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Coding Rule for Periodic Orbits in the One-dimensional Map

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A new coding rule for periodic orbits in unimodal one-dimensional maps is derived. The best-known example of a family of unimodal maps is the logistic map. The band merging is observed in the bifurcation diagram of the logistic map. Let a_m^k ($k \geq 1$) be the critical value at which 2^k-band merges into 2^{k-1}-band. At $a > a_m^0$, the diverging orbit appears and thus 1-band disappears. The relations $a_m^{k+1} < a_m^k$ for $k \geq 0$ hold. Let s_q be the code for periodic orbit of period q in the parameter interval $(a_m^1, a_m^0]$. Assume that the code s_q represented by symbols 0 and 1 is known. In the interval $(a_m^{k+1}, a_m^k]$, there exists the periodic orbit of period $2^k \times q$ ($k \geq 1$). Let its code be $s_{2^k \times q}$. Let $\mathcal{D}$ be the doubling operator defined by the substitution rules as $0 \Rightarrow 11$ and $1 \Rightarrow 01$. The following coding rule is derived. Operating k times of $\mathcal{D}$ to s_q, the code $s_{2^k \times q}$ is determined.

Key words: One-dimensional Map, Bifurcation Diagram, Coding Rule, Periodic Orbits, Doubling Operator

1. Introduction

In this paper, the coding rule for periodic orbits in unimodal one-dimensional maps is discussed. The best-known example of a family of unimodal maps is the logistic map. Let q be the period of periodic orbit. The periodic orbit is represented by a set of q symbols. This set is called code. The method using the code to classify periodic orbits has been introduced by Metropolis-Stein-Stein (Metropolis *et al.*, 1973). The kneading theory by Milnor-Thurston (Milnor and Thurston, 1988) inherits this method and it gives the useful method to calculate the topological entropy (see also Nagashima and Baba, 1999). In this paper, we use two symbols 0 and 1 to represent codes.

Using the bifurcation diagram displayed in Fig. 1 of the logistic map, we explain the problem discussed in this paper. In Fig. 1, for example, a natural number 3 means the periodic window of period-3 orbit. There exist infinitely many windows in the parameter interval $(a_m^1, a_m^0 = 4]$ where a_m^k ($k \geq 1$) is the critical value that 2^k ($k \geq 1$) bands merge into 2^{k-1} bands and a_m^0 is the critical value that one band disappears. For example, a window of period-3 orbit corresponds to the domain of a stability of period-3 orbit. We remark that the interval $(a_m^{k+1}, a_m^k]$ ($k \geq 0$) in the bifurcation diagram is 2^k-band.

Let us consider the window of period-3 orbit in 1-band. The origin of this window is the appearance of period-3 orbits with codes 001 and 011 which appear through the tangent bifurcation. In the bifurcation diagram, the stable periodic orbit with code 001 is observed. In the following, the periodic orbit with code 001 is abbreviated as the periodic orbit 001.

There exists the window of period-2×3 orbit (2×3 in Fig. 1) in the interval $(a_m^2, a_m^1]$. Our problem discussed in this paper is described as follows. "What is the coding rule to determine the code for period-2×3 orbit from that of period-3 orbit?" If we can determine the coding rule, using it, the codes for period-$2^k \times 3$ ($k \geq 2$) orbits in the interval $(a_m^{k+1}, a_m^k]$ are determined automatically. We can apply the coding rule to the periodic orbits included in the Sharkovskiĭ ordering (Sharkovskiĭ, 1964) (see Appendix A) in $(a_m^1, a_m^0]$ and also apply it to the periodic orbits not included in the Sharkovskiĭ ordering.

In Sec. 2, the notations used in this paper are introduced. In Sec. 3, Coding rule 3.1 for the period-doubling bifurcation is derived. In Sec. 4, Coding rule 4.1 as an answer to our problem is derived. In Sec. 5, the results are summarized.

2. Preparations

2.1 Code for periodic orbit

We introduce the logistic map f.

$$f \ : \ x_{n+1} = a x_n (1 - x_n). \tag{1}$$

Here $0 < a \leq 4$ and $0 \leq x_n \leq 1$. The fixed point P located at $x = 0$ is stable at $0 < a < 1$. A new fixed point Q appears at $a = 1$ and its position is $x = 1 - 1/a$ ($a > 1$). The fixed point Q occurs the period-doubling bifurcation at $a = 3$.

Here, we show the coding method introduced by Metropolis-Stein-Stein. Take the particular orbit starting from the initial point at $x = 1/2$ and coming back to the initial point. This type orbit is called the superstable periodic orbit. If the orbital point enters into the region satisfying $x > 1/2$, we give a symbol R. If the orbital point enters into the region satisfying $x < 1/2$, we give a symbol L. Suppose that the following coding is obtained.

$$1/2 \to R \to L \to R \to R \to 1/2. \tag{2}$$

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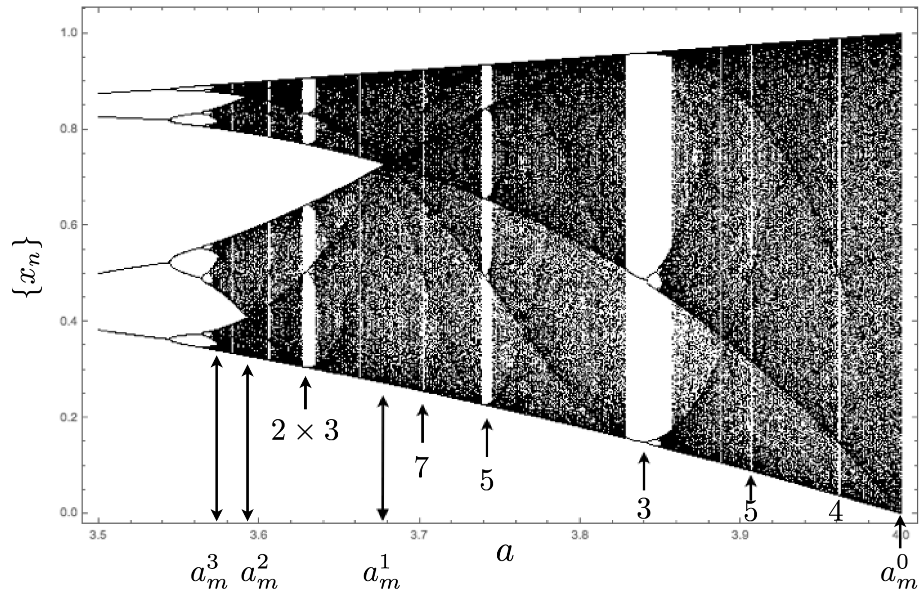

Fig. 1. Bifurcation diagram of the logistic map where $a_m^0 = 4$, $a_m^1 = 3.678573$, $a_m^2 = 3.592572$ and $a_m^3 = 3.574804$. A natural number 3 represents the window of period-3 orbit.

