Phase Transition with Noise for Time Series of JP¥/US$ Exchange Rate

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Any factor that exert the influence on the process of JP¥/US$ exchanges rate has been discussed widely by researchers in academic and business fields. In some cases, for instance, the cause of changes was explained by many macro-economical factors and they tried to explain by stochastic and statistical model. However, a heavy fall was necessarily expressed. We attempt to express the change of the JP¥/US$ exchange rate in addition to it by a phase transition model with noise. We carried out some empirical studies with a fractal analysis and statistical tests for verifying whether the series are random and generated by the stochastic model. We found that time series on difference of JP¥/US$ exchange rates satisfy these properties. We composed a stochastic differential equation as the model that might describe variability of the difference. We regressed the potential function in the sense of the time average on the stochastic differential equation with use of graphs of polynomials. This paper also presents the necessity of the number of order on the polynomials with consideration on stability of singular points for perturbation, what we call structural stability in Topology. With integration of the series reproduced by our model, we obtained series as numerical simulations of the JP¥/US$ exchange rate. We compare these series with the time series data of the JP¥/US$ exchange rate to evaluate our stochastic differential equation.
Key words: Central Rate, Stochastic Differential Equation (SDE), Stability on the Form of Probability Density Function

1. Introduction

According to Economics, it is said that the JP¥/US$ exchange rate is decided by demand and supply in the foreign exchange market for a day (Stiglitz, 1993). So many discussion are arisen on the factors exerts the changes of JP¥/US$ exchange rate in the finance and economic fields. To explain this, for example, Balance of trade and Interest rate difference were used. As the number of the factor increase on the regression, then multicolinearity occurs and the estimation fails. So far, we have discussed how many factors should describe the process of the JP¥/US$ exchange rate and the JP¥/US$ of the exchange rate seemed to be composed not of randomness but of chaos as reviewed below (Yoshimori *et al.*, 1999, 2003). We concluded that the exchange rate on time series date makes up the deterministic dynamical-system. But the method has not been established yet, which composes the dynamical equation for the time series data.

The author carried out empirical studies employing statistical analysis to address the problem of finding the number of variables describing the monthly JP¥/US$ exchange rate (Takada, 2013). Principal component analysis extracts essential variables in economic fundamentals such as and the money supply in Japan/US. We herein examine whether the foreign currency exchange determination can be described by a mathematical model with use of its structure and a noise term. The author assumes that the stochastic process indicates the changes of economic fundamentals.

Comparing time series data with the time series of their difference (differenced time series), it generally seems to be difficult to describe the differenced time series with use of a deterministic dynamical-system because the ruggedness increase on the differenced time series and their differentiability is lost (see Fig. 1). We proposed a method to construct a stochastic differential equation (SDE) on time series data with Markov property (Takada *et al.*, 2001). Our mathematical model, the SDE, is obtained as necessary condition with the data.

Our method treats entire of stochastic processes that generate observed data as a solution in this paper.

(i) Based on a certain assumption condition, we assume an entire set of thinkable processes.

(ii) We translate the fact on the data into a mathematical expression as an assumption.

(iii) The set is reduced to a subset of the entire set with the process (ii).

The subset is a family of stochastic processes. A mathematical model is extracted with recurrences of (ii) and (iii). It has logical significance because the model must be one obtained with this method, which generates the time series data. We call the method reducing method after the process (iii) in this paper.

Here, entire of stochastic processes contains not only

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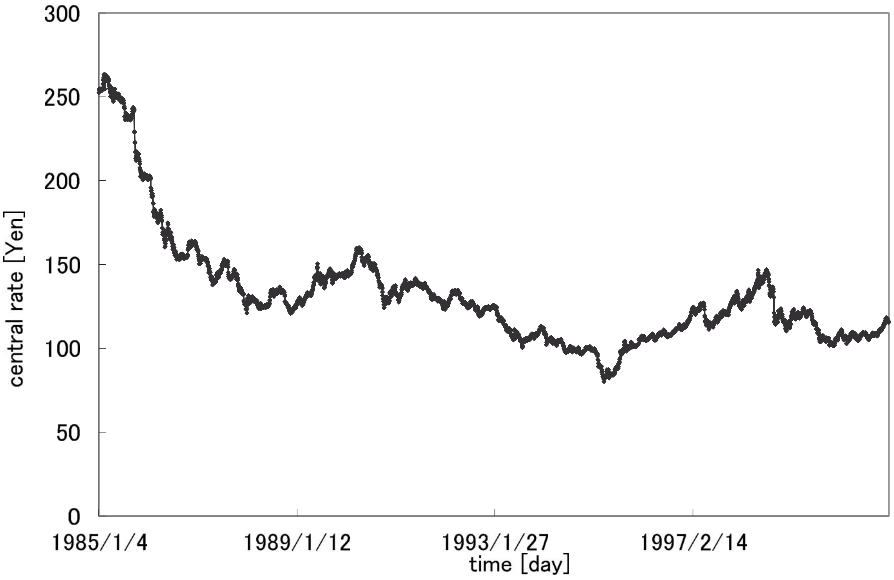

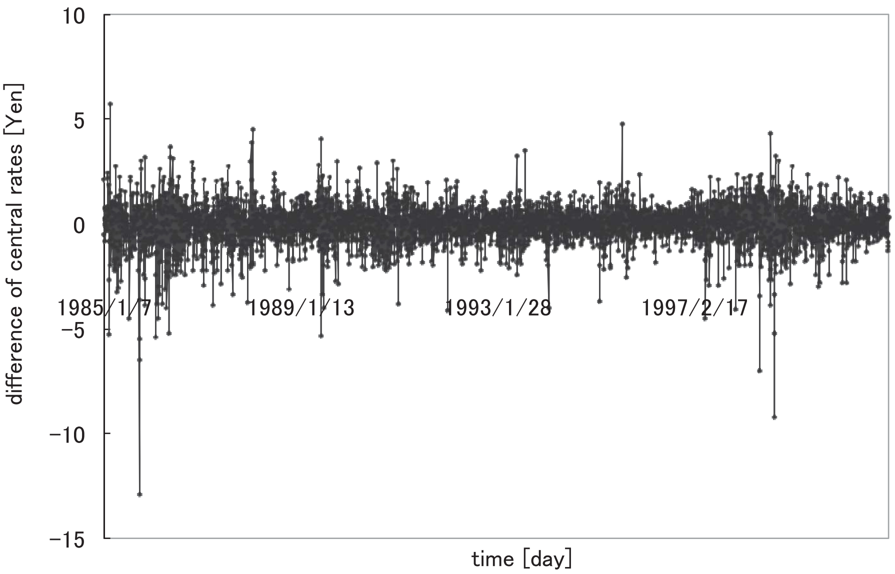

Fig. 1. This figure shows time series data on the US$/JP¥ (USD/JPY) exchange rate. (upper) Time series data on daily central rates, sampled from November 1, 1985, to January 31, 2001. (bottom) Time series data showing the differences in the central rates shown in upper.

