Stokes Flow around a Hypersphere in *n*-Dimensional Space and Its Visualization

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We derived the Stokes equations and velocity potential around a hyperspherical obstacle in n-dimensional space. The objectives of this study were to understand the hyperspace through the physics in the space and to bring the analytical solution of fluid flow in hyperspace for numerical simulation. The equations were obtained from the n-dimensional Navier-Stokes equation assuming the low Reynolds number flow. These were generalized formulae from a 3-dimensional system to an n-dimensional one. Our results show that the effect of the hyperspherical obstacle on the uniform flow is localized in higher dimensional spaces. We visualized the flow using the collections of hypersections.

Key words: Stokes Flow, Hypersphere, Hyperspace

1. Introduction

It is difficult to visualize hyperspace. As discussed in Abbott's "Flatland" [1], people occupying lower dimensional spaces struggle to comprehend the existence of higher dimensional spaces. In this paper, we consider fluid flow in n-dimensional space. By considering the flow of a fluid in a higher dimensional space we can gain further insight into the nature of hyperspace.

We derived the Stokes equations around a hypersphere in *n*-dimensional space. There are two reasons to consider fluid flow in hyperspace. First, understanding the physics of an *n*-dimensional system can help improve our understanding of the properties of hyperspace. Second, the analytical results for the velocity potential and the flow equations are useful for understanding the flow of a fluid in hyperspace. Furthermore, the analytical solutions of the equations for fluid flow give us an effective tool to examine the accuracy of numerical schemes.

We generalized the formulae for a 3-dimensional system to an *n*-dimensional one. This generalization is straightforward because the procedure for the derivation is the same as that for the 3-dimensional case [4]. Five quantities need to be derived to obtain the Stokes equations: 1) the velocity potential for uniform flow, 2) the velocity potential for the source doublet, 3) the velocity potential for the flow around a hypersphere, 4) the stream function for a perfect fluid, 5) the final form of the stream function for the Stokes flow.

It is known that the effect of obstacles for a Stokes flow is very different to that for a perfect fluid [4]. The effect of obstacles is broad for Stokes flow. The main difference between potential flow and Stokes flow is the absence or presence of viscosity. It is of interest whether such a broad effect of the viscosity is observed in higher dimensions or

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not.

2. Potential Flow

We start our treatment of the Stokes equations with the equation for the velocity potential in *n*-dimensional space Φ_n ,

$$\nabla_n^2 \Phi_n = 0, \tag{1}$$

where the operator ∇_n is defined by,

$$\nabla_n = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right). \tag{2}$$

This means that we focus on stable flow with a low Reynolds number [3]. Note that we use *n*-dimensional Cartesian coordinates and denote them (x_1, x_2, \dots, x_n) . The derivation of Eq. (1) is summarized in Appendix B. Using the velocity potential, the velocity field is defined by,

$$\mathbf{u}_n = \nabla_n \Phi_n. \tag{3}$$

Our goal is to obtain the exact form of the Stokes flow function around a hyperspherical obstacle in *n*-dimensional space. The situation is summarized as follows: A fluid flows parallel to the x_n axis from the negative x_n direction with a velocity U, and a hyperspherical obstacle of radius a is placed with its center at the origin. Consequently, the flow is axisymmetric with respect to x_n . We introduce cylindrical coordinates in *n*-dimensional space (y, θ_2 , θ_3 , \cdots , θ_{n-1} , x) and match the x axis of the cylindrical coordinate system with the x_n axis of the Cartesian coordinate system. We choose one of the rest coordinates (x_1, \dots, x_n) x_{n-1}), we suppose x_1 in the following, as the y axis. The y axis represents the radial distance from the origin of the (n-1)-dimensional spherical coordinate system and corresponds to a hypersection whose center is $(0, 0, \dots, 0, x)$ in n-dimensional space. Rest variables which represent azimuthal angles $(\theta_2, \theta_3, \dots, \theta_{n-1})$ in the cylindrical coordinate system can be ignored because of the symmetry. For