Here, a fraction 1/2 represents an initial point and the arrow ($\rightarrow$) means the orbital order. The code is determined as $RLRR$. The number of symbols in code does not accord with the length of period. Thus, in the kneading theory, a symbol C is added in front of this code and new code $CRLRR$ is defined. Symbol C means the center of interval.

Next, using $CRLRR$, we explain the minimum representation for code. The position represented by C is $x = 1/2$ and the mapping function has the maximum point at this point. This fact implies that next position represented by R (next symbol of C) is the maximum orbital point. In the unimodal map, the maximum orbital point is mapped to the minimum one. Thus, the position represented by L (next symbol of R) is the minimum one. We name the representation $LRRCR$ the minimum representation for code. In the following, we use the minimum representation for codes.

We use two symbols 0 and 1 in consideration of correspondence with the binary representation where a symbol 0 (1) means L (R). Thus, the code for P is 0 and that of Q is 1. Next, we give a meaning of symbol C. Two periodic orbits which appear through the tangent bifurcation constitute (0-1)-pair which means a pair of the stable and the unstable periodic orbits (Hall, 1994). In the two dimensional map, (0-1)-pair means the saddle-node pair. There exist periodic orbit with code where C is replaced by 0 and that with code where C is replaced by 1. The code $LRRCR$ means two periodic orbits 01101 and 01111. The set of these codes is an example of (0-1)-pair.

We comment on the codes for periodic orbits which appear through the period-doubling bifurcation. For example, let us consider the code 0111, which is the code for the daughter periodic orbit which appears through the period-doubling bifurcation of the period-2 orbit. We exchange a symbol 1 at the second-to-last to 0 and have new code 0101 which is the repetition of word 01. Thus, 0101 is meaningless as a code. The code for the periodic orbit which appears through the period-doubling bifurcation does not have a partner code of (0-1)-pair.

The code obtained here is the same as the code determined by the tent map T defined on [0, 1]. In the following, we explain this fact. The logistic map f at $a = 4$ is converted into the tent map T.

$$T \ : \ X_{n+1} = 1 - |2X_n - 1| \tag{3}$$

by the translation formula

$$x_n = \sin^2((\pi/2)X_n). \tag{4}$$

Thus, the logistic map at $a = 4$ and the tent map T are conjugate. Here, we take the orbit of logistic map at $a = 4$. If the orbit enters the interval [0, 1/2], the symbol is defined as 0. If the orbit enters the interval [1/2, 1], the symbol is defined as 1. For the point $x = 1/2$, we can use 0 or 1. This is originated from the fact that there are two representations to an irreducible fraction, for example, $1/2 =_{\bullet} 10^{\infty}$ and $1/2 =_{\bullet} 01^{\infty}$. Here, a symbol $_{\bullet}$ is a decimal point and the right hand sides are the binary representation.

For example, suppose that the code 011 is obtained. In the tent map, there exists the periodic orbit 011. Conversely, the periodic orbit in the tent map exists in the logistic map. From these facts, we can study the periodic orbit with a given code in the tent map. Translating the orbital points in the tent map by Eq. (4), we have the orbital points in the logistic map at $a = 4$. The orbital order of periodic points in the tent map is the same as that in the logistic map.

2.2 Block representation

First, we introduce two block symbols $E(2) = 01$ and $F(2) = 11$ (Yamaguchi and Tanikawa, 2009, 2016). Block symbol $E(2) = 01$ represents the code for the daughter periodic orbit which appears through the period-doubling bifurcation of Q. Block symbol $F(2) = 11$ is introduced for convenience sake and there is no periodic orbit represented by $F(2)$. Suppose that the periodic orbit of period

$q = 2n$ $(n \geq 2)$ is written by $E(2)$ and $F(2)$. We say the block symbol as block briefly. Since the first symbol of block $F(2)$ is 1, the first block of the minimum representation is $E(2)$.

Let us consider the block code that the number of blocks are greater than or equal to 2. The minimum representation begins with $E(2)F(2)$. In order to prove this fact, we confirm the large/small relation between $\bullet E(2)F(2) =_\bullet 0111$ and $\bullet E(2)E(2) =_\bullet 0101$. Translating them into the binary one, we obtain $\bullet 0101$ for $\bullet 0111$ and $\bullet 0110$ for $\bullet 0101$. The translation procedure is given in Appendix B. Since the relation $\bullet 0101 <_\bullet 0110$ holds, the claim is proved. In the following discussions, we use the abbreviated notations E and F.

2.3 Intervals and symbols

We explain the structure of bifurcation diagram displayed in Fig. 1. In the left side of Fig. 1, the accumulation of the period-doubling bifurcation is observed.

We decrease the parameter value of a from $a = a_m^0 = 4$. At the critical point $a = a_m^1 = 3.678573$, one band splits into two bands. At the critical point $a = a_m^2 = 3.592572$, two bands split into four bands. Let $a_m^\infty = 3.569945$ be the accumulation point of the band splitting. The critical value a_m^∞ is also the accumulation point of period-doubling bifurcation. Increasing a from a_m^∞, we can observe the band merging.

Next, we give the relation of the bifurcation diagram and the shape of mapping function. Figure 2(a) represents the shape of $f(x)$ at $a = a_m^0$ where $f(1/2) = 1$. The orbit starting from $x = 1/2$ reaches the fixed point P at $x = 0$. It is an example of superstable orbit.

The closed interval $\mathrm{Int}(A_0)$ is defined as $[1/2, 1]$ and $\mathrm{Int}(B_0)$ is defined as $[0, 1/2]$. We remark that $\mathrm{Int}(A_0)$ includes Q. The mapping function $f(x)$ is a unimodal function which has two monotonic branches. Let M_0 be the transition matrix representing the transitions between $\mathrm{Int}(A_0)$ and $\mathrm{Int}(B_0)$.

$$M_0 = \left(\begin{array}{c|cc} & A_0 & B_0 \\ \hline A_0 & 1 & 1 \\ B_0 & 1 & 1 \end{array} \right). \tag{5}$$

In M_0, A_0 (B_0) expresses $\mathrm{Int}(A_0)$ $(\mathrm{Int}(B_0))$. For example, the first row means that the image of $\mathrm{Int}(A_0)$ covers $\mathrm{Int}(A_0)$ and $\mathrm{Int}(B_0)$ once. The eigenvalue of M_0 is 2, and thus the topological entropy at $a = 4$ is $\ln 2$.

