

Distribution of the Closest Distance to a Rectangular Facility

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This paper presents an analytical expression for the closest distance to a finite size facility. The closest distance is the distance from customers to the closest point on a facility, and represents the accessibility of customers to the facility. The distributions of the rectilinear closest distance are derived for rectangular facilities of grid and random configurations. The distributions demonstrate how the density, size, and shape of facilities affect the closest distance. A numerical example shows that if the area taken up by facilities is constant, many small facilities are better than a few large facilities and that rectangular facilities are better than square facilities.

Key words: Location, Finite Size Facility, Rectilinear Distance, Average Distance

1. Introduction

Classical facility location models usually assume that facilities can be represented as points with no area. For some types of facilities, however, the size of facilities cannot be negligible compared to the size of the study region. Examples of such finite size facilities include parks, stadiums, and cemeteries.

The optimal location and shape of finite size facilities have been studied. Drezner (1986) addressed the Weber problem where both the facility and demand have circular shapes. Carrizosa *et al.* (1995) considered a more generalized case where the facility and demand are randomly distributed inside regions. Carrizosa *et al.* (1998) found the location and shape of a rectangular facility that minimize the average distance to the demand set. Brimberg and Wesolowsky (2000, 2002a, 2002b) formulated minisum and minimax location problems where both facilities and demand are represented by areas. Savaş *et al.* (2002) developed the facility placement problem for finding the location and orientation of a finite size facility in the presence of barriers. Kelachankuttu *et al.* (2007) and Sarkar *et al.* (2005) examined the placement of a rectangular facility in the presence of existing facilities. The former regarded facilities as barriers to travel, whereas the latter regarded facilities as generalized congested regions, where traveling is permitted at extra cost. Date *et al.* (2014) proposed an efficient procedure for the placement problem of a rectangular facility. Sarkar *et al.* (2007) considered the facility placement problem with a center objective. Miyagawa (2017b) developed a location model of a rectangular facility that considers both the accessibility of customers and the interference to travelers. The model was extended by Miyagawa (2017a) to determine the size of the facility.

In the location models of finite size facilities reviewed above, the closest distance has frequently been used as a cri-

terion of accessibility of customers to facilities. The closest distance is defined as the distance from customers to the closest point on a facility. Despite a large number of studies concerning the closest distance, few of them have provided analytical expressions for the closest distance. The analytical expressions are necessary to understand fundamental characteristics of the closest distance. For a single facility case, Miyagawa (2017b) obtained an analytical expression for the closest distance. The present paper considers the case of multiple facilities to examine the effect of the number of facilities on the closest distance.

In this paper, we derive the distribution of the closest distance to a finite size facility. The distribution shows how the closest distance is distributed in a study region, thereby describing the service level of facility location. The distribution will thus be useful for location analysis of finite size facilities. To obtain analytical expressions for the distribution, the distance is measured as the rectilinear distance and facilities are represented as rectangles of grid and random configurations. Although grid and random configurations of facilities may seem unrealistic, actual locations of facilities are somewhere between the two extremes. The theoretical results of the present method will give an insight into empirical analysis of actual locations.

The remainder of this paper is organized as follows. The next section derives the distributions of the closest distance for grid and random configurations of facilities. The following section provides a numerical example. The final section presents concluding remarks.

2. Distribution of the Closest Distance

Facilities are represented as rectangles with side lengths b_1 and b_2 ($b_1 \geq b_2$). Let r be the rectilinear distance from a randomly selected location in the study region excluding facilities to the closest point on the nearest facility. The rectilinear distance between two points (x_1, y_1) and (x_2, y_2) is defined as $|x_1 - x_2| + |y_1 - y_2|$. The rectilinear distance is a good approximation for the actual travel distance in cities

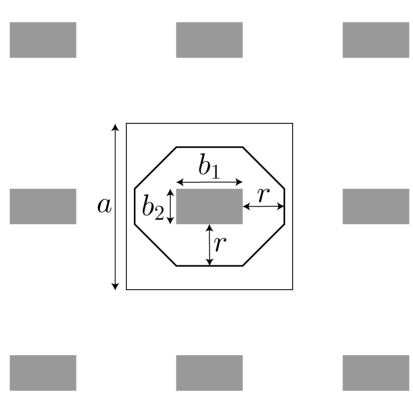


Fig. 1. Grid configuration of facilities.