this reason, we focus our consideration on the flow in x-y coordinates in the following. The relations among the coordinates are summarized in Appendix A. Here, $u_n(x, y)$ and $v_n(x, y)$ denote the x and y elements of the velocity in *n*-dimensional space, respectively. Then, the velocity field for the *n*-dimensional Cartesian coordinates is described by,

$$\mathbf{u}_{n}(x_{1}, x_{2}, x_{3}, \dots, x_{n}) = u_{n}(x_{n}, r_{n-1})\mathbf{e}_{n} + v_{n}(x_{n}, r_{n-1})\mathbf{e}_{r},$$
(4)
where $\mathbf{e}_{n} = (0, 0, \dots, 0, 1), \quad \mathbf{e}_{r} = (x_{1}, x_{2}, \dots, x_{n-1}, 0)/r_{n-1}, \quad \text{and} \quad r_{n-1} = \sqrt{x_{1}^{2} + x_{2}^{2} + \dots + x_{n-1}^{2}}.$

Consider a uniform flow whose magnitude of velocity is U. As mentioned above, the fluid flows parallel to the x axis from negative x to positive x. Therefore, the velocity potential of the flow is expressed by,

$$\Phi_n = Ux,\tag{5}$$

and the velocity field is,

$$u_n = \frac{\partial \Phi_n}{\partial x} = U, \ v_n = \frac{\partial \Phi_n}{\partial y} = 0.$$
 (6)

We introduce the velocity potential for the source doublet. As shown in Eq. (C.2), the velocity potential for a point source is described by,

$$\Phi_n = -\frac{m}{n-2} \frac{1}{r^{n-2}},$$
(7)

where $r = \sqrt{x^2 + y^2}$. Then we obtain the formula for the velocity potential for the source doublet at the origin. We also suppose that the dipole moment is parallel to the *x* axis from negative to positive *x*. That is,

$$\Phi_n = -\gamma \frac{x}{r^n},\tag{8}$$

as shown in Eq. (C.6). The negative sign indicates that the fluid flows from the field for negative x to the field for positive x. The coefficient γ is the strength of the dipole moment. Derivations of Eqs. (7) and (8) are summarized in Appendix C.

In the case of a perfect fluid, the potential function for the flow around the hyperspherical obstacle is described by the combination of the two velocity potentials: the uniform flow and the source doublet whose dipole moment is directed from positive x to negative x. That is,

$$\Phi_n = Ux + \gamma \frac{x}{r^n}.$$
(9)

Substituting $\gamma = Ua^n/(n-1)$ into the equation, we obtain the final form of the velocity potential for the flow of perfect fluid around a hypersphere as,

$$\Phi_n(x, y) = Ux + \frac{1}{n-1}U\left(\frac{a}{r}\right)^n.$$
 (10)

Consequently, the elements of the velocity field are,

$$u_n = \frac{\partial \Phi_n}{\partial x}$$

= $U\left(1 - \left(\frac{a}{r}\right)^n \left(\left(\frac{x}{r}\right)^2 - \frac{1}{n-1}\left(\frac{y}{r}\right)^2\right)\right), (11)$

$$v_n = \frac{\partial \Phi_n}{\partial y} = -U\left(1 + \frac{1}{n-1}\right) \left(\frac{a}{r}\right)^n \frac{xy}{r^2}.$$
 (12)



Fig. 1. *n* dependence of the maximum velocity of the potential flow.

The inner product of (x, y) and (u_n, v_n) obtained by substituting three relations r = a, $x = a \cos \phi$, and $y = a \sin \phi$, equals zero for any value of ϕ . This means that the flow does not penetrate the hypersphere. Therefore, we conclude that Eq. (10) describes the flow of a perfect fluid around a hypersphere of radius *a* whose center is located on the origin.