In Fig. 2(b), the functions $f^2(x)$ and $f^4(x)$ around $x = 1/2$ at $a = a_m^1$ are displayed. The superstable orbit displayed by arrowed line goes to the fixed point Q where the condition $f^4(1/2) = 1 - 1/a$ holds. From this condition, the critical value a_m^1 is determined. In Fig. 2(b), there are two intersection points of $f^2(x)$ and the diagonal line. The right intersection point is Q and the left one is α_0 which is the daughter orbital point which appears through the period-doubling bifurcation of Q. The other point α_1 is not displayed in Fig. 2(b) since it locates in the right side of Q. The period of daughter orbit is 2. We say it period-2 orbit. There are four intersection points of $f^4(x)$ and the diagonal line. New intersection points β_0 and $\beta_2 = f^2(\beta_0)$ are the daughter orbital points which appear through the period-doubling bifurcation of α_0. Note that β_0 is the minimum point in orbital points of period-2^2 orbit.

Let the interval sandwiched between dashed lines located in the region $x \leq 1/2$ $(x \geq 1/2)$ be $\mathrm{Int}(A_1)$ $(\mathrm{Int}(B_1))$. The interval including the fixed point α_0 of $f^2(x)$ is $\mathrm{Int}(A_1)$, $\mathrm{Int}(A_1)$ includes β_0 and $\mathrm{Int}(B_1)$ includes β_2. We remark that $\mathrm{Int}(A_k)$ and $\mathrm{Int}(B_k)$ for $k \geq 1$ include both end points (Nagashima and Baba, 1999).

The function $f^2(x)$ is a monotonic decreasing function in $\mathrm{Int}(A_1)$ and is a monotonic increasing one in $\mathrm{Int}(B_1)$. From the unimodal property of $f^2(x)$, the transition matrix M_1 representing the transitions between $\mathrm{Int}(A_1)$ and $\mathrm{Int}(B_1)$ is obtained as follows.

$$M_1 = \left(\begin{array}{c|cc} & A_1 & B_1 \\ \hline A_1 & 1 & 1 \\ B_1 & 1 & 1 \end{array} \right). \tag{6}$$

The eigenvalue of M_1 is 2, and thus the topological entropy at $a = a_m^1$ is $(1/2)\ln 2$.

In Fig. 2(c), the left intersection point of $f^4(x)$ and the diagonal line is β_0 and two intersection points γ_0 and γ_4 of $f^8(x)$ and the diagonal line are the daughter periodic points which appear through the period-doubling bifurcation of β_0. We say the periodic orbit of γ_0 and γ_4 period-2^3 orbit. Two points β_0 and γ_0 locate in $\mathrm{Int}(A_2)$, and α_0 and γ_4 in $\mathrm{Int}(B_2)$.

In Fig. 2(d), the left intersection point of $f^8(x)$ and the diagonal line is γ_0 and two intersection points δ_0 and δ_8 of $f^{16}(x)$ and the diagonal line are the daughter periodic points which appear through the period-doubling bifurcation of γ_0. Two points γ_0 and δ_0 locate in $\mathrm{Int}(A_3)$, and β_0 and δ_8 in $\mathrm{Int}(B_3)$.

The transition matrix M_k $(k \geq 1)$ at $a = a_m^k$ $(k \geq 0)$ is determined.

$$M_k = \left(\begin{array}{c|cc} & A_k & B_k \\ \hline A_k & 1 & 1 \\ B_k & 1 & 1 \end{array} \right). \tag{7}$$

The critical value a_m^k $(k \geq 0)$ is determined by the relation $f^{2^{k-1}\times 3+2}(1/2) = f^{2^k+2}(1/2)$ (see Appendix C). The eigenvalue of M_k is 2, and thus the topological entropy at $a = a_m^k$ $(k \geq 0)$ is $(1/2^k)\ln 2$. At a_m^∞, the topological entropy is zero.

If the orbital point by f^{2^k} locates in $\mathrm{Int}(A_k)$ $(\mathrm{Int}(B_k))$, let the symbol of orbital point be $\mathrm{Symb}(A_k)$ $(\mathrm{Symb}(B_k))$. These symbols are constructed by the symbols 0 and 1. For example, we consider the orbit in the parameter range $(a_m^{k+1}, a_m^k]$ $(k \geq 1)$. Suppose that x_0 exists in $\mathrm{Int}(A_k)$ or $\mathrm{Int}(B_k)$. The orbital point $x_{2^k} = f^{2^k}(x_0)$ does not escape from these intervals. This fact means that the code for periodic orbit by f^{2^k} is represented by $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$.

We give the remark about coding. If $\mathrm{Int}(A_0)$ is defined, $\mathrm{Symb}(A_0)$ is also determined. $\mathrm{Int}(B_0)$ and $\mathrm{Symb}(B_0)$ are also defined. At $a = 4$, their explicit representations by 0 and 1 are determined. We remark that $\mathrm{Symb}(A_0) = 1$ and $\mathrm{Symb}(B_0) = 0$. For the other symbols $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$ $(k \geq 1)$, the same facts hold. The length of $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$ represented by 0 and 1 is 2^k.

The structure of windows in the interval $(a_m^1, a_m^0]$ is similar to that in the interval $(a_m^{k+1}, a_m^k]$. Thus, if the periodic