Ornstein-Uhlembeck process but also many other various processes. We compose a SDE as the model that may describe variability of the difference of the JP¥/US$ exchange rate. We empirically study weakly stability on the differenced time series. We estimate a potential function in the sense of the time average on this SDE by the probability density function. We regress the potential function with use of graphs of polynomials. We lead that the potential function is not parabolic function which describes Ornstein-Uhlembeck process, but it must be forth order with our analysis for the differenced time series data of the JP¥/US$ exchange rate.

2. Historical and Time Series Data

We obtained time series data of the central rate from Bloomberg (see Fig. 1). The tick of the exchange rate is a market price with the most dealings for a certain interval. In this paper, we used here daily data.

Plaza Accord of September 19, 1985 and is believed historically important in international finance studies (Solomon, 1999). We obtained time series data of the central rate after the Agreement from Federal Reserved Bank of New York. The data $\left\{\varsigma_j\right\}_{j=1985.11.1}^{2001.1.31}$ were daily sampled from November 1, 1985 to January 31, 2001. The reason why we have chosen the beginning point of data on November 1, 1985 is to remove the reflex process by Plaza Agreement. The author assumes that variations of the exchange rate in this term before the Sept. 11 attacks are generated by a stationary process. The economic system would be affected by the sudden eruption of violence.

We estimated correlation dimensions of the attractor on which a dynamical system generates sampled time series of central rates and their difference. The difference of the central rates is defined as follows:

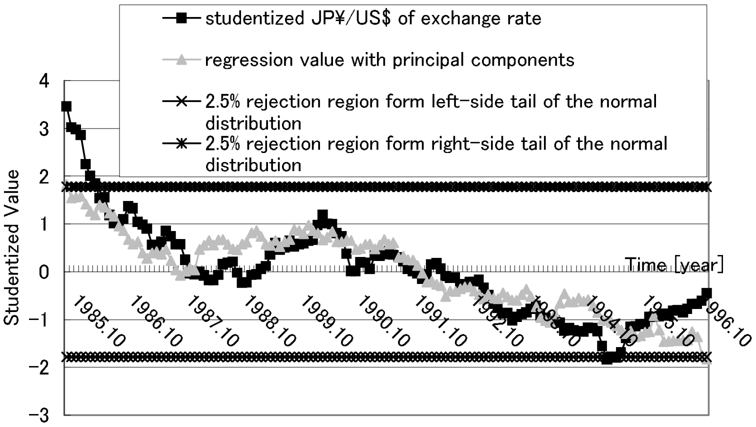

Fig. 2. This figure compares the central rate time series data and the series produced by the multiple regression model. The model uses two principle components of the Japanese Wholesale Price Index, the Japanese Gold and Foreign Exchange Reserves, the Japanese Short-term Interest Rate, the Japanese Current Balance, the U.S. Money Supply, and the U.S. Index of Industrial Production.

$$v_j = \varsigma_{j+1} - \varsigma_j \ \ (j = 1985.11.1 - 2001.1.30).$$

We assume the set of these days to be K at the following, when the central rate is sampled.

We analyzed the time series after the Agreement in this paper. We calculated the correlation dimension of each expected attractor on which a dynamical system generates time series of the central rate. We have concluded that it is necessary for two essential variables, principal components z_1, z_2,

$$\hat{\varsigma}_t = 1.58z_1(t) + 0.37z_2(t) \tag{1}$$

to regress the time series $\{\varsigma_j\}_{t\in K}$ with the multiple regression analysis where ς_t is a regression value at time t (Matsugi *et al.*, 2001). The determination coefficient on the multiple regression formula was more than 0.80. However, we consider squares of error sum at both sides of the interval remarkable (see Fig. 2); therefore the linear model is not enough to express the central rates. Our purpose of this study is what kind of mathematical models describe the variations of central rates and their difference.

The differenced time series seem to be complex in comparing with time series data of the central rates (Fig. 1). The variations on the difference of them are more furious than the time series data and seem to be random. The differential coefficients do not necessarily converge to bound values. We mathematically show the fact with calculation on their correlation dimension and run tests in next section. Based on the property of the differenced time series, we apply the method proposed to construct a SDE for a description of the JP¥/US$ exchange rate (Takada *et al.*, 2001). We also verify assumptions of the reducing method and evaluate the SDE with the numerical simulation here.

3. Empirical Study for Differenced Time Series and Their Randomness

We mathematically compare the time series data of the central rate with the differenced time series. We show that the later is more complex than the former with calculation of the correlation dimensions and statistics.

3.1 A method of fractal analysis

Using the embedology, we calculated the correlation dimension of the reconstructed attractor on which a dynamical system generates time series of JP¥/US$ exchange rates and their difference. We compose the following delay coordinates that are m dimension vector system:

$$\boldsymbol{x}_j = \begin{pmatrix} x_j & x_{j+\tau} & \cdots & x_{j+(m-1)\tau} \end{pmatrix}, \tag{2}$$

where τ is the sampling time and negative sign of τ, 2τ, ..., $(m–1)\tau$ mean delay time. An orbit with delay coordinates (2) is embedded in m dimensional phase space (see Appendix A). Assuming that a dynamical system on n dimensional compact manifold generates time series, it is possible to think that an attractor on the dynamical system can be reconstructed as the orbit if the time series is observed for a long while. According to Takens' theorem (Takens, 1981), a transform from the time series to the attractor is embedding on the condition of $m \geq 2n + 1$ (Appendix A). The attractor and the manifold are isomorphism.

It is important on the theorem mentioned above that the transformation Φ onto the delay coordinate is embedding. If the transformation Φ is embedding, it is immersion; therefore the fractal dimension of the attractor is preserved (Ikeguchi and Matozaki, 1996). Calculating the fractal dimension of the attractor with delay coordinates; we can analyze the geometrical structure of the manifold M.

The phase space in which we have embedded coordinates with delay time can be delimited with hyperspheres whose radius is assumed to be ε. The number of these hyperspheres is assumed to be $n(\varepsilon)$. The probability that the embedded points are contained in that sphere whose center is x_j is assumed to be P_j. The correlation dimension of the attractor with delay coordinates is defined as follows:

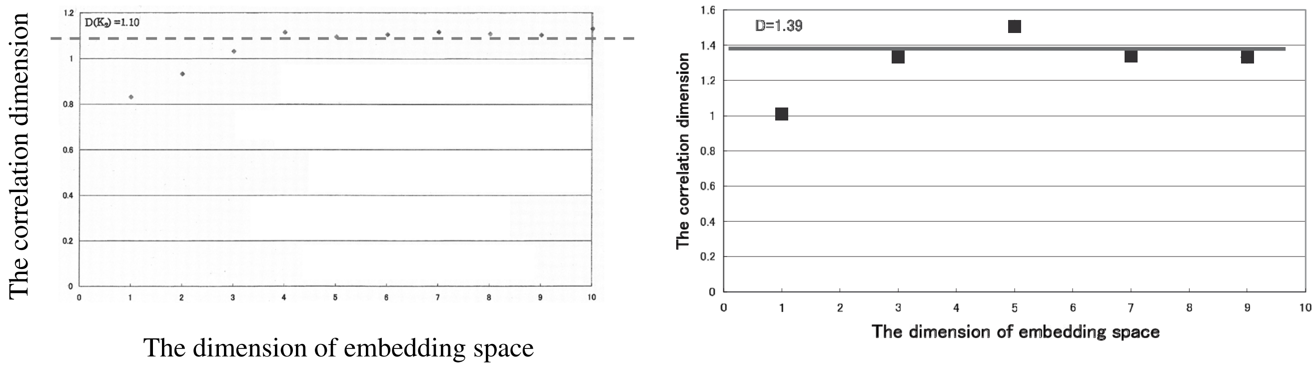

Fig. 3. This figure shows the relationship between the dimension of the embedding space and the correlation dimension of the reconstructed attractor on which a dynamic system generates the JP¥/US$ exchange rate time series. The time series are sampled on a daily basis in left and on a monthly basis in right.