The velocity along the y axis is obtained by the substitution of x = 0 into Eqs. (11) and (12). The fluid flows parallel to the x axis because v_n is zero on the axis. Then, the velocity becomes,

$$u_n = U\left(1 + \left(\frac{1}{n-1}\right)\left(\frac{a}{y}\right)^n\right). \tag{13}$$

From this equation, the velocity profile along the y axis has a maximum magnitude of u_n at $(0, \pm a)$ because $|y| \ge a$. That is,

$$u_n = \left(1 + \frac{1}{n-1}\right)U.$$
 (14)

This is also the maximum value of the magnitude of the velocity across the whole field. Figure 1 shows the *n* dependence of the velocity ratio u_n/U of Eq. (14). The value monotonically decreases and tends towards unity with increasing *n*.

3. Stokes Flow

We generalized the Stokes flow equations to *n*-dimensional space. As shown in Eq. (B.12), the equation of continuity in cylindrical coordinates is described by,

$$\frac{\partial u_n}{\partial x} + \frac{1}{y^{n-2}} \frac{\partial}{\partial y} \left(y^{n-2} v_n \right) = 0.$$
(15)

From this equation, we define the flow function Ψ_n as,

$$u_n = \frac{1}{y^{n-2}} \frac{\partial \Psi_n}{\partial y}, \ v_n = -\frac{1}{y^{n-2}} \frac{\partial \Psi_n}{\partial x}.$$
 (16)

This equation satisfies Eq. (15). Note that the multipliers are generalized from 1/y in the 3-dimensional system to $1/y^{n-2}$ in the *n*-dimensional system. Using a combination



Fig. 2. Comparisons of stream lines around a hypersphere between potential flow (L.H.S.) and Stokes flow (R.H.S.) for 3, 4, and 10 dimensions.

of these definitions and Eqs. (11) and (12), we derive the formula for the flow function as,

$$\Psi_n(x, y) = \frac{1}{n-1} U y^{n-1} \left(1 - \left(\frac{a}{r}\right)^n \right).$$
(17)

This function can also be expressed as,

$$\Psi_n(r,\theta) = \frac{1}{n-1} U\left(r^{n-1} - \frac{a^n}{r}\right) \sin^{n-1}\theta, \qquad (18)$$

by introducing the variable $\theta = \sin^{-1}(y/r)$.

In order to introduce the flow function for Stokes flow around a hypersphere, we add another term called the Stokes pole to Eq. (18). We write the new flow function as,

$$\Psi_n(r,\theta) = \left(\frac{1}{n-1}Ur^{n-1} - \frac{\beta}{r} - \lambda r\right)\sin^{n-1}\theta, \quad (19)$$

where the third term in the parentheses is the Stokes pole.

We set the boundary conditions such that the velocity on the hypersurface of the hypersphere is zero. The boundary conditions are,

$$\Psi_n(a,\theta) = 0, \tag{20}$$

for the tangential element and,

$$\left. \frac{\partial \Psi_n}{\partial r} \right|_{r=a} = 0, \tag{21}$$

for the normal element. Solving the simultaneous Eqs. (20) and (21) for β and λ , we obtain,

$$\beta = -\frac{n-2}{2(n-1)}Ua^n, \ \lambda = \frac{n}{2(n-1)}Ua^{n-2}.$$
 (22)

Finally, the flow function is derived by substitution of Eq. (22) into Eq. (19),

$$\Psi_n(r,\theta) = \frac{1}{2(n-1)} \cdot U\left(2r^{n-1} + (n-2)\frac{a^n}{r} - na^{n-2}r\right)\sin^{n-1}\theta.$$
(23)

The flow function for 4-dimensional space becomes,

$$\Psi_4(r,\theta) = \frac{1}{3r} U(r-a)^2 (r+a)^2 \sin^3 \theta, \qquad (24)$$

for example. Consequently, we obtained the velocity components of the field as,

$$u_{n}(x, y) = \frac{1}{y^{n-2}} \frac{\partial \Psi_{n}}{\partial y}$$

= $U\left(1 + \frac{n-2}{2} \left(\frac{a}{r}\right)^{n} - \frac{n}{2} \left(\frac{a}{r}\right)^{n-2} - \frac{n(n-2)}{2(n-1)} \left(\frac{a}{r}\right)^{n} y^{2} \left(\frac{1}{a^{2}} - \frac{1}{r^{2}}\right)\right),$ (25)

$$U_{n}(x, y) = -\frac{1}{y^{n-2}} \frac{1}{\partial x}$$
$$= U \frac{n(n-2)}{2(n-1)} \left(\frac{a}{r}\right)^{n-2} \left(\frac{a}{r} - 1\right) \frac{xy}{r^{2}}.$$
 (26)