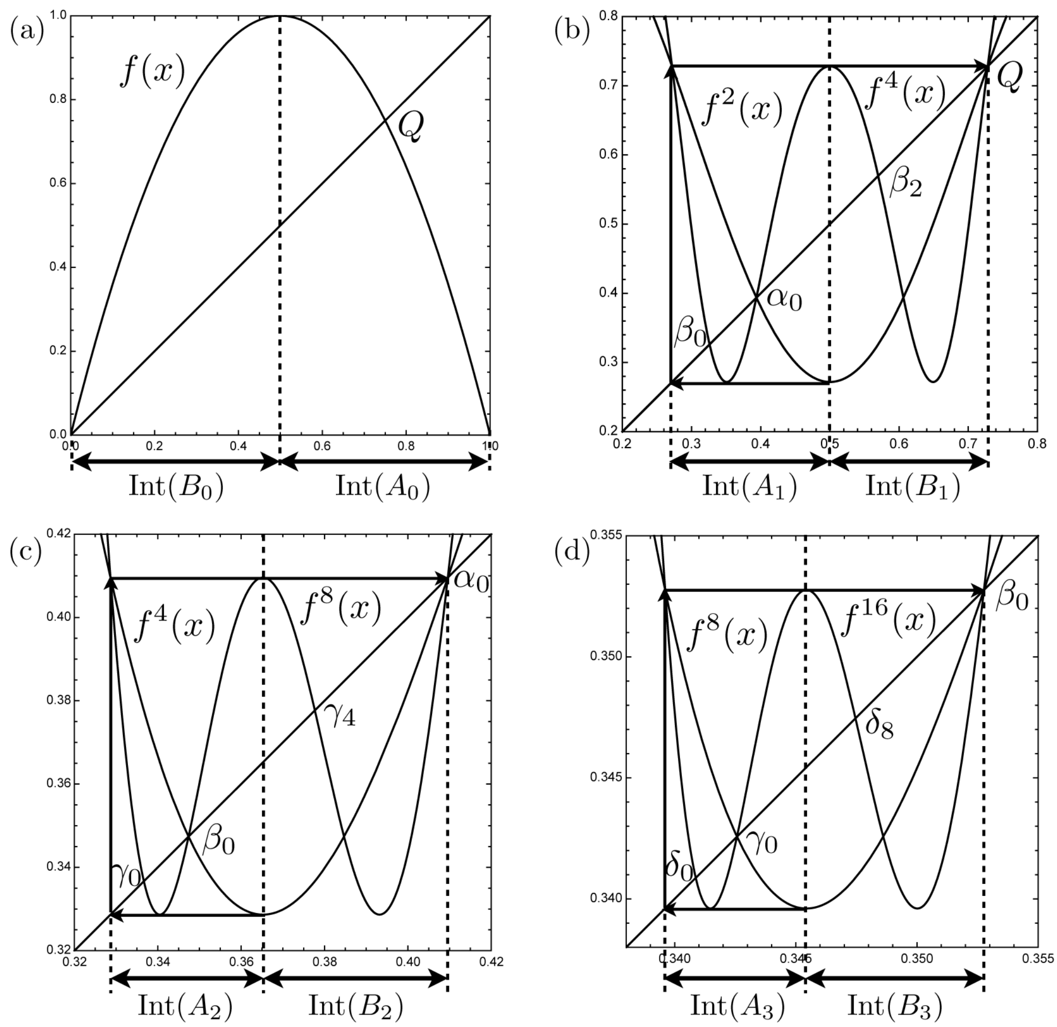

Fig. 2. Definition of two closed intervals $\mathrm{Int}(A_k)$ and $\mathrm{Int}(B_k)$ for $k = 0, 1, 2, 3$. (a) Situation at $a = a_m^0 = 4$. The point Q represents the fixed point at $x = 3/4$. (b) Situation at $a = a_m^1$. The orbital point α_0 is that of period-2 orbit and the orbital points β_0 and β_2 are those of period-2^2 orbit. (c) Situation at $a = a_m^2$. The orbital point β_0 is that of period-2^2 orbit and the orbital points γ_0 and γ_4 are those of period-2^3 orbit. (d) Situation at $a = a_m^3$. The orbital point γ_0 is that of period-2^3 orbit and the orbital points δ_0 and δ_8 are those of period-2^4 orbit. In (b), (c) and (d), the arrowed line represents the superstable orbit.

orbit of period q with code s_q exists in the interval $(a_m^1, a_m^0]$, the corresponding periodic orbit of period $2^k \times q$ with code $s_{2^k\times q}$ exists in the interval $(a_m^{k+1}, a_m^k]$. It is noted that s_q is represented by $\mathrm{Symb}(A_0)$ and $\mathrm{Symb}(B_0)$ and $s_{2^k\times q}$ is represented by $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$. Thus, our problem is renewed as Problem 2.1.

Problem 2.1. Derive the rule to determine $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$ from $\mathrm{Symb}(A_0) = 1$ and $\mathrm{Symb}(B_0) = 0$.

3. Coding Rule for the Period-doubling Bifurcation

Only the coding rule to determine codes for periodic orbits which appear through the period-doubling bifurcation of the fixed point Q has been known. In this section, we make clear the period-doubling bifurcation of Q and derive Coding rule 3.1. In oder to answer Problem 2.1, we need Coding rule 3.1.

Using Fig. 3, we explain the period-doubling bifurcation. Let two daughter periodic points appeared from Q be ξ_0 and ξ_1. We remark that these notations for period-2 orbit are different from those in Subsec. 2.3.

Orbital point ξ_0 moves to the region $x < 1/2$ across $x = 1/2$ with the increase in parameter a. If ξ_0 does not move to the region $x < 1/2$, the symbols of ξ_0 and ξ_1 are 1. The code 11 of period-2 orbit is obtained but it is repetition of the code $P_0 = 1$ of Q. This is a contradiction. As a result, one orbital point of daughter periodic points which is near to $x = 1/2$ moves to the region $x < 1/2$. Its symbol becomes 0. The code $P_1 = 01$ for period-2 orbit is determined.

Next, after the period-doubling bifurcation of period-2 orbit, new two daughter periodic points ζ_0 and ζ_2 are born from ξ_0. Let the point which is near to $x = 1/2$ be ζ_0. We increase the parameter a furthermore. The point ζ_0 moves to the region $x > 1/2$ across $x = 1/2$. Schematic orbit starting from ζ_0 is displayed in Fig. 4(a). We obtain the code 1101 for period-4 orbit. Its minimum representation is $P_2 = 01 \cdot 1 \cdot 1$ and it is rewritten as $P_2 = P_1 P_0 P_0$. The notation $P_1 P_0 P_0$ means that we write the code for $P_1 P_0 P_0$ in this order.

New daughter periodic points η_0 and η_4 appear from ζ_0 of period-4 orbit. Let the point which is near to $x = 1/2$ be η_0.

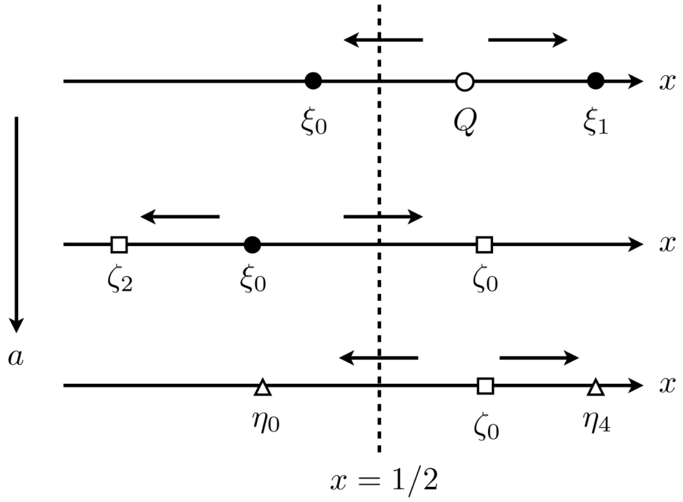

Fig. 3. The parameter a increases from the upper figure to the lower one. The orbital points ξ_0 and ξ_1 are the daughter periodic points appeared from Q. The arrows represent the direction of movement of ξ_0 and ξ_1 when a increases. The orbital points ζ_0 and ζ_2 are the daughter periodic points appeared from ξ_0, and η_0 and η_4 are those appeared from ζ_0.

