The Kepler Triangle and Its Kin

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The Kepler triangle, also known as the golden right triangle, is the right triangle with its sides of ratios '$1 : \phi^{1/2} : \phi$,' where ϕ denotes the golden ratio. Also known are the silver right triangle and the square-root-three ϕ right triangle. This study introduces the generalised golden right triangle, which have sides of lengths closely related to ϕ and the Fibonacci numbers, F_n: '$(F_{n-2})^{1/2} : \phi^{n/2} : (F_n)^{1/2}\phi$' for any natural number n. This formalism covers all the known ϕ-related right triangle, i.e., the Kepler triangle and its kin. As n tends to infinity, the ratios of the sides go to '$\phi^{-1} : 5^{1/4} : \phi$.' Our model plays an important role in the classroom to study the golden ratio, the Fibonacci numbers and the Pythagorean theorem.

Key words: Golden Ratio, Fibonacci Sequence, Pythagorean Theorem

1. Introduction

The point that Herodotus (485–425BCE) writes about King Khufu's pyramid in his Histories, vol. II, article 124 (Löhmer, 2006), is '... For the making of the pyramid itself there passed a period of twenty years; and the pyramid is square, each side measuring eight hundred feet, and the height of it is the same. ...' But modern mathematicians have fabricated the scientific urban-legend: 'the triangle in the side of the pyramid equals the square of the height.' Then we have the relation related to the golden ratio ϕ: (half the base, the vertical height, the height of the triangle) $= (1, \phi^{1/2}, \phi)$, i.e., the Kepler triangle or the golden right triangle. The actual measurements support the idea in this urban legend. Johannes Kepler (1571–1630) is not an inventor of the golden right triangle. He just mentioned it in his letter to Prof Michael Mästlin dated 30th October 1597 (Frisch, 1858, pp. 34–38).

The aim of this study is to generalise the golden right triangle, which has sides of lengths closely related to ϕ and the Fibonacci numbers. Our formalism covers all the known ϕ-related right triangles, i.e., the golden right triangle, the silver right triangle and the square-root-three ϕ right triangle (Olsen, 2002, 2006, pp. 42–43); these are the Kepler triangle and its kin. The ultimate golden right triangle, as the parameter $n \to \infty$, is derived as (the short leg, the long leg, the hypotenuse) $\to (\phi^{-1}, 5^{1/4}, \phi)$. This result is absolutely novel.

2. Theory

Definition I: the golden ratio

As shown in Fig. 1, the segment AB is divided by C such that

$$\mathrm{AB} : \mathrm{BC} = \mathrm{BC} : \mathrm{CA}.$$

The golden ratio is defined by BC, designated by ϕ, to CA as unity. Hence the above-mentioned relation is rewritten as follows.

$$1 + \phi : \phi = \phi : 1,$$

or

$$\phi^2 = \phi + 1.$$

The solution is given by

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

Definition II: the Fibonacci sequence

The sequence is defined by the recursion formula,

$$F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 3,$$

subject to the initial condition: $F_1 = F_2 = 1$.

We also extend the Fibonacci sequence towards the non-positive generation. Since $F_2 = F_1 + F_0$, then

$$F_0 = 0.$$

Accordingly

$$F_{-1} = 1,$$

for $F_1 = F_0 + F_{-1}$.

Below we often make use of the Binet's formula (e.g., Olsen, 2006, p. 53):

$$F_n = \frac{1}{\sqrt{5}}\{\phi^n - (-1)^n \phi^{-n}\} \quad \text{for} \quad n \geq 1.$$

Lemma I: the Fibonacci number by the ratio of the irrational numbers

$$F_n = \frac{\phi^n + F_{n-2}}{\phi^2} \quad \text{for} \quad n \geq 1. \tag{1}$$

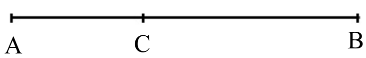

Fig. 1. The golden section: BC $= \phi$; CA $= 1$.

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[Proof]

We shall show the truth by use of the Binet's formula.

$$\frac{\phi^n + F_{n-2}}{\phi^2}$$
$$= \phi^{n-2} + \phi^{-2}\frac{1}{\sqrt{5}}\{\phi^{n-2} - (-1)^{n-2}\phi^{2-n}\}$$
$$= \frac{1}{\sqrt{5}}\{\sqrt{5}\phi^{n-2} + \phi^{n-4} - (-1)^{n-2}\phi^{-n}\}$$
$$= \frac{1}{\sqrt{5}}\{(\phi^2 - \phi^{-2})\phi^{n-2} + \phi^{n-4} - (-1)^n\phi^{-n}\}$$
$$(\because \phi^2 - \phi^{-2} = \sqrt{5})$$
$$= \frac{1}{\sqrt{5}}\{\phi^n - \phi^{n-4} + \phi^{n-4} - (-1)^n\phi^{-n}\}$$
$$= \frac{1}{\sqrt{5}}\{\phi^n - (-1)^n\phi^{-n}\}$$
$$= F_n.$$

[Q.E.D.]