Figure 2 shows comparisons of stream lines for a potential flow (Eqs. (11) and (12)) and a Stokes flow (Eqs. (25) and (26)) in 3, 4, and 10 dimensions. The effect of the hyperspherical obstacle on the uniform flow tends to be localized as the number of dimensions increases in both cases. The obstacle affects the field more widely in the case of Stokes flow (presence of viscosity) than in the case of potential flow in the same dimension. The effect, however, is localized as the number of dimensions increases.

We considered the velocity profiles on the y axis of the Stokes flow for different dimensions. For those cases, v_n equals zero so that the fluid flows parallel to the x axis. The equation for the velocity profile is,

$$u_{n}(0, y) = U\left(1 - \frac{n-2}{2(n-1)}\left(\frac{a}{y}\right)^{n} - \frac{n}{2(n-1)}\left(\frac{a}{y}\right)^{n-2}\right).$$
(27)



Fig. 3. Velocity profiles along the *y* axis of the Stokes flow for different dimensions.

Figure 3 shows the velocity profile along the y axis obtained from Eq. (27). An abrupt increase in velocity is observed near the hypersurface of the hypersphere for high dimensions. This indicates that the effect of hyperspherical obstacles decreases with an increasing number of dimensions.

4. Visualization

We tried to visualize the 4-dimensional flow using Abbott's description of a sphere [1]. He explained the shape of a sphere by the continuous change of a circle's radius to a fictional inhabitant of a two-dimensional flatland. Similar to the Abbott method, we made sections of 3- and 4dimensional spaces and displayed the velocity field in each section. The velocity field is obtained from Eq. (4). The field is described by,

$$\mathbf{u}_{3}(x_{1}, x_{2}, x_{3}) = \begin{pmatrix} v_{3}\left(x_{3}, \sqrt{x_{1}^{2} + x_{2}^{2}}\right) \frac{x_{1}}{\sqrt{x_{1}^{2} + x_{2}^{2}}} \\ v_{3}\left(x_{3}, \sqrt{x_{1}^{2} + x_{2}^{2}}\right) \frac{x_{2}}{\sqrt{x_{1}^{2} + x_{2}^{2}}} \\ u_{3}\left(x_{3}, \sqrt{x_{1}^{2} + x_{2}^{2}}\right) \end{pmatrix}^{T},$$
(28)

for 3-dimensional flow and,

$$\mathbf{u}_{4} = \begin{pmatrix} v_{4} \left(x_{4}, \sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}} \right) \frac{x_{1}}{\sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}}} \\ v_{4} \left(x_{4}, \sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}} \right) \frac{x_{2}}{\sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}}} \\ v_{4} \left(x_{4}, \sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}} \right) \frac{x_{3}}{\sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}}} \\ u_{4} \left(x_{4}, \sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}} \right) \frac{x_{3}}{\sqrt{x_{1}^{2} + x_{2}^{2} + x_{3}^{2}}} \end{pmatrix}$$

$$(29)$$

for 4-dimensional flow.

Figure 4 shows the velocity fields for Stokes flow in 3dimensional space obtained using Eqs. (25), (26) and (28). The fluid flows from left to right. Each figure shows the velocity field for $x_2 = 0.0, 0.3, 0.6$, and 0.9. The color



Fig. 4. 3-dimensional Stokes flow on 2-dimensional planes. a: $x_2 = 0.0$, b: $x_2 = 0.3$, c: $x_2 = 0.6$, and d: $x_2 = 0.9$.

denotes the magnitude of the x_2 element of the velocity field. In other words, the color represents the amount of fluid that goes to or from another dimension. For $x_2 = 0.0$, the fluid flows along the section so that the color of all arrows in this section are same. On the other hand, the fluid also flows into and from another dimension (x_3) so that the color is dependent on the position. Red denotes a large magnitude for the velocity and obviously the area of red increases with increasing x_2 .