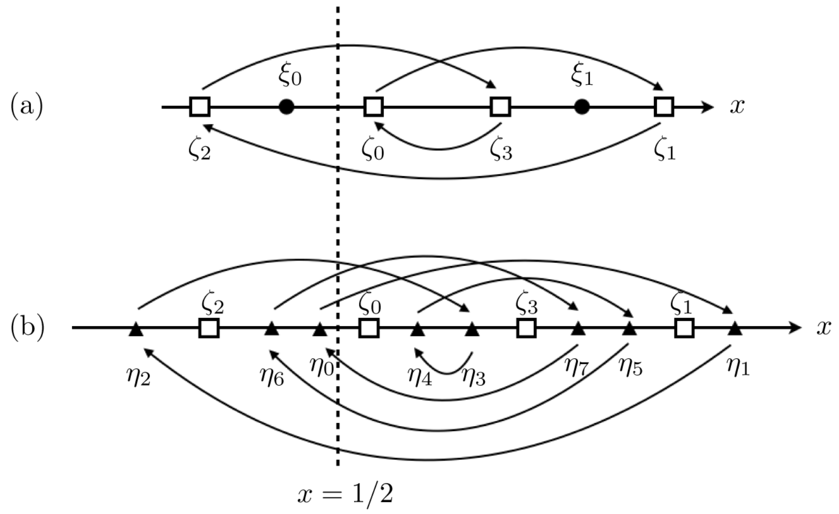

Fig. 4. (a) Period-4 orbit. (b) Period-8 orbit.

Schematic orbit starting from η_0 is displayed in Fig. 4(b). The point η_0 moves to the region $x < 1/2$ across $x = 1/2$. The code for period-8 orbit starting from η_0 is 01011101. Its minimum representation is $P_3 = 0111 \cdot 01 \cdot 01 = P_2 P_1 P_1$. Thus, Coding rule 3.1 is obtained.

Coding rule 3.1. Let P_0 be the code for Q and P_k ($k \geq 1$) be the code for daughter periodic orbit which appears through the period-doubling bifurcation of Q. The code P_{k+1} ($k \geq 1$) is determined by the recursive rule as

$$P_{k+1} = P_k P_{k-1} P_{k-1} \tag{8}$$

where $P_0 = 1$ and $P_1 = 01$.

Here, we define the doubling operator $\mathcal{D}$.

Definition 3.2. The doubling operator $\mathcal{D}$ is defined.

$$\mathcal{D} \ : \ 0 \Rightarrow 11 \equiv F, \ 1 \Rightarrow 01 \equiv E. \tag{9a}$$

$$\mathcal{D} \ : \ E \Rightarrow EF, \ F \Rightarrow EE. \tag{9b}$$

Here, the notation $0 \Rightarrow 11$ means the replacing 0 with 11.

Using $\mathcal{D}$, we rewrite Coding rule 3.1 in Substitution rule 3.3.

Substitution rule 3.3. Operating k times of $\mathcal{D}$ to P_0, the code $\widehat{P_k}$ is determined. After rewriting $\widehat{P_k}$ in the minimum representation, P_k is obtained.

4. Coding Rule as an Answer to Problem 2.1

4.1 Coding rule

In this subsection, we derive Coding rule 4.1 which is an answer to Problem 2.1. First, we take out the periodic orbits related to period-3 orbit from the Sharkovskiĭ ordering.

$$3 \rhd 2 \times 3 \rhd 2^2 \times 3 \rhd 2^3 \times 3 \rhd \cdots . \tag{10}$$

Here, $3 \rhd 2 \times 3$ means the fact that period-3 orbit implies the existence of period-2×3.

The window of period-3 orbit is in 1-band, that of period-2×3 in 2-band, and so on. The two codes for period-3

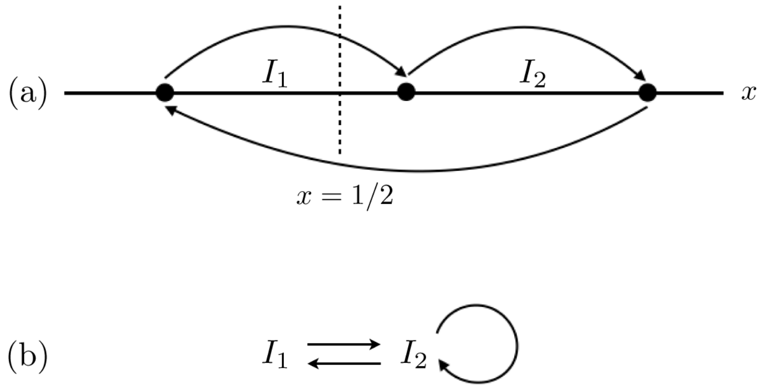

Fig. 5. (a) Period-3 orbit 011. Interval I_1 includes $x = 1/2$. (b) Štefan diagram C_3.

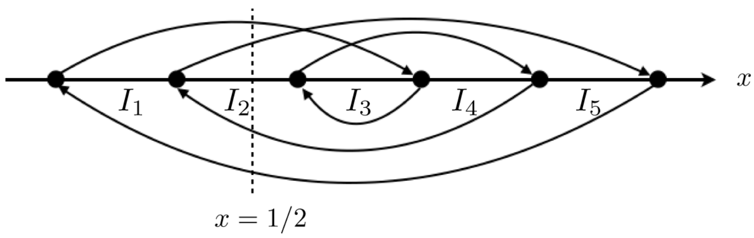

Fig. 6. Period-6 orbital points $011101 = EFE$. Interval I_2 includes $x = 1/2$.

orbits are 011 and 001. Here, we use the code 011. Note that we obtain the same results mentioned below even if the code 001 is used. Using the tent map, we confirm the order relations of orbital points of period-3 orbit 011 and display schematic orbit in Fig. 5(a) where we place the orbital points to the equal distance. Two intervals I_1 and I_2 are defined in Fig. 5(a) where I_1 and I_2 include both end points. We name the graph representing the transitions between I_1 and I_2 Štefan diagram (Štefan, 1977, Devaney, 2003). Here, we call the diagram displayed in Fig. 5(b) C_3. From C_3, it is easy to see the existence of period-6 orbit starting from I_1 and coming back to I_1.

$$I_1 \to I_2 \to I_2 \to I_2 \to I_1 \to I_2 \to I_1. \tag{11}$$

Here, the relation $I_1 \to I_2$ means that the image of I_1 covers I_2.

Since the interval I_1 includes $x = 1/2$, we can use 0 and 1 for the orbit in I_1. On the other hand, the interval I_2 locates in the region satisfying $x > 1/2$. The symbol of the orbital point in I_2 is 1. We obtain the code $01^3 01$ of period-6 orbit. The other period-6 $01^3 11$ also exists. These constitute (0-1)-pair. Using blocks E and F, $01^3 01$ is represented as EFE and 01^5 as EFF. Just after the tangent bifurcation, period-6 orbit EFE is unstable and that with EFF is stable just after the appearance (see Appendix D). Thus, all periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation in the interval $(a_m^2, a_m^1]$ are coded by two symbols Symb(A_1) and Symb(B_1).

