thus, we buy x/S_1 shares at price S_1 and hold this position until $t = 2$. Clearly, the value of our portfolio at $t = 2$ is given by*

$$V_2 = \frac{x}{S_1} S_2.$$

Since no capital has been added or withdrawn between [0, 2], this portfolio strategy must be included in any reasonable definition of a self-financing portfolio. However, this strategy is not self-financing in the language of Elliott and Van Der Hoek (2003). To prove this result, we must show that

$$x\frac{S_2}{S_1} \neq x + \int_0^2 h_u^1 S_u \diamond dw_u^H.$$

In fact, even the expected values of the terms in the above expression are unequal. For the right-hand side, we have

$$E\left(x + \int_0^2 h_u^1 S_u \diamond dw_u^H\right) = x,$$

whereas for $H \neq 1/2$, the expected value of the left-hand side is

$$\begin{aligned} E\left(\frac{x}{S_1} S_2\right) &= x E\left(\exp\left\{w_2^H - \frac{1}{2}2^{2H}\right\}\right. \\ &\qquad \left.\exp\left\{-w_1^H + \frac{1}{2}1^{2H}\right\}\right), \\ &= x \exp\left\{-\frac{1}{2}(2^{2H} - 1)\right\} E\left(\exp\left\{w_2^H - w_1^H\right\}\right), \\ &= x \exp\left\{-\frac{1}{2}(2^{2H} - 1)\right\} \exp\left\{\frac{1}{2}|2 - 1|^{2H}\right\}, \\ &= x \exp\left\{1 - 2^{2H-1}\right\} \neq x, \end{aligned}$$

where $S_t = s_0 \exp\left\{w_t^H - \frac{1}{2}t^{2H}\right\}$.

Appendix D. Dickey and Fuller Test

Let $X_1, \ldots, X_n$ be observations from the AR(1) model

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + Z_t, \quad Z_t \sim \text{iid}(0, \sigma^2), \quad \text{(D.1)}$$

where $|\phi_1| < 1$ and $\mu = \text{E}[X_t]$. For large n, the maximum likelihood estimator $\hat{\phi}_1$ of ϕ_1 is approximately $N(\phi_1, (1 - \phi_1^2)/n)$. If a unit root exists, this normal approximation is invalidated (even asymptotically), so cannot be used to test the unit root hypothesis $H_0 : \phi_1 = 1$ vs. $H_1 : \phi_1 < 1$. To construct a test for H_0, we write the model (D.1) as

$$\nabla X_t = X_t - X_{t-1} = \phi_0^* + \phi_1^* X_{t-1} + Z_t, \quad Z_t \sim \text{iid}(0, \sigma^2),$$

where $\phi_0^* = \mu(1 - \phi_1)$ and $\phi_1^* = \phi_1 - 1$. Now let $\hat{\phi}_1^*$ be the ordinary least squares estimator of ϕ_1^* found by regressing ∇X_t on 1 and X_{t-1}. The estimated standard error of $\hat{\phi}_1^*$ is

$$\widehat{\text{SE}}\left(\hat{\phi}_1^*\right) = s\left(\sum_{t=2}^{n}(X_{t-1} - \bar{X})^2\right)^{-1/2},$$

where $s^2 = \sum_{t=2}^{n}\left(\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* X_{t-1}\right)^2/(n-3)$ and $\bar{X}$ is the sample mean of $X_1, \ldots, X_{n-1}$. Dickey and Fuller derived the limit distribution of the t-ratio $\hat{\tau}_\mu := \hat{\phi}_1^*/\widehat{\text{SE}}\left(\hat{\phi}_1^*\right)$ as $n \to \infty$ under the unit root assumption $\phi_1^* = 0$, which allows a test of the null hypothesis $H_0 : \phi_1 = 1$. The details are provided in Brockwell and Davis (2002).

References

Andersen, T., Bollerslev, T., Diebold, F. and Ebens, H. (2001) The distribution of realized stock return volatility, *Journal of Financial Economics*, **61**, 43–76.

Baillie, R. T. (1996) Long memory processes and fractional integration in econometrics, *Journal of Econometrics*, **73**, 5–59.