Figure 5 shows the velocity fields for Stokes flow in 4dimensional space using the same method. The figures show the hypersections of $x_3 = 0.0, 0.3, 0.6$, and 0.9. The color denotes the magnitude of the x_3 element of the velocity field. For the 3-dimensional case, the fluid goes to or comes from another dimension. The fluid moves to another dimension (x_4) in front of the sphere and returns behind the sphere. We can see how such phenomena disperse in 3dimensional space in the figures. The redirection to another dimension is dispersed across the 3-dimensional space so that the total amount of fluid that is redirected to other dimensions is large compared to that for 3-dimensional flow. As a result, fluid that is relatively far from the obstacle does not move between dimensions to avoid it.

5. Hagen-Poiseuille Flow

We also obtained the velocity profile for a Hagen-Poiseuille flow (the velocity profile in the hypercylinder) in *n*-dimensional space. We supposed that the hypercylinder consisted of a hyperspherical section whose radius is *a* and a longitudinal wall. We denote the velocity distribution of the fluid $u_n(r)$, where *r* is the distance to the axis of the hypercylinder. The velocity on the wall is zero so that $u_n(\pm a)$ is zero. The length of the hypercylinder is *l* and the pressure difference between the input and output is Δp . From Eq. (B.3), the equation for u(r) is expressed as,

$$\frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{du_n}{dr}\right) = -\frac{\Delta p}{\mu l},\tag{30}$$

where $\Delta p/l$ represents a pressure gradient. We introduce a new notation $k = \Delta p/\mu l$ in the following. The solution to this equation for the given boundary conditions is,

$$u_n(r) = \frac{k}{2(n-1)}(a^2 - r^2).$$
 (31)



Fig. 5. 4-dimensional Stokes flow on the 3-dimensional hypersection. a: $x_3 = 0.0$, b: $x_3 = 0.3$, c: $x_3 = 0.6$, and d: $x_3 = 0.9$.

This solution indicates that the velocity profile in the hypercylinder is always quadratic and the dimension is the only coefficient of the polynomial. Figure 6 shows the profiles for the same value of k. The velocity distribution tends to be smaller and flatter with increasing dimensions. Note that the pressure has dimensions of force per unit facet so that the coefficient itself has different interpretations depending on n.

6. Discussion

The effect of the hyperspherical obstacle is localized with increasing number of dimensions n, for both the potential flow and the Stokes flow. For this reason, we conclude that the effect of a hyperspherical obstacle decreases with increasing dimensions. This is because the fluid can avoid the obstacle by moving between dimensions. The visualization of the flow in 4-dimensional space revealed how such evasion occurs. Similar behavior is observed in the Hagen-Poiseuille flow.

The visualization of the flow in four-dimensional space, however, does not improve our understanding of hyperspace. The information obtained from the visualization of



Fig. 6. Velocity profiles for Hagen-Poiseuille flow for different *n*.

four-dimensional Stokes flow can be predicted based on three-dimensional one. One of the reasons is that we consider an axisymmetric flow which shows the same characteristics for different axes. Vector plots in 3-dimensional space are not useful because vectors on the near side of the obstacle hide those on the far side. Furthermore, this method cannot be applied to the cases where the number of dimensions is higher than four.