Using the tent map, we confirm the order of period-6 (2×3) orbital points with block code EFE and display them in Fig. 6. Five intervals I_k ($k = 1, 2, \cdots, 5$) are defined and the transition matrix M_6 among them are obtained.

$$M_6 = \left(\begin{array}{c|ccccc} & I_1 & I_2 & I_3 & I_4 & I_5 \\ \hline I_1 & 0 & 0 & 0 & 1 & 1 \\ I_2 & 0 & 0 & 0 & 0 & 1 \\ I_3 & 0 & 0 & 1 & 1 & 0 \\ I_4 & 0 & 1 & 0 & 0 & 0 \\ I_5 & 1 & 0 & 0 & 0 & 0 \end{array} \right). \tag{12}$$

For example, the first row means that transitions from I_1 to I_4 and to I_5 are permitted.

We can construct Štefan diagram C_6 displayed in Fig. 7. The orbit starting from I_3, passing through I_4 and coming back to I_3 does not exist. Thus, I_3 is deleted in Fig. 7. We name Fig. 7 the simplified Štefan diagram. We define the long cycle and the short one in C_6. Let the long cycle

$$I_1 \to I_4 \to I_2 \to I_5 \to I_1 \tag{13}$$

be R_o and the short one

$$I_1 \to I_5 \to I_1 \tag{14}$$

be R_s. Here, we define $R_i = R_s R_s$. The length of R_o is the same as that of R_i.

Let s_2 be the symbol of orbital point in I_2. Remember the fact that I_2 extends over the regions satisfying $x < 1/2$ and $x > 1/2$. Thus, there exist the periodic orbit satisfying the condition $s_2 = 0$ and that satisfying the condition $s_2 = 1$. For our purpose, we choose the periodic orbit satisfying the condition $s_2 = 1$. Thus, we obtain $R_o = 0111$ and $R_i = 0101$.

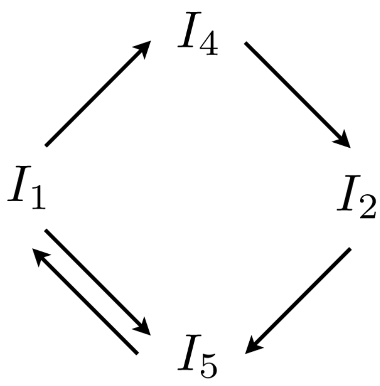

Fig. 7. The simplified Štefan diagram C_6 where I_3 is deleted.

Using blocks E and F, R_o is represented as $EF \equiv \mathrm{Symb}(A_2)$ and R_i as $EE \equiv \mathrm{Symb}(B_2)$. We also have the following relations.

$$\mathrm{Symb}(A_2) = \mathrm{Symb}(A_1)\mathrm{Symb}(B_1), \tag{15a}$$

$$\mathrm{Symb}(B_2) = \mathrm{Symb}(A_1)\mathrm{Symb}(A_1). \tag{15b}$$

All periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation in the interval $(a_m^3, a_m^2]$ are coded by two symbols $\mathrm{Symb}(A_2)$ and $\mathrm{Symb}(B_2)$.

From the simplified Štefan diagram C_6, we obtain that the periodic orbit $R_\mathrm{o}R_\mathrm{i}R_\mathrm{o}$ exists. Its code is represented as $EF \cdot EE \cdot EF$. This implies the existence of period-12 ($2^2 \times 3$) orbit. The partner of (0-1)-pair is $EF \cdot EE \cdot EE$.

Using the tent map, we confirm the order of periodic orbit $EF \cdot EE \cdot EF$ and display them in Fig. 8. Eleven intervals are defined and the transition matrix M_{11} among them are obtained.

$$M_{12} = \left(\begin{array}{c|ccccccccccc} & I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 & I_9 & I_{10} & I_{11} \\ \hline I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ I_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ I_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ I_6 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ I_7 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_9 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_{10} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_{11} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{16}$$

The simplified Štefan diagram C_{12} is displayed in Fig. 9. In C_{12}, let the long cycle

$$I_1 \to I_7 \to I_5 \to I_{10} \to I_2 \to I_8 \to I_4 \to I_{11} \to I_1 \tag{17}$$

be R_o, and the short cycle

$$I_1 \to I_7 \to I_5 \to I_{11} \to I_1 \tag{18}$$

be R_s. Here, we define $R_\mathrm{i} = R_\mathrm{s}R_\mathrm{s}$.

The interval I_4 extends over the regions satisfying $x < 1/2$ and $x > 1/2$ Thus, we choose the periodic orbit satisfying the condition that the symbol of orbital point in I_4 is 0. Thus, we obtain $EFEE \equiv \mathrm{Symb}(A_3)$ for the block code for R_o and $EFEF \equiv \mathrm{Symb}(B_3)$ for that of R_i. $\mathrm{Symb}(A_3)$ and $\mathrm{Symb}(B_3)$ are represented by using $\mathrm{Symb}(A_2)$ and $\mathrm{Symb}(B_2)$.

$$\mathrm{Symb}(A_3) = \mathrm{Symb}(A_2)\mathrm{Symb}(B_2), \tag{19a}$$

$$\mathrm{Symb}(B_3) = \mathrm{Symb}(A_2)\mathrm{Symb}(A_2). \tag{19b}$$

All periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation in the interval $(a_m^4, a_m^3]$ are coded by two symbols $\mathrm{Symb}(A_3)$ and $\mathrm{Symb}(B_3)$. Summarizing the results obtained above, we obtain Coding rule 4.1. Coding rule 4.1 is proved in Subsec. 4.2.

Coding rule 4.1. Let $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$ be the symbols in the interval $(a_m^{k+1}, a_m^k]$ where $k \geq 0$. $\mathrm{Symb}(A_k)$ and $\mathrm{Symb}(B_k)$ are determined by the following coding rules.

$$\mathrm{Symb}(A_0) = 1 \equiv P_0, \tag{20a}$$

$$\mathrm{Symb}(B_0) = 0, \tag{20b}$$

$$\mathrm{Symb}(A_1) = E \equiv P_1, \tag{20c}$$

$$\mathrm{Symb}(B_1) = F, \tag{20d}$$

$$\mathrm{Symb}(A_k) = \mathrm{Symb}(A_{k-1})\mathrm{Symb}(B_{k-1}) \equiv P_k \ (k \geq 2), \tag{20e}$$

$$\mathrm{Symb}(B_k) = \mathrm{Symb}(A_{k-1})\mathrm{Symb}(A_{k-1}) \equiv P_{k-1}P_{k-1} \ (k \geq 2). \tag{20f}$$

Using the doubling operator $\mathcal{D}$, Coding rule 4.1 is renewed as Substitution rule 4.2.