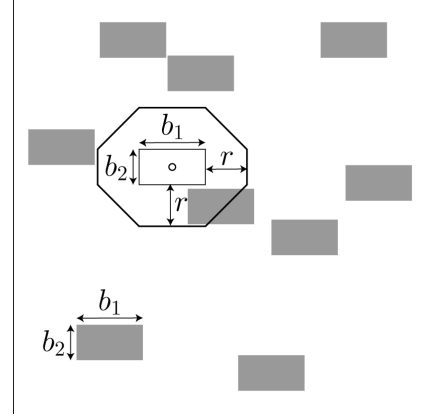


Fig. 2. Random configuration of facilities.

with a grid road network (Love and Morris, 1979; Brimberg *et al.*, 2007; Griffith *et al.*, 2012). In this section, we derive the distributions of the rectilinear closest distance r for grid and random configurations of facilities.

Suppose first that facilities are regularly distributed on a square grid with spacing a ($a \geq b_1$), as shown in Fig. 1. The density of facilities (the number of facilities per unit area) and the proportion of facility area (the area taken up by facilities per unit area) are expressed as $\rho = 1/a^2$ and $\lambda = b_1 b_2 / a^2 = \rho b_1 b_2$, respectively.

Let $F(r)$ be the cumulative distribution function of the closest distance indicating the probability that the closest distance is less than or equal to r . Then, $F(r)$ is given by

$$F(r) = \frac{S(r)}{S}, \quad (1)$$

where S and $S(r)$ are the area of the study region and the area of the region such that the closest distance is less than or equal to r in the study region, respectively. The study region can be confined to the region where a facility is the nearest, which is given by the square with side length a including the facility, as shown in Fig. 1. The area of the study region is then $S = a^2 - b_1 b_2$. The region such that the closest distance is less than or equal to r is given by an octagon, as shown in Fig. 1. The area of this buffer region around the facility is

$$S(r) = \begin{cases} 2r(b_1 + b_2 + r), & 0 < r \leq \frac{a}{2} - \frac{b_1}{2}, \\ 2ar - \frac{1}{2}(a - b_1)(a - b_1 - 2b_2), & \frac{a}{2} - \frac{b_1}{2} < r \leq \frac{a}{2} - \frac{b_2}{2}, \\ a^2 - b_1 b_2 - \frac{1}{2}(2a - b_1 - b_2 - 2r)^2, & \frac{a}{2} - \frac{b_2}{2} < r \leq a - \frac{b_1}{2} - \frac{b_2}{2}. \end{cases} \quad (2)$$

Substituting S and $S(r)$ into Eq. (1) and differentiating with respect to r , we have the following probability density func-

tion

$$f(r) = \frac{dF(r)}{dr} = \begin{cases} \frac{2\rho(2r+b_1+b_2)}{1-\rho b_1 b_2}, & 0 < r \leq \frac{1}{2\sqrt{\rho}} - \frac{b_1}{2}, \\ \frac{2\sqrt{\rho}}{1-\rho b_1 b_2}, & \frac{1}{2\sqrt{\rho}} - \frac{b_1}{2} < r \leq \frac{1}{2\sqrt{\rho}} - \frac{b_2}{2}, \\ \frac{2\sqrt{\rho}}{1-\rho b_1 b_2} \{2 - \sqrt{\rho}(2r + b_1 + b_2)\}, & \frac{1}{2\sqrt{\rho}} - \frac{b_2}{2} < r \leq \frac{1}{\sqrt{\rho}} - \frac{b_1}{2} - \frac{b_2}{2}, \end{cases} \quad (3)$$

which we call the distribution of the closest distance. From $f(r)$, we have the average closest distance

$$E = \int_0^{1/\sqrt{\rho} - b_1/2 - b_2/2} r f(r) dr = \frac{2 - 2\sqrt{\rho}(b_1 + b_2) + \rho(b_1^2 + b_2^2)}{4\sqrt{\rho}(1 - \rho b_1 b_2)}. \quad (4)$$