Appendix A. Coordinates

We introduced three types of coordinates: Cartesian, spherical, and cylindrical. We denote the Cartesian coordinates by (x_1, x_2, \dots, x_n) , the spherical coordinates by $(r_n, \theta_2, \dots, \theta_n)$, and the cylindrical coordinates by $(r_{n-1}, \theta_2, \dots, \theta_{n-1}, x_n)$. Note that the cylindrical coordinates consist of a combination of the (n - 1)-dimensional hyperspherical coordinates, because a cylinder in an *n*-dimensional system consists of a hyperspherical section and its length. The relation between the Cartesian coordinates and spherical coordinates is summarized [2] by,

$$r_{k-1} = r_k \sin \theta_k, \ x_k = r_k \cos \theta_k, \ x_1 = r_2 \sin \theta_2, \quad (A.1)$$

where r_k is defined by,

$$r_k^2 = \sum_{i=1}^k x_i^2.$$
 (A.2)

Figure A.1 shows the relation between the cylindrical coordinates and the variables in 2-dimensional and 3-dimensional space, for example. It is notable that the Cartesian coordinates and the cylindrical ones are identical in a case of 2-dimensional space because of the lack of azimuthal angles.

The formula for the Laplacian depends on both *n* and the coordinate system. We denote the Laplacian for the Cartesian coordinates by ∇_n^2 , $\nabla_{n,S}^2$ for the spherical coordinates, and $\nabla_{n,C}^2$ for the cylindrical coordinates. For example, the Laplacian for the spherical coordinates in 4-dimensional space is written as,

$$\nabla_{4,\mathrm{S}}^{2} = \frac{1}{r_{4}^{3}} \frac{\partial}{\partial r_{4}} \left(r_{4}^{3} \frac{\partial}{\partial r_{4}} \right) + \frac{1}{r_{4}^{2} \sin^{2} \theta_{4} \sin^{2} \theta_{3}} \left(\frac{\partial^{2}}{\partial \theta_{2}^{2}} \right) \\ + \frac{1}{r_{4}^{2} \sin^{2} \theta_{4} \sin \theta_{3}} \left(\frac{\partial}{\partial \theta_{3}} \left(\sin \theta_{3} \frac{\partial}{\partial \theta_{3}} \right) \right) \\ + \frac{1}{r_{4}^{2} \sin^{2} \theta_{4}} \left(\frac{\partial}{\partial \theta_{4}} \left(\sin^{2} \theta_{4} \frac{\partial}{\partial \theta_{4}} \right) \right).$$

In general, the Laplacian for the *n*-dimensional spherical coordinates is written as,

$$\nabla_{n,\mathbf{S}}^{2} = \frac{1}{r_{n}^{n-1}} \frac{\partial}{\partial r_{n}} \left(r_{n}^{n-1} \frac{\partial}{\partial r_{n}} \right) + \frac{1}{r^{2}} \Delta_{S^{n-1}}, \qquad (A.3)$$

where $\Delta_{S^{n-1}}$ is the spherical Laplace operator in (n-1)dimensions [2]. The spherical Laplace operator consists of variables of azimuthal angles and does not include r_n . For cylindrical coordinates, the Laplacian is a combination of those for (n-1)-dimensional spherical coordinates and for one dimensional Cartesian coordinates,

$$\nabla_{n,\mathbf{C}}^2 = \nabla_{n-1,\mathbf{S}}^2 + \frac{\partial^2}{\partial x_n^2}.$$
 (A.4)

Appendix B. Basic Formulae

The formulae for continuity and the Navier-Stokes equations can be generalized to *n*-dimensional space as,

$$\nabla_n \cdot \mathbf{u}_n = 0, \tag{B.1}$$

and,

$$\frac{\partial \mathbf{u}_n}{\partial t} + (\mathbf{u}_n \cdot \nabla_n) \,\mathbf{u}_n = -\frac{1}{\rho} \nabla_n p + \frac{\mu}{\rho} \nabla_n^2 \mathbf{u}_n + \frac{1}{\rho} \mathbf{K}, \quad (B.2)$$

where ρ is the density, p is the pressure, μ is the viscosity, and **K** is the external force. We used the following assumptions: low Reynolds number, steady flow, and absence of external force. These are the same as the assumptions for the 3-dimensional Stokes flow. Under these assumptions, the equations are simplified to,