Substitution rule 4.2. Suppose that the code s_q for the periodic orbit of period q in the interval $(a_m^1, a_m^0]$ is known. Let $s_{2^k \times q}$ be the code for period-$2^k \times q$ orbit in the interval $(a_m^{k+1}, a_m^k]$. Applying k times of the doubling operator $\mathcal{D}$ to s_q, the code $s_{2^k \times q}$ is determined.

Using Coding rule 4.1, we have Proposition 4.3.

Proposition 4.3. Let s be the code of periodic orbit which is appeared through the period-doubling bifurcation or the tangent bifurcation in $(a_m^{k+1}, a_m^k]$ $(k \geq 1)$. The code s is represented by $\mathrm{Symb}(A_m)$ and $\mathrm{Symb}(B_m)$ $(m = k, k-1, \cdots, 0)$.

We give two remarks. For example, the period-2 orbit exists in $(a_m^3, a_m^2]$. This is not the periodic orbit which is appeared through the period-doubling bifurcation in $(a_m^3, a_m^2]$. Thus, its code is not represented by $\mathrm{Symb}(A_2)$ or $\mathrm{Symb}(B_2)$.

Let $s_{2\times 3}$ be the code of periodic orbit which is appeared through the tangent bifurcation in $(a_m^2, a_m^1]$. The code $s_{2\times 3}$ is represented by $\mathrm{Symb}(A_1)$ and $\mathrm{Symb}(B_1)$. Using Coding rule 4.1, we obtain that $s_{2\times 3}$ is represented by $\mathrm{Symb}(A_0)$ and $\mathrm{Symb}(B_0)$.

Finally, we show two examples of how to use of Substitution rule 4.2. We pay attention to the period-5 orbit locating between the windows of period-3 orbit and period-4 orbit. This is not included in the Sharkovskiĭ ordering. The codes

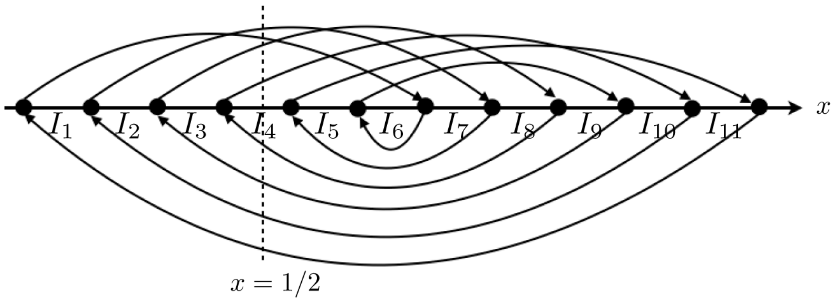

Fig. 8. Periodic orbit $011101010111 = EF \cdot EE \cdot EF$. Interval I_4 includes $x = 1/2$.

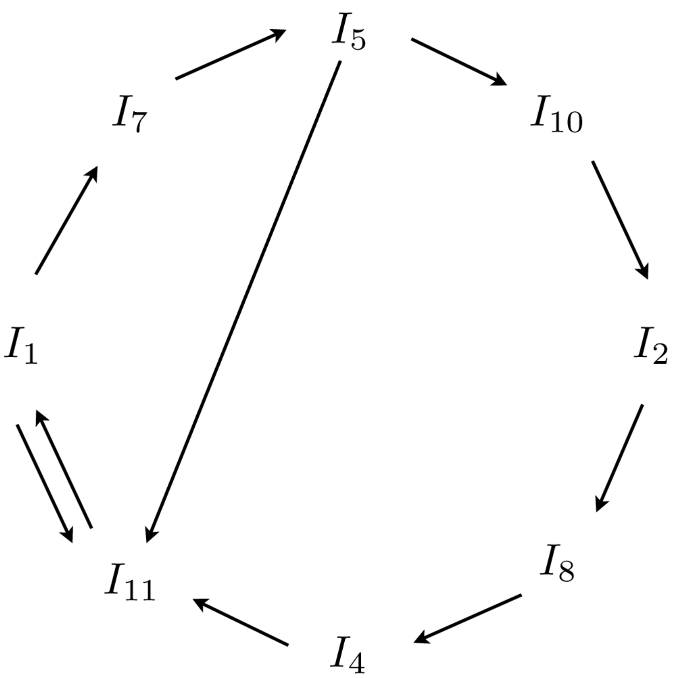

Fig. 9. The simplified Štefan diagram C_{12} where I_3, I_6 and I_9 are deleted.

are 00111 and 00101. First, we apply $\mathcal{D}$ to the stable periodic orbit 00111. The code for period 2×5 is $EFFEE$, and that of period $2^2 \times 5$ is $EF \cdot EE \cdot EE \cdot EF \cdot EF$. Next, we apply $\mathcal{D}$ to the unstable periodic orbit 00101. The code for period 2×5 orbit is $EFFEF$, and that of period $2^2 \times 5$ orbit is $EF \cdot EE \cdot EE \cdot EF \cdot EE$. By numerical calculation, the correctness of codes obtained here is confirmed.

4.2 Proof of Coding rule 4.1

First, we give the proof of Eqs. (20c) and (20d). Next, we give the proof of Eqs. (20e) and (20f).

Proof of Eqs. (20c) and (20d). We consider the situation after the accumulation of period-doubling bifurcations. Therefore, there exist the orbital points β_0 and β_2 of period-4 orbit appeared from α_0 of period-2 orbit. Suppose that the point β_0 locates in the left side of α_0. The point $\beta_0 \in$Int(A_1) is the minimum point of period-4 orbit. On the other hand, the point β_2 locates in the right side of α_0 and in Int(B_1). The point β_2 move to Int(B_1) (see the proof of Coding rule 3.1). From Coding rule 3.1, the code for orbit starting from β_0 is $P_2 = P_1P_0P_0 = 01 \cdot 11$. The former part 01 represents Symb$(A_1) = E$ and thus the latter one 11 represents Symb$(B_1) = F$. This means that the word for orbit from β_2 to β_0 is 11. Thus, we have Symb$(A_1) = E$ and Symb$(B_1) = F$. (Q.E.D.)

Proof of Eqs. (20e) and (20f). Assume that the daughter periodic points γ_0 and γ_4 appear from β_0 of period-4 orbit (see Fig. 2(c)). Suppose that γ_0 (γ_4) locates in the left (right) side of β_0 and γ_4 moves to Int(B_2). From Coding rule 3.1, the code for the periodic orbit starting from γ_0 is $P_3 = P_2P_1P_1 = EF \cdot EE$. The point $f^4(\gamma_0)$ enters into Int(B_2), and $f^4(\gamma_4)$ comes back to Int(A_2). Thus, the former part EF represents Symb(A_2) and the latter one EE represents Symb(B_2). Two symbols are the super-blocks constructed by E and F.