$$D_2 = \lim_{\varepsilon \to 0} \frac{\log\left(\sum_{i=1}^{n(\varepsilon)} P_i^2\right)}{\log \varepsilon} = \lim_{\varepsilon \to 0} \frac{\log C(\varepsilon)}{\log \varepsilon}, \quad (3)$$

where $C(\varepsilon)$ is the correlation integral (Appendix A).

In general, it seems to be difficult to calculate the theoretical limit on Eq. (3) because of the resolution- limit of the observation. For instance, the resolution-limit is 0.05 Yen as a lower bound on the amount of dealings of this exchange rate. Based on this formula (3), we calculated regression coefficients $D_2(m)$ as the correlation dimension of the attractor with delay coordinates on each embedding space with the following method and obtained the presumed correlation dimension for it.

Calculation Method (Sano and Sawada, 1985) for each embedding space whose dimension is m.
1) We integrated $C(\varepsilon)$, the correlation integration.
2) We regressed a line on the figure of $\log\varepsilon$ versus $\log C(\varepsilon)$ at the range $-4 \le \log C(\varepsilon) \le -2$. This regression coefficient is assumed to be $D_2(m)$.
3) The calculated $D_2(m)$ is saturated in the embedding space with enough high dimension ($m \ge 2n + 1$) from the theorem. The saturated value is assumed to be a correlation dimension D_2.

3.2 Results of fractal analysis

With this Calculation Method, we estimated $D_2(m)$ in each m dimensional embedding space (m = 1, ..., 10). Fractal dimension of the attractor could be calculated with increase of the dimension of embedding space because the embedding space can sufficiently include the attractor. Based on the theorem (Appendix A), $D_2(m)$ are saturated and we can accurately estimate the correlation dimension of the attractor as the partial manifold in higher dimensional embedding space. If the time series were generated by a dynamical system on 2-dimensional compact manifold, we can estimate the correlation dimension of it in 5-dimensional embedding space at least.

In Fig. 3.1, we show the relationship between dimensions of the embedding space and the reconstructed attractor on which a dynamical system generates the time series of the JP¥/US$ exchange rate. $D_2(m)$ converged to a bounded value 1.10 in more than 5 dimensional embedding spaces. That is, it is possible for 2 independent variables at most to describe the time series. In addition, we show the relationship between dimensions of the embedding space and the attractor on which a dynamical system generates another time series sampled monthly in Fig. 3.2 to verify the regularity on the time series of the JP¥/US$ exchange rate in accordance with the following experience rule.

Empirical Law (Ramsey and Yuan) If the measured values of the correlation dimension increase with an increase in the number of sampling data, the time series can be considered to be random numbers and if the values decrease, we can believe that they are generated by a chaos system.

As a result, $D_2(m)$ converged to a bounded value 1.39 in more than 5 dimensional embedding spaces. We admitted that the measured values of the correlation dimension decreased as the number of sampling grew. We have believed that a chaos system generates the time series. That is, it is mathematically suggested that a deterministic equation can describe the time series of the JP¥/US$ exchange rate.

We also show the relationships between dimensions of the embedding space and the attractor on which a dynamical system generates the differenced time series of the central rates. In Fig. 4, we can compare the relationship on the time series sampled daily with monthly. Both $D_2(m)$ did not saturate in less than 10 dimensional embedding spaces. We also admitted that the measured values of the correlation dimension increased as the number of sampling grew. We have concluded that a stochastic process generates the differenced time series.

3.3 Statistical tests

With statistical tests, we verify whether we consider time series of JP¥/US$ exchange rates and their difference to be random. We use run tests and rank tests that are non-parametric methods (Appendix C). The run test is used for verification of hypothesis "the time series is random". A significant point of verification on the number

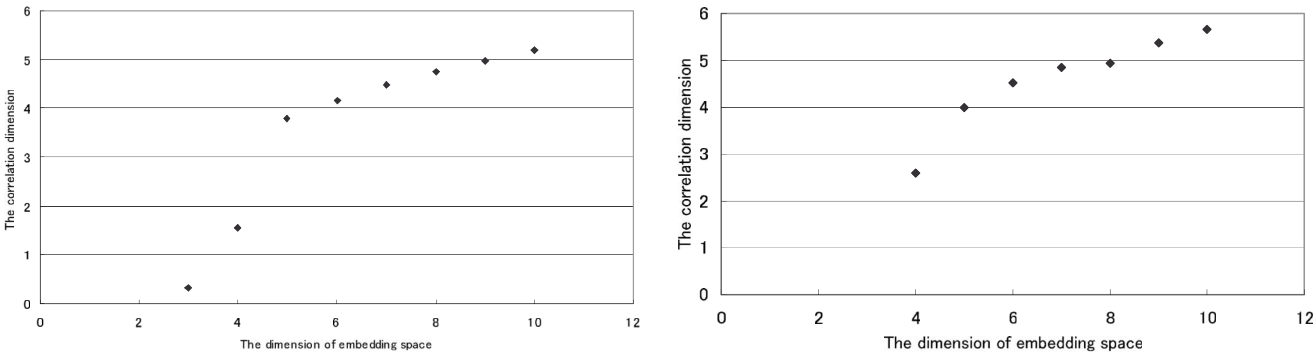

Fig. 4. This figure shows the relationship between the dimension of the embedding space and the correlation dimension of the attractor on which a dynamic system generates the differenced time series of the central rates. The differenced time series are sampled on a daily basis in left and on a monthly basis in right.

of run is introduced to the run test. This method is competent for a large number of samples. The rank test is used for verification of the same hypothesis "the time series is random without tendency of increase or decrease at a uniform pace". Both statistical test values conform to normal distributions. The reference value is 1.96 for a significant level 0.05.

We calculated the statistical test values of the time series sampled monthly and daily, four kinds of the time series, with statistical test values z_L, z_Q. As the values are shown in Table 1, the null hypothesis was rejected except for the differenced time series of the central rates with rank tests. Especially, the null hypothesis was also rejected on the differenced time series sampled monthly with run tests.

Methods of statistical tests could be applied for time series of the JP¥/US$ exchange rates and their difference. We could suggest randomness on the differenced time series data sampled daily with use of plural methods.

4. Stochastic Model on Differential Process of JP¥/US$ Exchange Rate

In this section, we review the reducing method mentioned in Section 2 and construct a mathematical model for time series data. We assume that there is a process generating the time series of the JP¥/US$ exchange rate. We obtain the SDE for description of the process. We also verify assumptions of the reducing method.

