The Metallic Right-Triangles

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(Received June 24, 2020; Accepted October 19, 2020)

The Kepler triangle and its kin are systematically described by the unique formalism (Sugimoto, 2020). The present study affords the super-set of the formalism covering the above-mentioned triangles and other new triangles based on the metallic means and the generalised Fibonacci sequences adjoined to those means. The super-set is given by (the short leg, the long leg, the hypotenuse) $= (G_{n-2}^{1/2}(m), m^{1/2}\Phi^{n/2}(m), G_n^{1/2}(m)\Phi(m))$, where $G_n(m)$ and $\Phi(m)$ designate the nth generalised Fibonacci sequence and the metal mean of the metal number 'm', respectively. The initial members of the golden, silver and bronze right-triangles are presented.
Key words: Metallic Means, Generalised Fibonacci Sequences, Pythagorean Theorem

1. Introduction

I solved the generalisation problem about the Kepler triangle (Sugimoto, 2020). My formalism covers 'all' the known specimens of the Kepler triangle and its kin. The second specimen is sometimes called 'the silver right-triangle.' The 'silver' set of (the short leg, the long leg, the hypotenuse) assumes $(1, \Phi^{3/2}, 2^{1/2}\Phi)$, where Φ denotes the golden ratio. I didn't and still don't feel fit to call this triangle as the silver right-triangle. The silver ratio is '$1 + 2^{1/2}$', whilst this triangle has $2^{1/2}$ only. This unfit feeling drives me to explore higher-dimensional formalism of the metallic right-triangles.

2. Theory

Definition I: the metallic means
The metallic mean $\Phi(m)$ of the metal number 'm' is defined by

$$\Phi(m) = m + \frac{1}{\Phi(m)}, \tag{1}$$

where m is a natural number. The specific value is given by

$$\Phi(m) = \frac{m + \sqrt{m^2 + 4}}{2}. \tag{2}$$

Note that another solution of (1) is given by $-1/\Phi(m)$.

In case $m = 1$, we obtain the golden mean or ratio.

$$\Phi(1) = \frac{1 + \sqrt{5}}{2} = \Phi.$$

In case $m = 2$, we obtain the silver mean or ratio.

$$\Phi(2) = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2}.$$

In case $m = 3$, we obtain the bronze mean or ratio.

$$\Phi(3) = \frac{3 + \sqrt{13}}{2},$$

and so on.

Definition II: the generalised Fibonacci sequence of the metal number 'm'
By the recursion formula below, we define the generalised Fibonacci sequence $G_n(m)$ adjoined to the metallic mean of the metal number 'm.'

$$G_n(m) = mG_{n-1}(m) + G_{n-2}(m), \tag{3}$$

with the initial values, $G_1(m) = 1$ and $G_2(m) = m$.

Next, we determine the non-positive generations.

$$G_0(m) = 0,$$

because $G_2(m) = m$ and $G_1(m) = 1$.

$$G_{-1}(m) = 1,$$

because $G_1(m) = 1$ and $G_0(m) = 0$.

Theorem I: the general solution of $G_n(m)$
The general solution of $G_n(m)$ is given by

$$G_n(m) = \frac{1}{\sqrt{m^2 + 4}} \left\{ \Phi^n(m) - \left(-\frac{1}{\Phi(m)} \right)^n \right\}. \tag{4}$$

[Proof]
Rewriting (3), we obtain the following relation.

$$\frac{G_n(m)}{G_{n-1}(m)} = m + \frac{1}{G_{n-1}(m)/G_{n-2}(m)}.$$

Therefore, we may guess

$$\lim_{n\to\infty} \frac{G_n(m)}{G_{n-1}(m)} = \Phi(m),$$

doi:10.5047/forma.2020.007

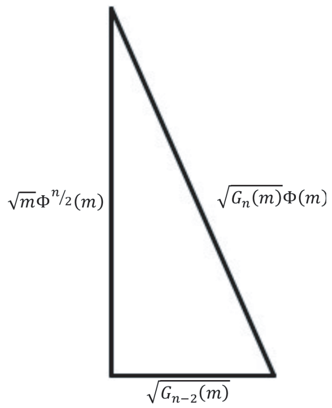

Fig. 1. The *n*th metallic right-triangle of the metal number '*m*' (a generic image).

because of (1).