$$\mu \nabla_n^2 \mathbf{u}_n = \nabla_n p. \tag{B.3}$$

The homogeneous solution to Eq. (B.3) can be written in terms of the velocity potential Φ_n ,

$$\mathbf{u}_n = \nabla_n \Phi_n = \left(\frac{\partial \Phi_n}{\partial x_1}, \frac{\partial \Phi_n}{\partial x_2}, \cdots, \frac{\partial \Phi_n}{\partial x_n}\right), \qquad (B.4)$$

where Φ_n satisfies the Laplace equation,

$$\nabla_n^2 \Phi_n = \frac{\partial^2 \Phi_n}{\partial x_1^2} + \frac{\partial^2 \Phi_n}{\partial x_2^2} + \dots + \frac{\partial^2 \Phi_n}{\partial x_n^2} = 0.$$
(B.5)

This is the same as the equation presented at the beginning of this paper, Eq. (1).

We consider the equation of continuity B.1 for the cylindrical coordinates. We introduce new notations for the variable ξ_i whose relations to other variables is defined by,

$$(\xi_1, \xi_2, \cdots, \xi_n) = (r_{n-1}, \theta_2, \theta_3, \cdots, \theta_{n-1}, x_n).$$
 (B.6)

Using the new variables, the equation of continuity is expressed as,

$$\nabla_n \cdot \mathbf{u}_n = \frac{1}{\prod_{i=1}^n h_i} \sum_{i=1}^n \left(\frac{\partial}{\partial \xi_i} \left(\frac{\prod_{k=1}^n h_k}{h_i} v_i \right) \right), \quad (B.7)$$

where v_i is the *i*-th element of the velocity in the cylindrical coordinate system and h_k is defined by,

$$h_k = \sqrt{\sum_{i=1}^n \left(\frac{\partial x_i}{\partial \xi_k}\right)^2}.$$
 (B.8)

For $3 \le k \le n - 1$, h_k^2 is obtained by,

$$h_k^2 = \sum_{i=1}^n \left(\frac{\partial x_i}{\partial \xi_k}\right)^2 = \sum_{i=1}^k \left(\frac{\partial x_i}{\partial \theta_k}\right)^2$$
$$= \sum_{i=1}^{k-1} \left(\frac{\cos \theta_k}{\sin \theta_k} x_i\right)^2 + \left(-\frac{\sin \theta_k}{\cos \theta_k} x_k\right)^2$$
$$= \left(\frac{\cos \theta_k}{\sin \theta_k}\right)^2 \sum_{i=1}^{k-1} x_i^2 + (r_k \sin \theta_k)^2$$
$$= \left(\frac{\cos \theta_k}{\sin \theta_k}\right)^2 r_{k-1}^2 + r_k^2 \sin^2 \theta_k$$
$$= r_k^2 (\cos^2 \theta_k + \sin^2 \theta_k) = r_k^2.$$
(B.9)

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Fig. A.1. Definitions of the cylindrical coordinates and the variables in the cylindrical coordinates.

Using similar procedures, we obtain the equations for h_i ,

$$h_1 = 1, h_2 = r_2, h_3 = r_3, \dots, h_{n-1} = r_{n-1}, h_n = 1.$$

(B.10)

For the axisymmetric case, the right-hand side of Eq. (B.7) is written as,

$$\frac{1}{r_{n-1}^{n-2}f(\theta_{1},\cdots,\theta_{n-2})} \left\{ \frac{\partial}{\partial r_{n-1}} \left(\frac{r_{n-1}^{n-2}f(\theta_{1},\cdots,\theta_{n-2})}{h_{1}}v_{1} \right) + \frac{\partial}{\partial x_{n}} \left(\frac{r_{n-1}^{n-2}f(\theta_{1},\cdots,\theta_{n-2})}{h_{n}}v_{n} \right) \right\}$$

$$= \frac{1}{r_{n-1}^{n-2}} \frac{\partial}{\partial r_{n-1}} \left(r_{n-1}^{n-2}v_{1} \right) + \frac{\partial v_{n}}{\partial x_{n}}, \qquad (B.11)$$

where $f(\theta_1, \dots, \theta_{n-2})$ is $\prod_{i=2}^{n-2} (h_i/r_{n-1})$ and is not dependent on r_{n-1} or x_n . Therefore, under the notation used in this study, the equation of continuity for the *n*-dimensional axisymmetric flow is written as,

$$\frac{1}{y^{n-2}}\frac{\partial}{\partial y}\left(y^{n-2}v_n\right) + \frac{\partial u_n}{\partial x} = 0.$$
 (B.12)