Suppose next that facilities are randomly distributed with density ρ . Note that facilities can overlap in this case. The proportion of facility area λ is expressed as the probability that a randomly selected location in the study region lies inside at least one facility. If a location lies inside a facility, the rectangle with side lengths b_1 and b_2 centered at the location contains the center of the facility. Thus, λ is the probability that the rectangle with side lengths b_1 and b_2 contains the center of at least one facility. The probability that a region of area S contains exactly x randomly distributed points, denoted by $P(x, S)$, is given by the Poisson distribution

$$P(x, S) = \frac{(\rho S)^x}{x!} \exp(-\rho S), \quad (5)$$

where ρ is the density of points (Clark and Evans, 1954). Since the center of facilities is also randomly distributed, we have

$$\begin{aligned} \lambda &= 1 - P(0, b_1 b_2) \\ &= 1 - \exp(-\rho b_1 b_2). \end{aligned} \quad (6)$$

The proportion of facility area λ is important if some facilities overlap because the total construction cost of facilities

Table 1. Density, size, and shape of facilities.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
ρ	1.0	1.2	1.4	1.0	1.0	1.0	1.0
b_1	0.3	0.3	0.3	0.4	0.5	$0.3\sqrt{2}$	$0.3\sqrt{3}$
b_2	0.3	0.3	0.3	0.4	0.5	$0.3/\sqrt{2}$	$0.3/\sqrt{3}$

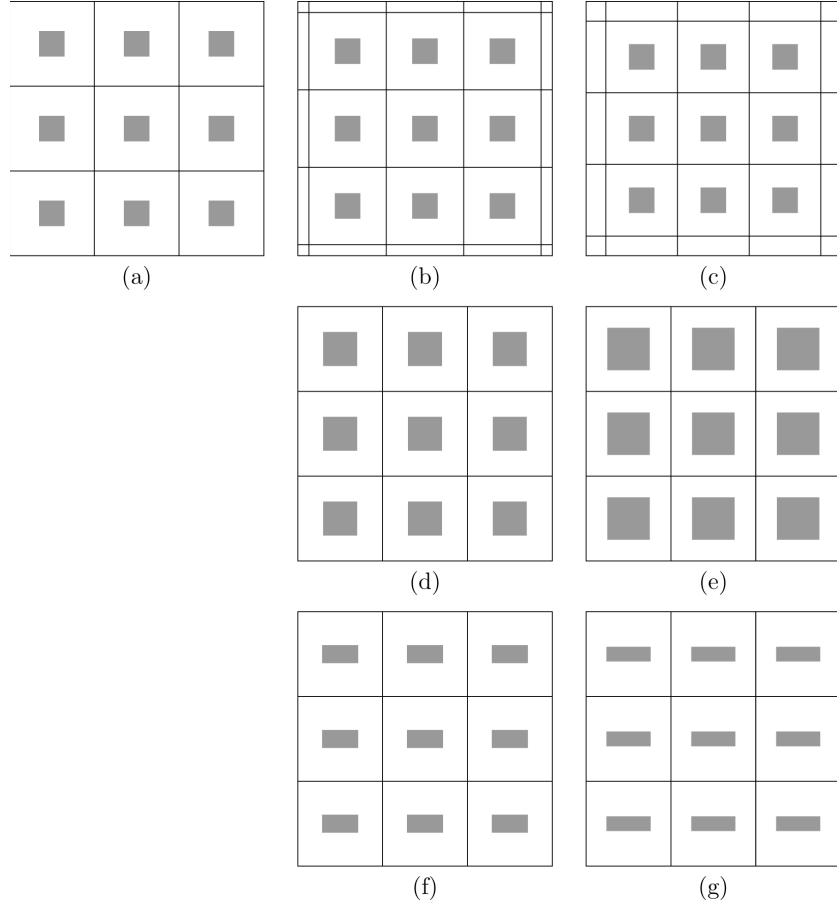


Fig. 3. Seven grid patterns of facilities: The square around a facility represents the nearest region of the facility.