Bardet, J.-M. and Bertrand, P. (2007) Identification of the multiscale fractional brownian motion with biomechanical applications, *Journal of Time Series Analysis*, **28**, 1–52.

Beran, J. (1994) *Statistics for Long-memory Processes*, Vol. 61 of *Monographs on Statistics and Applied Probability*, Chapman and Hall, New York.

Beran, J., Yuanhua, F., Ghosh, S. and Kulik, R. (2013) *Long-Memory Processes: Probabilistic Properties and Statistical Methods*, Springer, 2013 edn.

Biagini, F., Hu, Y., Øksendal, B. and Zhang, T. (2008) *Stochastic Calculus for Fractional Brownian Motion and Applications (Probability and Its Applications)*, Springer London, softcover reprint of hardcover 1st ed. 2008 edn.

Björk, T. and Hult, H. (2005) A note on wick products and the fractional black-scholes model, *Finance and Stochastics*, **9**, 197–209, doi:10.1007/s00780-004-0144-5.

Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities, *Journal of political economy*, **81**, 637.

Brockwell, P. and Davis, R. (1991) *Time Series: Theory and Methods: Theory and Methods*, Springer series in statistics, Springer-Verlag.

Brockwell, P. and Davis, R. (2002) *Introduction to Time Series and Forecasting*, Springer.

Chen, Y., Sun, R. and Zhou, A. (2008) An overview of fractional order signal processing (FOSP) techniques, in *2007 Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference, DETC2007*, Vol. 5 PART B, pp. 1205–1222.

Coeurrjolly, J. F. (2000) Simulation and identification of the fractional brownian motion: A bibliographical and comparative study, *Journal of Statistical Software*, **5**, 1–53.

Das, S. and Pan, I. (2011) *Fractional Order Signal Processing: Introductory Concepts and Applications*, SpringerBriefs in Applied Sciences and Technology, Springer.

Davie, A. M. (2008) Differential equations driven by rough paths: An approach via discrete approximation, *Applied Mathematics Research eXpress*, **2008**.

Dickey, D. A. and Fuller, W. A. (1979) Distribution of the estimators for autoregressive time series with a unit root, *Journal of the American Statistical Association*, **74**, 427–431.

Duncan, T. E., Hu, Y. and Pasik-Duncan, B. (2000) Stochastic calculus for fractional brownian motion I. theory, *SIAM Journal on Control and Optimization*, **38**, 582–612.

Elliott, R. J. and Van Der Hoek, J. (2003) A general fractional white noise theory and applications to finance, *Mathematical Finance*, **13**, 301–330.

Geweke, J. and Porter-Hudak, S. (1983) The estimation and application of long memory time series models, *Journal of Time Series Analysis*, **4**, 221–238.

Gradinaru, M. and Nourdin, I. (2009) Milstein's type schemes for fractional sdes, *Ann. Inst. H. Poincar Probab. Statist.*, **45**, 1085–1098.

Granger, C. W. J. (1980) Long-memory relationships and the aggregation of dynamic models, *Econometrics*, **14**, 227–238.

Granger, C. W. J. and Joyeux, R. (1980) An introduction to long-memory time series models and fractional differencing, *J. Time Ser. Anal.*, **1**, 15–29.

Gubinelli, M. (2004) Controlling rough paths, *Journal of Functional Analysis*, **216**, 86–140.

Hansen, P. R. and Lunde, A. (2005) A realized variance for the whole day based on intermittent high-frequency data, *Journal of Financial Econometrics*, **3**, 525–554.

Hosking, J. R. (1981) Fractional differencing, *Biometrika*, **68**, 165–176.

Hu, Y. and Øksendal, B. (2003) Fractional white noise calculus and applications to finance, *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, **06**, 1–32.

Hurst, H. (1951) Long-term storage capacity of reservoirs, *Transactions of the American Society of Civil Engineers*, **116**, 770–799.

Kloeden, P. E. and Platen, E. (2010) *Numerical Solution of Stochastic Differential Equations (Stochastic Modelling and Applied Probability)*, Springer.