4.1 Review the reducing method to construct a stochastic differential equation

We showed the following reducing method to construct a mathematical model, on time series data (Takada *et al.*, 2001). The model is based on the stochastic theory. Definition on this reducing method is as follows.

Definition 2 This reducing method covers a continuous stochastic process on state space Ω. We assume a conditional probability to be $P(x|y, t)$ in such case as the random process $X(t)$ satisfies conditions of $X(0) = y(\in \Omega)$ and $X(t) = x(\in \Omega)$.

Assumption 1 The described stochastic process is one of the Markov processes (Markovian).

Table 1. This table shows the results of run tests and rank tests used to determine the randomness in each time series. The figures are the statistical test values of the time series and of the differenced time series, sampled on a monthly and a daily basis, respectively.

Time series/tests	Rank	Run
Time series sampled monthly	–8.42	–10.63
Time series sampled daily	–45.03	–57.20
Differenced variations sampled monthly	1.00	–2.17
Differenced variations sampled daily	1.54	1.85

Assumption 2 $X(t)$ is not an anomalous process that rapidly extends far in a short time.

Assuming Markov property and Physical demand on the process $X(t)$ mentioned above, the process must be described by the Fokker-Plank equation (Appendix B). On the contrast, the SDE:

$$\frac{dv(t)}{dt} = a(v) + b(v)F(t), \tag{4}$$

corresponds to the Fokker-Plank equation (FPE) with method of the stochastic Liouville equation (Kubo, 1963). This FPE goes over into normalized FPE with permutation of variables in accordance of Stratonovich's rule or condition of $b \equiv 1$. With this condition, this normalized stochastic differential equation (NSDE) corresponds to a normalized FPE with calculation of moments of transition probability uniquely (Goel and Richter, 1978). The problem finding the NSDE corresponding to a given FPE has only one solution though it does not have a unique solution in the family of Eq. (4) or general stochastic differential equations (Appendix B). The following relationship was shown in the reducing method to construct the NSDE for description of time series obtained with discreet observation of the process $X(t)$.

$$g(v) = C\exp[-2U(v)]. \tag{5}$$

This formula shows the relationship between a stationary probability density function $g(v)$ on the FPE and time-

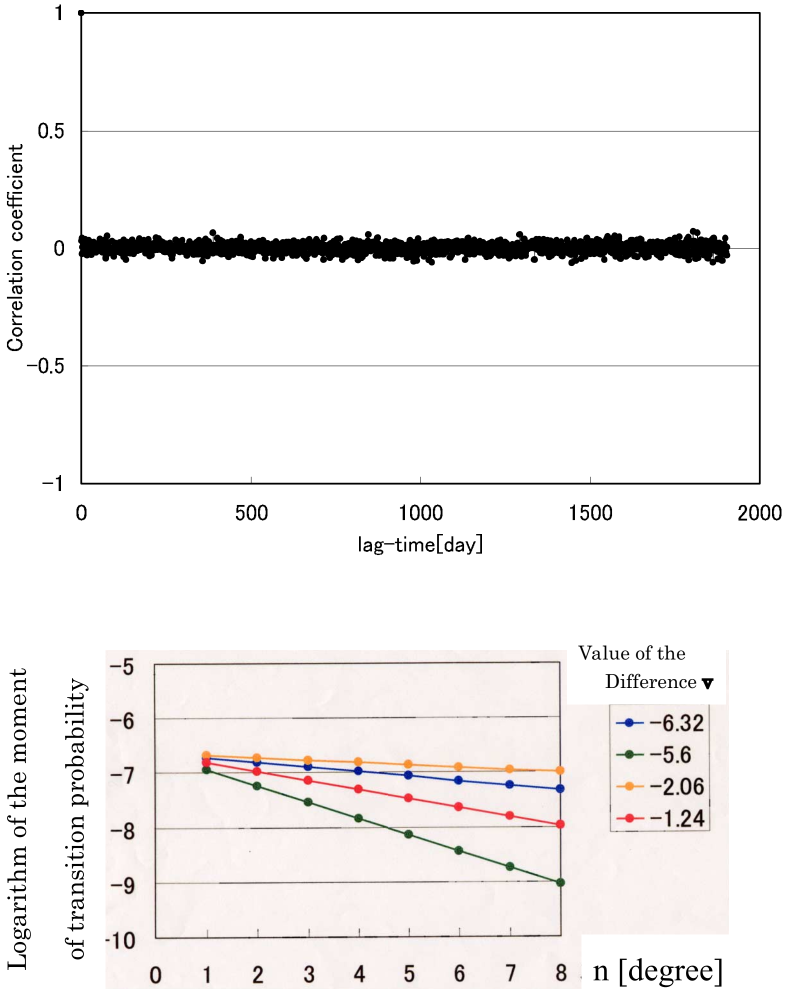

Fig. 5. We empirically study the differenced time series of the central rate, sampled on a daily basis. The autocorrelation function decreases exponentially, and falls below $1/e$ (upper). Typical numerical integrations of the moments of the transition probability converge to zero as n increases (bottom).

averaged potential function $U(v)$ on Eq. (4) (Appendix B). Therefore, the following proposition is obtained by differentiating both sides of Eq. (5) (Takada *et al.*, 2001).

Theorem 1 The number of stationary points and their positions z on the potential function $U(v)$ and on the stationary probability density function $g(v)$ are corresponding.

Many of data that expected to be described with a SDE seem to be in a certain dynamical equilibrium state. The stationary probability density function $g(v)$ can be obtained with a normalized histogram of time series practically. With Proposition 1, the form of the graph of $\log g(v)$ corresponds to the form of the graph of the potential function $U(v)$ in the meaning of time average on Eq. (4) that describes the process. We can consider NSDE to be the mathematical model for description of them, which is constructed with the time series data.

4.2 Empirical Analysis of assumptions with differenced time series data

The above-mentioned method can be applied to a construction of the mathematical model corresponding to the time series on differential of the central rates. We verify suitability for the assumptions in Subsection 4.1 with the differenced time series data $\{v_t\}_{t\in K}$. However, it is necessary for us to normalize from $\{v_t\}_{t\in K}$ into $\{\tilde{v}_t\}_{t\in K}$ by the following transformation:

$$v_t \mapsto \tilde{v}_t = \frac{v_t - \langle v_t \rangle}{\sigma(\dot{v}_t)}, \tag{6}$$

to construct the SDE (4) as $b = 1$ (NSDE), that is a mathematical model for description of time series. $\sigma(\dot{v}_t)$ is a standard deviation on accelerations of the central rates.

It has been calculated that the stationary autocorrelation function on Markovian decays exponentially (Gardiner, 1983). Assumption 1 and 2 are examined with each verification:

Empirical Analysis of Assumption 1 The autocorrelation function of time series $\rho_{vv}(k)$ decreases exponentially and falls below $1/e$ after that, where k means lag-time.