Table 2. Proportion of facility area.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Grid	0.090	0.108	0.126	0.160	0.250	0.090	0.090
Random	0.086	0.102	0.118	0.148	0.221	0.086	0.086

depends on the area rather than the number of facilities. If the closest distance from a location to a facility is less than or equal to r , the buffer region around the rectangle (blank rectangle in Fig. 2) centered at the location contains the center of the facility, as shown in Fig. 2. The cumulative distribution function $F(r)$ is thus the probability that the buffer region contains the center of at least one facility. Using the Poisson distribution (5), we have

$$F(r) = 1 - P(0, 2(r + b_1 + b_2)r) = 1 - \exp\{-2\rho r(r + b_1 + b_2)\}. \quad (7)$$

Differentiating $F(r)$ with respect to r yields

$$f(r) = 2\rho(2r + b_1 + b_2) \exp\{-2\rho r(r + b_1 + b_2)\}. \quad (8)$$

The average closest distance is

$$E = \int_0^\infty r f(r) dr = \frac{1}{2\sqrt{\rho}} \exp\left\{\frac{\rho(b_1 + b_2)^2}{2}\right\} \operatorname{erfc}\left\{\frac{\sqrt{\rho}(b_1 + b_2)}{\sqrt{2}}\right\}, \quad (9)$$