Theorem I: the generalised golden right triangle

We generalise the golden right triangle of (the short leg, the long leg, the hypotenuse) by

$$(\sqrt{F_{n-2}}, \phi^{n/2}, \sqrt{F_n}\phi) \quad \text{for} \quad \forall n \geq 1,$$

as shown in Fig. 2. Then the known ϕ-related right triangles, i.e., the Kepler triangle and its kin, are all covered by the formula above.

In case $n = 1$, the Kepler triangle, the golden right triangle, is defined by the set of $(1, \phi^{1/2}, \phi)$.

In case $n = 3$, the silver right triangle is defined by the set of $(1, \phi^{3/2}, 2^{1/2}\phi)$.

In case $n = 4$, Olsen's square-root-three ϕ right triangle (Olsen, 2002) is defined by the set of $(1, \phi^2, 3^{1/2}\phi)$.

We should note that, in case $n = 2$, the triangle is degenerated to the segment of the set $(0, \phi, \phi)$.

We may call the ultimate golden right triangle, as n tends to infinity. Although the set itself is divergent, we can determine the ratios amongst the sides. By use of the Binet's formula, we retain the leading terms of the sides such that

$$(\sqrt{F_{n-2}}, \phi^{n/2}, \sqrt{F_n}\phi) \to (5^{-\frac{1}{4}}\phi^{\frac{n-2}{2}}, \phi^{\frac{n}{2}}, 5^{-\frac{1}{4}}\phi^{\frac{n}{2}}\phi)$$
$$= (\phi^{-1}, 5^{1/4}, \phi)5^{-1/4}\phi^{n/2},$$

as $n \to \infty$. The set of values in the parentheses above last satisfies the Pythagorean theorem, because

$$\phi^2 = \sqrt{5} + \phi^{-2}.$$

[Proof]

Due to Eq. (1) of Lemma I, the generalised golden right triangle satisfies the Pythagorean theorem:

$$F_n\phi^2 = \phi^n + F_{n-2}. \quad (2)$$

We confirm the uniqueness of the generalisation about the combination between integers and the golden ratio. Suppose the integer sequences a_n and b_n, completely different

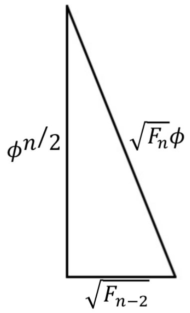

Fig. 2. The generalised golden right triangle (a generic image).

from the Fibonacci sequence, exist for the generalisation, then these must satisfy the relation such that

$$a_n\phi^2 = \phi^n + b_n. \quad (3)$$

Taking difference between Eqs. (2) and (3), we obtain the result below.

$$(F_n - a_n)\phi^2 = F_{n-2} - b_n.$$

The left-hand side of the equation above is irrational, whilst the right-hand side is integral. Therefore, the equation above holds true, if and only if $F_n - a_n = 0$ and $F_{n-2} - b_n = 0$. That is, $a_n = F_n$ and $b_n = F_{n-2}$. Hence the generalisation is unique.

[Q.E.D.]

3. Conclusion

It is our major success to generalise the definition of the Kepler triangle and its kin by use of the recursion formula of ϕ and the Fibonacci numbers:

$$\text{(the short leg, the long leg, the hypotenuse)}$$
$$= (\sqrt{F_{n-2}}, \phi^{n/2}, \sqrt{F_n}\phi) \quad \text{for} \quad \forall n \geq 1.$$

This formalism covers all the known ϕ-related right triangles, i.e., the Kepler triangle and its kin. In case $n = 2$, the triangle is degenerated to the segment of the length ϕ. As n tends to infinity, the ultimate golden right triangle is found to have the ratios of the sides:

$$\text{(the short leg, the long leg, the hypotenuse)} \to (\phi^{-1}, 5^{1/4}, \phi).$$

Our definition becomes a nice classroom model to explore the relation amongst the golden ratio, the Fibonacci numbers and the Pythagorean theorem.

References

Christian Frisch ed. (1858) Joannis Kepleri Astronomia Opera Omnia, vol. I, XIV pp. & 672 pp.

Franz Löhmer (2006) Herodotus' Book II of the Histories, https://www.cheopus-pyramide.ch/Khufu-pyramid/Herodotus.html

Scott Olsen (2002) The Indefinite Dyad and the Golden Section: Uncovering Plato's Second Principle, Nexus Network Journal, vol. 4, no. 1, http://www.nexusjournal.com/GA-v4n1-Olsen.html

Scott Olsen (2006) The Golden Section - Nature's Greatest Secret (Walker & Co., N.Y.), 58 pp.