Empirical Analysis of Assumption 2 The numerical

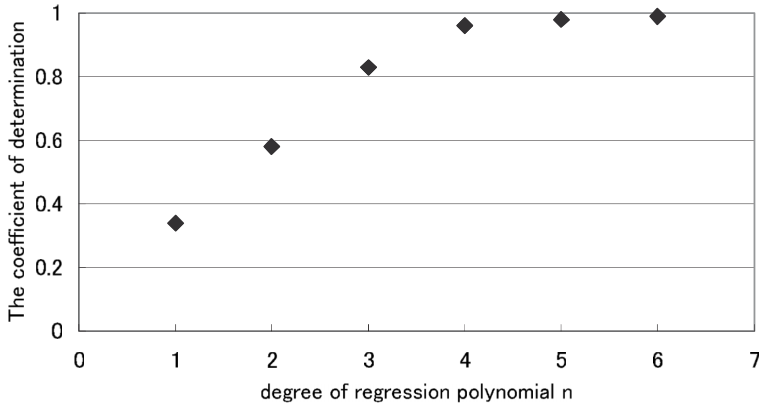

Fig. 6. We regress the logarithmic histograms on the differenced time series with the graphs of each polynomial of degree n. We show the relation between n and the coefficient of determination R on each regression of the logarithmic histogram.

integration of the moments of transition probability:

$$M_n(v) = \sum_{u \in \Omega} (u - \tilde{v})^n P_{u,\tilde{v}}(t) W / \tau \tag{7}$$

$$\text{s.t. } P_{u,\tilde{v}}(\tau) = \left. \frac{\#\{(V_t = \tilde{v}) \cap (V_{t+\tau} = u)\}}{\#(V_t = \tilde{v})} \right|_{t \in K},$$

converges zero as n increases, where character # means "the number of", W is the minimal scale of the dealing (0.05 ¥/$) and τ is the sampling time (1 [day]).

The results of empirical analysis of Assumption 1 and Assumption 2 are shown in Fig. 5. Then, we could believe that Assumption 1 and 2 were suitable to the obtained data. We also assumed that the differenced time series were a kind of Markov process. In the following, we treat the differential process of the JP¥/US$ exchange rate as a Markov process.

4.3 Construction of the approximate mathematical model

We assumed that the process of JP¥/US$ exchange rate was in a certain dynamical equilibrium state. Based on the reducing method in Subsection 4.1, the mathematical model can be constructed. Before we lead a mathematical model on the process of the JP¥/US$ exchange rate with the reducing method, we have standardized time series $\{v_t\}_{t\in K}$. These standardized time series are assumed to be $\{\tilde{v}_t\}_{t\in K}$. We have regressed logarithmic histograms on this differenced time series $\{\tilde{v}_t\}_{t\in K}$ with graphs of each polynomial of degree n. We calculated the coefficient of determination R (Kendall and Stuart, 1958). The following NSDE can be led approximately as a mathematical model on the process of difference of the JP¥/US$ exchange rates with the regression polynomial:

$$\frac{d\tilde{v}}{dt} = \frac{1}{2} \sum_{k=1}^{n} k a_k \tilde{v}^{k-1} + F(t), \tag{8}$$

where $F(t)$ is the standardized fluctuating force generated with the Gauss type stochastic process. a_k are regression coefficients of the polynomial for k (natural numbers) on the regression with the least square method. The relation between n and R on each regression of the logarithmic histogram is shown in Fig. 6. R tended to 0.9 at n = 4 and was sufficiently large there.

Demand on Stochastic

It must be more than 0.9; the value R shows the suitability for the regression curve and the relative importance on correlations of different magnitudes (Robert and James, 1969).

Demand on Geometry

The time-averaged potential function should be structurally stable with consideration of the perturbation exerted on the control-system.

With the relation obtained by differentiating both sides of Eq. (B.4), we could believe that each time-averaged equilibrium space on the NSDE was approximated sufficiently with the graph of a polynomial of degree 3 (Appendix B). Based on the Demand on Geometry, the time-averaged potential function must be approximated by the polynomial of 4 degree because the regression polynomial that is more than 5 degree has degenerate singular points or more than 3 minimal points.

The logarithmic histogram on the time series $\{\tilde{v}_t\}_{t\in K}$ and its typical regression with the graph of a polynomial of degree 4 is shown in Fig. 7. Here, coefficients of the regression polynomial were as follows:

$$\frac{d\tilde{v}(t)}{dt} = \frac{1}{2} \sum_{k=1}^{4} k a_k \tilde{v}^{k-1} + F(t), \tag{9}$$

where $(a_1\ a_2\ a_3\ a_4) = (0.18\ {-0.43}\ {-0.12}\ {-0.0070})$.

5. Simulations and Evaluation

We can easily formulate a difference equation into the first-order differential equation; therefore, Eq. (9) is an advantageous expression for the numerical computation. The computations give numerical solutions of the SDE,

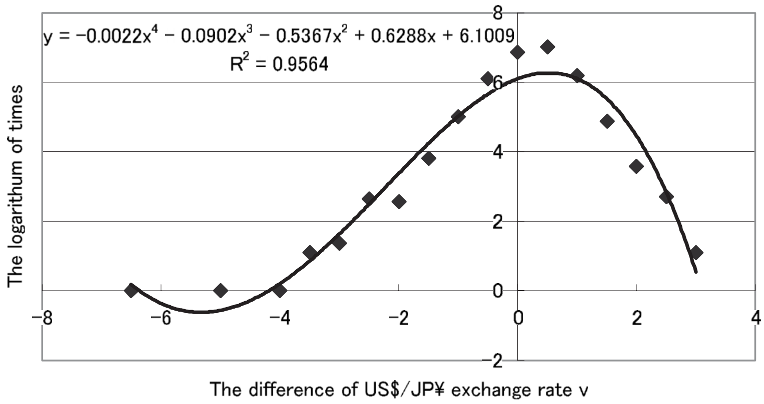

Fig. 7. We show the logarithmic histogram of the differenced time series and its typical regression with the graph of a polynomial of degree 4.

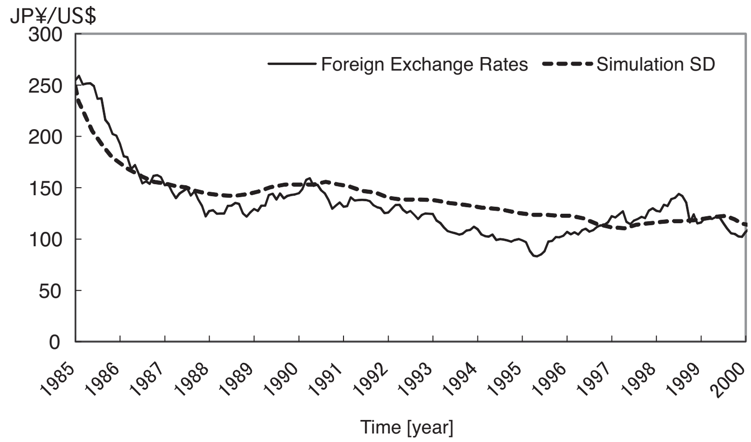

Fig. 8. We compare the central rate time series and the time series generated by the numerical simulation.

which are not probability density functions but movement of the variable. We can compare the solutions with the time series data and verify the reducing method to construct the SDE.