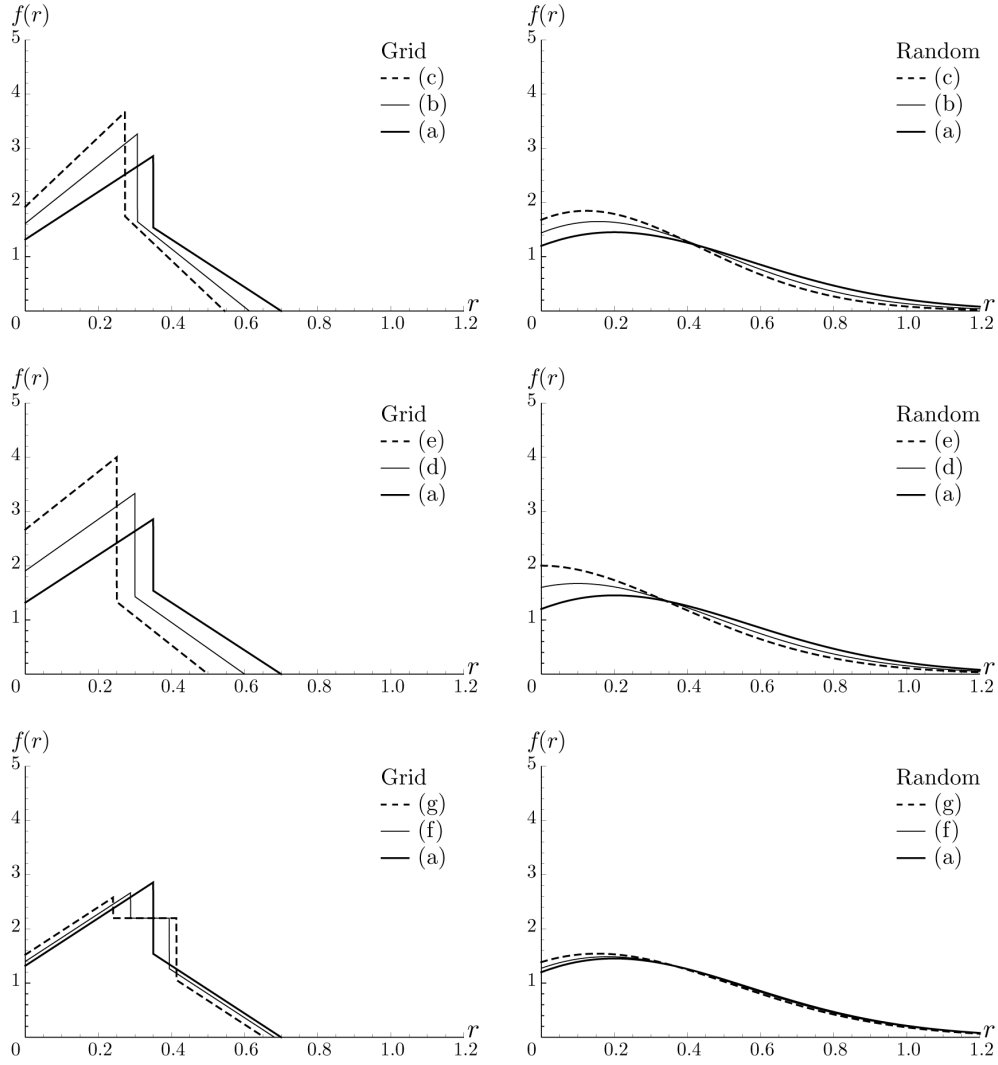


Fig. 4. Distribution of the closest distance.

Table 3. Average closest distance.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Grid	0.269	0.231	0.201	0.214	0.167	0.262	0.251
Random	0.412	0.363	0.326	0.366	0.328	0.402	0.389

where $\text{erfc}(x)$ is the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) dt. \quad (10)$$

3. Numerical Example

As a numerical example, we consider seven combinations of the density, size, and shape of facilities, as shown in Table 1. The seven patterns for the grid configuration are depicted in Fig. 3, where the square around a facility represents the nearest region of the facility. Comparing patterns (a)(b)(c), (a)(d)(e), and (a)(f)(g) shows how the density, size, and shape of facilities affect the closest distance, respectively. The proportions of facility area for the grid and random configurations are shown in Table 2. Note that the proportion for (f) and (g) is the same as that for (a). It

can be seen that the proportion for the random configuration is smaller than that for the grid configuration because some facilities overlap in the random configuration.

The distributions of the closest distance are shown in Fig. 4. It can be seen that the density and size of facilities have a similar effect on the distribution and that the shape of facilities has little effect particularly for the random configuration. The average closest distance is shown in Table 3. As the density, size, and width of facilities increase, the average closest distance decreases. Rectangular facilities are then better than square facilities. As expected, the average distance for the grid configuration is smaller than that for the random configuration. Less expected is that the average distance for (c) is smaller than that for (d), even though the proportion of facility area for (c) is smaller than that for (d).

It follows that if the area taken up by facilities in a city is constant, many small facilities are better than a few large facilities.

4. Conclusions

This paper has derived the distributions of the closest distance to a rectangular facility of grid and random configurations. The analytical expressions for the distributions are useful for location analysis of finite size facilities as follows. First, they give an estimate for the service level of actual facility location. By comparing the distributions, we can evaluate the efficiency of actual locations. For example, if the closest distance for an actual location is much greater than that for the random configuration, relocating or reshaping some facilities should be considered. Second, they demonstrate how the density, size, and shape of facilities affect the closest distance. Note that finding these relationships by using empirical approaches requires computation of the distance for various combinations of the parameters. These relationships help planners to determine the number, size, and shape of facilities to achieve a certain level of service. Finally, they provide all the information about the closest distance. The average and standard deviation of the closest distance are obtained from the distribution. The average can be used as a criterion of efficiency, whereas the standard deviation can be used as a criterion of equity.

An interesting topic for future work is to incorporate the hierarchy consisting of several different sizes of facilities. Addressing other configurations and shapes of facilities would also be interesting.

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