The daughter periodic orbit with code $P_{k+1} = A_kB_k$ has its orbital point in Int(A_k) and Int(B_k). Coding rule 3.1 guarantees that the super-blocks Symb(A_k) and Symb(B_k) are determined by Eqs. (20e) and (20f). (Q.E.D.)

5. Concluding Remarks

The parameter interval $(a_m^{k+1}, a_m^k]$ in the bifurcation diagram is defined. The code of periodic orbit in $(a_m^{k+1}, a_m^k]$

is coded by Symb(A_k) and Symb(B_k). Coding rule 4.1 to determine Symb(A_k) and Symb(B_k) from Symb(A_0) = 1 and Symb(B_0) = 0 is derived and its correctness is proved. We can apply Coding rule 4.1 to the periodic orbits in the unimodal maps.

For example, in the window of period-3 orbit, the bifurcation processes similar to the original bifurcation diagram in $a \in (0, 4]$ are observed. For periodic orbits in the window of period-3 orbit, we can consider the same problem discussed in this paper.

Interval $(a_m^{k+1}, a_m^k]$ is regarded as a small world where the basic words are Symb(A_k) and Symb(B_k). This concept came out of the symbol dynamics naturally. It is needed to reconsider the relation of periodic orbit and code. As a result, we may provide new concept or interpretation of the bifurcation diagram.

Acknowledgments. Author would like to thank the referee for helpful comments.

Appendix A. Sharkovskiĭ ordering

Theorem A was proved by Sharkovskiĭ (Sharkovskiĭ, 1964).

Theorem A. Consider the following ordering on the set of natural numbers (Sharkovskiĭ ordering):

$$
\begin{gathered}
3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \\
2 \times 3 \rhd 2 \times 5 \rhd 2 \times 7 \rhd 2 \times 9 \rhd \cdots \\
2^2 \times 3 \rhd 2^2 \times 5 \rhd 2^2 \times 7 \rhd 2^2 \times 9 \rhd \cdots \\
2^3 \times 3 \rhd 2^3 \times 5 \rhd 2^3 \times 7 \rhd 2^3 \times 9 \rhd \cdots \\
\cdots \rhd 2^4 \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.
\end{gathered}
\tag{A.1}
$$

Let f be a one-dimensional continuous map from interval to itself. If f has a period-n orbit and the relation $n \rhd m$ in the Sharkovskiĭ ordering holds, then f has a period-m orbit.

Appendix B. Translation procedure

We introduce the translation procedure from code for the tent map ($X_{n+1} = 1 - |2X_n - 1|$) to that of the binary map ($X_{n+1} = 2X_n \pmod 1$) (Yamaguchi and Tanikawa, 2016).

Procedure B. Let $w = s_1 s_2 s_3 \cdots s_q$ be a given code. If the parity of w is even, we prepare $s = s_1 s_2 \cdots s_q$ and $t = t_1 t_2 \cdots t_q$ where $t_1 = s_1$. If the parity of w is odd, we prepare $s = ww$. We rewrite the suffices as $s = s_1 s_2 \cdots s_{2q}$ and prepare $t = t_1 t_2 \cdots t_{2q}$ where $t_1 = s_1$.

If the parity of w is even, for $2 \le k \le q$, determine t_k by the following rules (a) or (b). If the parity of w is odd, for $2 \le k \le 2q$, determine t_k by the following rules (a) or (b). After t is determined, output t.

(a) If $\Sigma_{i=1}^{k-1} s_j$ is odd, we let $t_k = 1 - s_k$.
(b) If $\Sigma_{i=1}^{k-1} s_j$ is even, we let $t_k = s_k$.

Appendix C. How to determine the critical value a_m^k

All orbital points exist in the region $[x_{\min}, x_{\max}]$ where $x_{\max} = f(1/2)$ (≤ 1) and $x_{\min} = f^2(1/2)$ (≥ 0). Here, we use the fact that the maximum point $x_{\max}$ is mapped to the minimum point $x_{\min}$ in the unimodal map. In the logistic map, $f(1/2)$ gives the maximum point.

First, we derive the equation to determine a_m^1 $(k = 1)$. In Fig. 2(b), $x_{\min}$ is the left edge of Int(A_1). The relations $f^2(x_{\min}) = x_Q$, $x_{\min} = f^2(1/2)$ and $f(x_Q) = x_Q$ hold. In these equations, x_Q represents the position of Q. Using these relations, we obtain the equation to determine a_m^1.

$$f^5(1/2) = f^4(1/2). \tag{C.1}$$

The critical value a_m^1 determined by Eq. (C.1) is equal to that by $f^4(1/2) = x_Q = 1 - 1/a$. In fact, the left hand side of Eq. (C.1) is rewritten as $f^5(1/2) = f(f^4(1/2)) = f(x_Q) = x_Q$.

Next, we derive the equation to determine a_m^2 $(k = 2)$. In Fig. 2(c), $x_{\min}$ is the left edge of Int(A_2). Let x_{α_0} be the position of α_0. From the relations $f^4(x_{\min}) = x_{\alpha_0}$, $x_{\min} = f^2(1/2)$ and $f^2(x_{\alpha_0}) = x_{\alpha_0}$, we have the equation to determine a_m^2.

$$f^8(1/2) = f^6(1/2). \tag{C.2}$$

We derive the equation to determine a_m^3 $(k = 3)$. In Fig. 2(d), $x_{\min}$ is the left edge of Int(A_3). Let x_{β_0} be the position of β_0. We have the relations $f^8(x_{\min}) = x_{\beta_0}$, $x_{\min} = f^2(1/2)$ and $f^4(x_{\beta_0}) = x_{\beta_0}$ and obtain the equation to determine a_m^3.

$$f^{14}(1/2) = f^{10}(1/2). \tag{C.3}$$

Repeating this procedure, the following equation to determine a_m^k $(k \ge 1)$ is derived.

$$f^{2^{k-1} \times 3 + 2}(1/2) = f^{2^k + 2}(1/2). \tag{C.4}$$

Appendix D. Parity of code and the stability of periodic orbit

We define the parity of code and give the stability of periodic orbit (Hall, 1994).

Definition D.1. If the number of 1 included in the code is even (odd), the parity of code is even (odd).

Property D.2.
(i) Suppose that the parity of code s is even. The periodic orbit with code s is unstable.
(ii) Suppose that the parity of code s is odd. The periodic orbit with code s is stable just after the appearance and occurs the period-doubling bifurcation.

Property D.2 is applicable to all periodic orbits which are appeared through the period-doubling bifurcation or the tangent bifurcation.

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