5.1 Numerical simulations of our mathematical model

We proposed a construction method of the mathematical model on time series data with Markov property and obtained a SDE as a mathematical model. To be concrete, we applied the method to the differenced time series data of the central rate $\{\tilde{v}_t\}_{t\in K}$ and constructed a mathematical model (9) of $\{\tilde{v}_t\}_{t\in K}$. In this section, we calculate the numerical simulation on Eq. (9) and evaluate the SDE obtained for the description of the JP¥/US$ exchange rate time series in Subsection 4.3. The SDE cannot describe the central rates but the difference of them; therefore it is necessary to integrate with a time valuable for the description. We reproduce the central rate with the integration.

The initial condition −1.12 was given by the normalized difference of the time series data of the central rate. We used pseudo random number series obtained by the linear congruential method (Lehmer, 1951). Series of the pseudo random numbers whose domain was [0, 1) were standardized respectively because of the standard deviation on the Gaussian white noise. We introduced these series into the white noise terms on the difference equation that we rewrote Eq. (9) into. We calculated the difference equation with the Runnge-Kutta-Gill formula by the time step 1. We transformed values on the time series obtained with this method in accordance of the inverse transformation of Eq. (6). Thus, we regenerate the differenced time series of the central rates $\{\hat{v}_t\}_{t\in K}$ and integrated them on each time step to compare the time series data of the central rate. That is, we regressed variations of the central rate with use of the following series:

$$\hat{\varsigma}_{j+1} = \varsigma_{1985.11.1} + \sum_{i=1985.11.1}^{j} \hat{v}_i \quad (j = 1985.11.1 - 2001.1.30). \tag{10}$$

We show this time series obtained with the method proposed in this paper and time series of the central rate (Fig. 8). We compare these series and evaluate the NSDE (9) in the next section.

5.2 Evaluation for the mathematical model

In the previous researches, the time series on the central rate were described by the multiple regressions with economic variables. We compare these previous regression models with the SDE (9) here.

Their coefficient of determination R^2 is about 0.72 with-

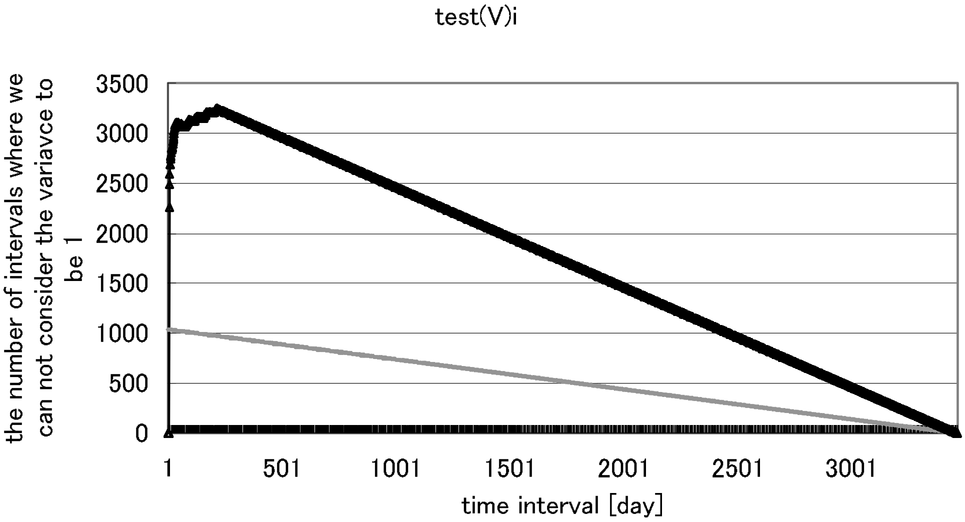

Fig. 9. This figure shows that the inequalities $(V)_i < 2.24$ do not hold for any M, except for values expressed for a period of less than a week.

out spick functions that express unexpected political speech (Moriyama, 1999). We showed our value R^2 in Section 2. Obviously, the magnitude for the explanation of the regression model had been improved by our regression model with used of two essential components, the first principal component of six economic variables and so on. We show the comparison on the time series of the central rate with the time series reproduced by our regression model on Fig. 1. We found that the error at both ends was comparatively large on those figure. We reflected that the linear sum has been mostly used as regression models in economics and believed that we should try to compose the non-linear model for description on the time series of the central rate.

We obtained the mathematical model for the process on the differential of the central rate $v(t)$ in accordance of Assumption 1 and Assumption 2. The mathematical model is a SDE. We evaluate the magnitude for the explanation with the SDE here. We composed the SDE (9) that describes the process on the difference of the JP¥/US$ exchange rate. The first term on the right hand of the SDE is nonlinear function that depends on the autonomic variable. We specially emphasize on the non-linearity of the autonomic variable on this SDE.

The time series of the central rate $\left\{\hat{\varsigma}_j\right\}_{j=1985.11.1}^{2001.1.30}$ were reproduced by the integration with a variable t (see Fig. 8). We found that reproduced variations were smooth in comparison of the time series data of the central rate. We believed that the smoothness was caused by the integration (10). But 0.88 was obtained as the correlation coefficient between $\left\{\hat{\varsigma}_j\right\}_{j=1985.11.1}^{2001.1.30}$ and $\left\{\varsigma_j\right\}_{j=1985.11.1}^{2001.1.30}$ in our result. We also concluded that the errors at both ends were less than the errors with use of the multiple regression formula (1). We can expect depreciation of yen against the dollar in recently well with the SDE (9).

6. Discussion

The JP¥/US$ exchange rate time series is not assumed to be random but the differenced time series is with calculation of the correlation dimensions and so forth in Section 3; therefore, we obtained the SDE for the description of the differenced time series of the JP¥/US$ exchange rate with the reducing method.

6.1 Evidence of SDE

We believed the control system on the normalized difference of the JP¥/US$ exchange rates in a certain dynamical equilibrium state. According to empirical studies in Section 3, we have thought that its process is one of weakly stationary processes; therefore we could estimate the time-averaged potential function on a SDE by the probability density function. We have believed that the SDE is appropriate as a mathematical model for their difference. We could also show the appropriateness by the numerical simulation with use of our SDE (9). Phase transition occurred on our time-averaged potential by the noise generated with the SDE. We believed that this phase transition explained the sudden yen's appreciation after the Plaza Accord.

We attempt to show evidence that the process is one of stationary processes with use of a test (S) (Appendix C). This test was proposed by Okabe and Inoue (1994) and Okabe and Nakano (1991). However, the test (S) has not been obtained by mathematical theory but many experiments using the data of which regularity and structure were given. We have calculated values of the left side on each test $(M)_i$, $(V)_i$, $(O)_i$, which corresponds to time interval. Inequalities except $(V)_i < 2.24$ were satisfied for any length of time intervals (Fig. 9). This result also gave us evidence that the process on the normalized time series of difference of JP¥/US$ exchange rates is one of weakly stationary processes. The test (S) was accepted with that differenced time series data, of which the time interval was less than a week. That is, we could believe that the SDE was appropriate as the mathematical model on the differenced time series for a week. We may have to construct mathematical rules on connection of the stochastic differential equations as a mathematical model over a week. This mathematical model might be a deterministic equation as we claimed (Takada *et al.*, 1999).