Kozlowski, A. (2012) Option prices under the fractional black-scholes model, *The Wolfram Demonstrations Project (retrieved on 2012-01-5)*.

Kumar, D. (2014) *Long Memory in the Volatility of Indian Financial Market: An Empirical Analysis Based on Indian Data*, Anchor Academic Publishing.

Künsch, H. (1987) Statistical aspects of self-similar processes, in *Proceedings of the 1st World Congress of the Bernoulli Society, Vol. 1 (Tashkent, 1986)*, pp. 67–74, VNU Sci. Press, Utrecht.

Lo, A. (1991) Long-term memory in stock market prices, *Econometrica*, **59**, 1279–313.

Lyons, T. and Qian, Z. (2003) *System Control and Rough Paths (Oxford Mathematical Monographs)*, Oxford University Press, U.S.A.

Mandelbrot, B. (1972) Statistical methodology for nonperiodic cycles: From the covariance to R/S analysis, *Annals of Economic and Social Measurement*, **1**, 259–290.

Mandelbrot, B. B. and Van Ness, J. W. (1968) Fractional Brownian motions, fractional noises and applications, *SIAM Review*, **10**, 422–437.

Masuda, H. and Morimoto, T. (2012) Optimal weight for realized variance based on intermittent high-frequency data, *Japanese Economic Review*, **63**, 497–527.

Minotani, C. (2001) *Kinyu deta no tokei bunseki*, Toyo Keizai Shinposha, Tokyo (in Japanese).

Mishura, Y. and Shevchenko, G. (2008) The rate of convergence for euler approximations of solutions of stochastic differential equations driven by fractional brownian motion, *Stochastics An International Journal of Probability and Stochastic Processes*, **80**, 489–511.

Neuenkirch, A. and Nourdin, I. (2007) Exact rate of convergence of some approximation schemes associated to sdes driven by a fractional brownian motion, *Journal of Theoretical Probability*, **20**, 871–899, doi:10.1007/s10959-007-0083-0.

Neuenkirch, A., Tindel, S. and Unterberger, J. (2010) Discretizing the fractional lvy area, *Stochastic Processes and their Applications*, **120**, 223–254.

Norros, I., Valkeila, E. and Virtamo, J. (1999) An elementary approach to a girsanov formula and other analytical results on fractional brownian motions, *Bernoulli*, **5**, 571.

Peters, E. E. (1994) *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics (Wiley Finance)*, Wiley, 1 edn.

Reisen, V. A. (1994) Estimation of the fractional difference parameter in the arfima(p,d,q) model using the smoothed periodogram, *Journal of Time Series Analysis*, **15**, 335–350.

Robinson, P. M. (1995) Gaussian semiparametric estimation of long range dependence, *Ann. Statist.*, **23**, 1630–1661.

Rogers, L. C. G. (1997) Arbitrage with fractional brownian motion, *Mathematical Finance*, **7**, 95–105.

Rostek, S. (2009) *Option Pricing in Fractional Brownian Markets (Lecture Notes in Economics and Mathematical Systems)*, Springer Berlin Heidelberg, 1st edition. edn.

Rostek, S. and Schöbel, R. (2013) A note on the use of fractional brownian motion for financial modeling, *Economic Modelling*, **30**, 30–35.

Shimotsu, K. (2010) Exact local whittle estimation of fractional integration with unknown mean and time trend, *Econometric Theory*, **26**, 501–540.

Shimotsu, K. and Phillips, P. C. B. (2005) Exact local whittle estimation of fractional integration, *Ann. Statist.*, **33**, 1890–1933.

Shiryayev, A. (1998) On arbitrage and replication for fractal models, research report 20, maphysto, Tech. rep., Department of Mathematical Sciences, University of Aarhus.

Sottinen, T. and Esko, V. (2003) On arbitrage and replication in the fractional blackscholes pricing model, *Statistics and Risk Modeling*, **21**, 93–108.

Teverovsky, V., Taqqu, M. S. and Willinger, W. (1999) A critical look at lo's modified r/s statistic, *Journal of Statistical Planning and Inference*, **80**, 211–227.