Appendix C. Single Source and Source Doublet

We consider a point source of strength m, located at the origin. The behavior of the point source is dependent on the sign of m: it is a source for m > 0 and a sink for m < 0. The potential function for the point source is obtained from the Laplace equation in n-dimensional spherical coordinates Eq. (A.3) by,

$$\frac{1}{r_n^{n-1}}\frac{d}{dr_n}\left(r_n^{n-1}\frac{d\Phi_n}{dr_n}\right) = 0.$$
 (C.1)

The potential function for a point source depends on only the radial variable r_n so that the partial differential in Eq. A.3) is replaced by the ordinary differential and the second term vanishes. The solution to Eq. (C.1) is,

$$\Phi_n = -\frac{m}{n-2} \frac{1}{r_n^{n-2}}.$$
 (C.2)

We acquired the n-dimensional source doublet whose dipole direction is in the positive x direction. The potential is the limit of the sum of the potentials of point source A and point sink B when the distance between the two points goes to zero. We located A and B on the x_n axis: negative x_n for the source and positive x_n for the sink. The distance from the origin to each point is ϵ , and their locations in Cartesian coordinates are $(0, 0, \dots, -\epsilon)$ and $(0, 0, \dots, \epsilon)$, respectively. First, we take arbitrary point P and denote the distances of the point source and the point sink from P as r_{nA} and r_{nB} , respectively. The velocity potential at P is written as,

$$\Phi_n = -\frac{m}{n-2} \frac{1}{r_{\rm nA}^{n-2}} + \frac{m}{n-2} \frac{1}{r_{\rm nB}^{n-2}} = \frac{m}{n-2} \frac{r_{\rm nB}^{n-2} - r_{\rm nA}^{n-2}}{r_{\rm nA}^{n-2} r_{\rm nB}^{n-2}}.$$

Using the theorem of cosines, r_{nA} and r_{nB} are approximated by,

$$r_{nA}^{2} = r_{n}^{2} + \epsilon^{2} + 2r_{n}\epsilon\cos\theta \simeq r_{n}^{2}\left(1 + \frac{2\epsilon}{r_{n}}\cos\theta\right), \quad (C.4)$$
$$r_{nB}^{2} = r_{n}^{2} + \epsilon^{2} - 2r_{n}\epsilon\cos\theta \simeq r_{n}^{2}\left(1 - \frac{2\epsilon}{r_{n}}\cos\theta\right), \quad (C.5)$$

where r_n is the distance of the point P from the origin and θ is the angle between the x_n axis and the position vector for P. We substitute these equations into Eq. (C.3) and take the limit as $\epsilon \to 0$, where $\mu = 2m\epsilon$ is constant. Then we obtain the formula for the velocity potential of the source doublet as,

$$\Phi_{n} = \lim_{\epsilon \to 0} \left(-\frac{\mu}{2\epsilon(n-2)} \left(\frac{r_{n}^{n-2}(2\frac{\epsilon}{r_{n}}(n-2)\cos\theta)}{(1-\frac{\epsilon^{2}}{r_{n}^{2}}(n-2)^{2}\cos^{2}\theta)r_{n}^{2n-4}} \right) \right)$$
$$= -\mu \frac{1}{r_{n}^{n-1}}\cos\theta = -\mu \frac{x}{r_{n}^{n}}, \tag{C.6}$$

and we use the formula $(1 + a\epsilon)^k \simeq 1 + ka\epsilon$ for small ϵ .

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