We will be able to obtain a new economic rule interpreting the mathematical model.

6.2 Condition on the SDE

In general, the SDE can be also written as a first-order differential equation that depends on a random variable v and fluctuating force $F(t)$ in the inhomogeneous term:

$$\frac{dv(t)}{dt} = G(v, F(t)),$$

where $F(t)$ is standardized fluctuating force generated with a Gauss type stochastic process. The problem finding the SDE corresponding to a given FPE does not have a solution uniquely. However, the solution is unique if we limit the SDE as a type of following additive formula, which is described as a sum of a function on the variable z and the fluctuating force.

$$\frac{dv(t)}{dt} = \hat{a}(v) + F(t). \tag{11}$$

Goel and Richter (1978) have shown that this SDE corresponds to a normalized Fokker Plank equation (FPE) that describes a regular stochastic process with calculation of moments of transition probability in accordance of Stratonovich's rule. The SDE is originally an element of the set of the normalized Fokker Plank equations that are solvable on a certain condition.

The integral in accordance of Stratonovich's rule is, what we call, Stratonovich's integral (Appendix A). The advantage of this integral is to be able to treat infinitesimal calculus as ordinal Newtonian derivative or Liemann integral. Its defect is not to be able to assume that time-average of the second term on left side $\langle b(v)W(t)\rangle$ equals to zero. However, the time-average is equivalent to zero with Ito integral (Appendix A). The following equation is a relationship between Stratonovich's and Ito derivative.

$$(S)dx = \left(a - \varepsilon b\frac{\partial b}{\partial v}\right)dt + b(v)dB(t) \tag{12}$$

where $(S)dx$ is the former derivative (Suzuki, 1994). A problem of rule-of-calculation appears if the coefficient of the white noise term depends on the random variable $b(v)$ on Eq. (4). Stratonovich derivative is equivalent in Ito derivative on Eq. (11).

6.3 Presumption for the mathematical model

Based on the Eq. (5), we assumed the logarithm of observed probability distribution as the solution of the FPE for the difference of JP¥/US$ exchange rates. Based on the time series data, we have led that the potential function in the sense of the time average is not a parabolic function that describes the OU process, but it must be expressed by a polynomial whose order is larger than 4. Moreover, the potential function must be forth order if it is structural stable. We mention the necessity of the number of its order with the consideration on the structural stability.

Thus, we obtained the SDE as the mathematical model for time series data that were assumed to be random. We concluded the SDE (9) for the description on the differenced time series of JP¥/US$ exchange rates. However, according to the result of the test (S) mentioned in Subsection 6.1, we could determine time-averaged potentials per moving intervals of which the length was less than a week. We have believed that each time-averaged potential function is different. It seems to be possible for the potentials to fluctuate with the ravages of time. Static potential function on our SDE might be only a mathematical approximation although the SDE described the sudden yen's appreciation after the Plaza Accord. Based on the SDE constructed in this paper, we will try to construct the theory to obtain a family of differential equations that do not describe the difference but the exchange rate.

We have believed that the linear congruential method is useful because this method is most fast generator on the series of the pseudo random numbers and widely used as the generator. But the problems of this method were pointed out. For instance, they considered the period on the pseudo random number to be short and the species on the series fewer. We attempt to introduce Maximum-length linearly recurring sequences into the noise term on the difference equation that we rewrote Eq. (9) into for improvement of numerical simulation. We believe that we can compare the time series data with the expectation-path obtained from these numerical simulations and evaluate the SDE for the description of them.

In the next step, we adopt economic fundamentals, especially the money supply in Japan, into the mathematical model of the exchange rate for recent decades because current government, in an attempt to produce a weaker yen and a period of high stock prices in collaboration with the Governor of the Bank of Japan, are using the money supply. A recent regression model could be obtained, composed of the principal components in macroeconomic factors for t = 1985.10 to 2010.12. (Takada, 2013). The author compared three coefficients of principal components in this recent regression model with those in the previous one. In particular, the second and the third principal components showed remarkable changes; therefore, it can be proposed that foreign currency exchange determination might be described by a non-equilibrium system.

7. Conclusion

We could conclude here differenced time series of JP¥/US$ exchange rates to be a kind of Markov processes. We constructed the time-averaged potential function on the SDE for variations of the differenced time series data. We could reproduce the time series of the JP¥/US$ exchange rates with our SDE. We regressed the time series data of JP¥/US$ exchange rate with time integral of the values reproduced by numerical simulation of the SDE. As the result of their comparison, we could improve the mathematical model for description of the JP¥/US$ exchange rate.

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Appendix A: Enumeration of Definitions and Theorem

We enumerate definition and theorem to use in this paper. First, we discussed a condition of the stochastic differential equation (SDE) with rules of calculation. We mention these definitions here. We assume $n+1$ division of an interval to be $a \equiv t_0, t_1, \ldots, t_n \equiv b$. Δ_n expresses a maximum value of definition $|t_{k+1} - t_k|$ for any k.

Definition A.1 (S)

$$\int_a^b f(B(t))dB(t) = \lim_{\substack{n\to\infty \\ \Delta_n \to 0}} \sum_{k=0}^{n-1} f\left(\frac{B_{k+1}+B_k}{2}\right)\left(B_{k+1}-B_k\right)$$

Definition A.2 (I)

$$\int_a^b f(B(t))dB(t) = \lim_{\substack{n\to\infty \\ \Delta_n \to 0}} \sum_{k=0}^{n-1} f(B_k)\left(B_{k+1}-B_k\right),$$

where the function f is a element of $\mathbf{C}^\infty(a, b)$ and $B_k = B(t_k)$. We sum the left value of each interval with the definition A.2. The left value $f(B_k)$ is independent on the interval $B_{k+1} - B_k$. Thus, the time average of their product $\langle f(B_k)(B_{k+1} - B_k)\rangle$ goes over into zero.

The following definition and theorem are useful for Fractal analysis in Section 2.

Definition A.3 The correlation integral is described by the following expression;

$$C(\varepsilon) = \frac{1}{N^2}\sum_{i,j=1}^{N} \Theta\left(\varepsilon - \left|X_i - X_j\right|\right),$$

where Θ is Heviside's function.

Theorem A.4 (Takens, 1981) It is assumed that n dimensional compact manifold M is given. A mapping $\Phi_{(\phi,g)}$: $M \to \mathbf{R}^{2n+1}$ on the pair (ϕ, g), i.e.,

$$\Phi_{(\phi,g)} = g(x), g(\phi(x)), \cdots, g\left(\phi^{2n}(x)\right) \tag{A.1}$$

is a embedding, where ϕ, g are C^2-function and smooth.

One to one correspondence exists between M and $2n + 1$ dimensional Euclidean space. The pair (ϕ, g) by which $\Phi_{(\phi,g)}$ is assumed to be the embedding mapping is dense in the function space (Tsuda, 1999). We can apply this theorem if we consider that ϕ is a diffeomorphic mapping on M, ϕ^t is flow on the dynamical system, and g shows the observed value; therefore Φ on Eq. (A.1) transforms from each point on the manifold M onto each delay coordinate.

Appendix B: The Method to Construct the Stochastic Differential Equation

Takada *et al*. (2001) showed the theory to construct the stochastic differential equation (SDE) for description of time series observed the process $X(t)$ discreetly with the Assumption 1 and Assumption 2. If these are assumed, the stochastic process $X(t)$ can be described by the Fokker-Plank equation (FPE). This stochastic differential equation goes over into the normalized Fokker-Plank equation

$$\begin{aligned}\frac{\partial g(v|v_0,t)}{\partial t} &= -\frac{\partial}{\partial z}\left\{a(v)g(v|v_0,t)\right\} + \frac{1}{2}\frac{\partial^2}{\partial z^2}g(v|v_0,t) \\ &\equiv -\frac{\partial}{\partial z}J(v|v_0,t),\end{aligned} \tag{B.1}$$

where $g(v|v_0, t)$ is a probability density function transformed from $P(x|y, t)$ and v_0 is the initial condition with the permutation of variables (Goel and Richter, 1978). A strict stationary solution of Eq. (B.1) can be obtained by Harken (1975). The stationary solution of Eq. (B.1) is as follows:

$$g(v) = \left\{C - 2J(\infty)\int^v \exp\left[-2\int^\zeta a(\xi)d\xi\right]d\zeta\right\}\exp\left[2\int^v(\xi)d\xi\right], \tag{B.2}$$

where the stationary value $J(v|v_0, \infty) \equiv J(\infty)$ is a constant and C is a normalization factor defined with the formula: $\int_{-\infty}^{\infty} g(\xi)d\xi = 1$. Equation (B.1) can be led by calculating moments of transition probability

$$M_n(u) = \lim_{\tau\to 0}\frac{1}{\tau}\int_\Omega (v-u)^n P(v|u,\tau)dv \tag{B.3}$$

on the process that a stochastic differential equation (SDE) describes (Goel and Richter, 1978). The problem finding the SDE corresponding to a given FPE does not have a solution uniquely. However, the solution is unique if we limit the additive formula (11). With the operation of the SDE corresponding to the FPE (B.1), the second term of the right-hand side on Eq. (11) goes to 0 if we take time averages of both sides of this equation. With Stratonovich's rule (Stratonovich, 1963), Eq. (11) can be rewritten into an ordinary differential equation (ODE) on the time average of the random variable v. $a(v) = 0$ is equilibrium space of the ODE. On Eq. (11), $a(v) = 0$ means the equilibrium space in the sense of the time average. Hence, the space integral of the coefficient function $a(v)$:

$$U(v) = -\int^v a(\xi)d\xi \tag{B.4}$$

is a potential function in the sense of the time average on

Eq. (11). With this formula (B.4), the stationary solution (B.2) can be rewritten as the Eq. (5) under a natural boundary condition such as $J(\pm\infty) = 0$.

Appendix C. Stochastic Tests

We discussed whether our theory could cover the time series of JP¥/US$ exchange rate with some stochastic tests. We give their method as follows.

Run test: A value of each point of time series is compared to a median of the time series, and the value is written as;

a) a symbol "+" for a larger value

b) a symbol "-" for a smaller value

where the continuous symbols + or - are called a run. The number of the continuous the symbol + is presented as m and - is n. Length of the run L is defined as length of the continuous group of the same symbol. If the length of the run is too long, it shows that observed values makes the same type of groups. Otherwise, if the different symbols (+ and –) appears alternatively and the length of the run is too short, it shows that the series have regularity. These series of the observed value are not random series. Then, we think verification of the randomness of the time series with a null hypothesis "the time series is not random". If the time series are random, the value L satisfies the following relations (Minotani, 2000).

$$E[L] = \frac{2mn}{m+n} + 1 \tag{C.1}$$

$$V[L] = \frac{2mn(2mn - m - n)}{(m+n)^2(m+n-1)} \tag{C.2}$$

where $E[L]$ is an average, $V[L]$ is a standard deviation of the random value L. If m and n are larger than 20 at least, they can assume that the statistical test value $Z_L = (L - E[L])/\sqrt{V[L]}$ with a normal approximation of L composes a standard distribution.

Rank test: We think over the following pairs to verify whether the time series $X(t)$ monotonously increase or decrease.

$$X(i) > X(j)$$

where $i < j$. We assume the number of the pairs to be Q. According to easy consideration, the time series increase at uniform pace if $Q = 0$ and decrease if $Q = {}_NC_2$. The graph of the time series is linear in these cases. In general, there are N! cases as permutations on the set $\{X(t_1), ..., X(t_N)\}$. We can obtain a stochastic distribution which depends on Q. The average and standard deviation were calculated by Kendall and Stuart (1958) with use of Bernoulli's number and cumulants of the distribution.

$$E[Q] = \frac{1}{4}N(N-1): V[Q] = \frac{1}{72}N(N-1)(2N+5).$$

We use statistical test value $Z_Q = (Q - E[Q])/\sqrt{V[Q]}$ standardized by these factors $E[Q]$, $V[Q]$. It is said that this distribution goes over into normal distribution if $N > 10$ (Kendall and Stuart, 1958).

Test (S): We consider difference of the time series to be fluctuating force. We calculate the mean and standard deviation on the time series of the difference to standardize the time series of the difference. Each distribution of the following statistical test values $(M)_i$, $(V)_i$, $(O)_i$ is assumed to be a standard distribution by the center limit theorem and low of a large number if their variations are random and there are a number of data on the time series.

$$(M)_i = \frac{1}{\sqrt{M+1}}\left|\sum_{k=0}^{M}\xi_i(k)\right|,$$

$$(V)_i = \left|\sum_{k=0}^{M}\left(\xi_i(k)^2 - 1\right)\Big/\sqrt{\sum_{k=0}^{M}\left(\xi_i(k)^2 - 1\right)^2}\right|,$$

$$(O)_i = \frac{1}{\sqrt{L_{n,m}^{(1)}} + \sqrt{L_{n,m}^{(2)}}}\left|\sum_{k=m}^{M-n}\xi_i(k)\xi_i(n+k)\right|,$$

where M is time interval (analytic interval) and n is lag time. $L_{n,m}{}^{(1)} + L_{n,m}{}^{(2)}$ expresses the data length of the analytic interval (Table 2). Their average, standard deviation and correlation coefficients go over into 0, 1 and 0 respectively. Then, Okabe and Inoue (1994) and Okabe and Nakano (1991) proposed the following test (S) to verify whether the time series were obtained by observation of a stable process.

Test (S) ratios are less than 0.2, 0.3 and 0.2 respectively, that are rejected against each standard $(M)_i < 1.96$, $(V)_i < 2.24$, $(O)_i < 1.96$ for all i.

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Note added in proof. The author found a misprint in the notation of Eq. (A.1) on page S51. The correct expression is as follows:

$$\Phi_{(\phi,g)} = \left(g(x) \;\; g\big(\phi(x)\big) \;\; \cdots \;\; g\big(\phi^{2n}(x)\big) \right).$$

In addition, Table 2 on line 18, right column of page S52 should be deleted due to the lack of the table in the original manuscript.