Hence it is worthwhile to assume the general solution of $G_n(m)$ as a superposition of two solutions to (1), $\Phi(m)$ and $-1/\Phi(m)$ such that

$$G_n(m) = C_1\Phi^n(m) + C_2\left(-\frac{1}{\Phi(m)}\right)^n,$$

where C_1 and C_2 are the constants to be determined by use of the necessary conditions.

Since $G_0(m) = 0$, $C_1 + C_2 = 0$. That is $C_2 = -C_1$.

Since $G_1(m) = 1$, $C_1 = (m^2+4)^{-1/2}$ because of (2). Therefore, we arrive at (4).

Then we shall confirm the validity of (4) by direct application to (3). We start from the right-hand side of (3).

$$\begin{aligned}
&mG_{n-1}(m) + G_{n-2}(m)\\
&= \frac{1}{\sqrt{m^2+4}}\left\{m\Phi^{n-1}(m) - m\left(-\frac{1}{\Phi(m)}\right)^{n-1}\right.\\
&\qquad \left. +\Phi^{n-2}(m) - \left(-\frac{1}{\Phi(m)}\right)^{n-2}\right\}\\
&= \frac{1}{\sqrt{m^2+4}}\left\{\left(m+\frac{1}{\Phi(m)}\right)\Phi^{n-1}(m)\right.\\
&\qquad \left. -(m-\Phi(m))\left(-\frac{1}{\Phi(m)}\right)^{n-1}\right\}\\
&= \frac{1}{\sqrt{m^2+4}}\left\{\Phi(m)\Phi^{n-1}(m)\right.\\
&\qquad \left. -\left(-\frac{1}{\Phi(m)}\right)\left(-\frac{1}{\Phi(m)}\right)^{n-1}\right\} \qquad (\because (1))\\
&= \frac{1}{\sqrt{m^2+4}}\left\{\Phi^n(m) - \left(-\frac{1}{\Phi(m)}\right)^n\right\}\\
&= G_n(m).
\end{aligned}$$

Thus, we arrive at the left-hand side of (3). [Q.E.D.]

Corollary

$$\Phi^2(m) - \frac{1}{\Phi^2(m)} = m\sqrt{m^2+4}. \qquad (5)$$

It is trivial because of (4) for $n = 2$.

Definition III: the metallic right-triangles

We define the metallic right-triangles by the set of the sides,

(the short leg, the long leg, the hypotenuse)

$$= \left(\sqrt{G_{n-2}(m)},\ \sqrt{m}\Phi^{n/2}(m),\ \sqrt{G_n(m)}\Phi(m)\right), \qquad (6)$$

as shown in Fig. 1.

Remark

In case $n = 2$ for any m, the triangles defined above become degenerated to the segments.

(the short leg, the long leg, the hypotenuse)

$$= (0,\ \sqrt{m}\Phi(m),\ \sqrt{m}\Phi(m)). \qquad (\because G_0(m) = 0)$$

Theorem II: Pythagorean Theorem to the metallic right-triangles

The triangles, defined in *Definition III* above, satisfy the Pythagorean Theorem in the following form.

$$G_n(m)\Phi^2(m) = m\Phi^n(m) + G_{n-2}(m). \qquad (7)$$

[Proof]

We start from the right-hand side of (7).

$$\begin{aligned}
&m\Phi^n(m) + G_{n-2}(m)\\
&= m\Phi^n(m) + \frac{1}{\sqrt{m^2+4}}\left\{\Phi^{n-2}(m) - \left(-\frac{1}{\Phi(m)}\right)^{n-2}\right\}\\
&= \frac{1}{\sqrt{m^2+4}}\left\{m\sqrt{m^2+4}\Phi^n(m)\right.\\
&\qquad \left. +\Phi^{n-2}(m) - \left(-\frac{1}{\Phi(m)}\right)^{n-2}\right\}\\
&= \frac{1}{\sqrt{m^2+4}}\left\{m\sqrt{m^2+4}\Phi^{n-2}(m)\right.\\
&\qquad \left. +\Phi^{n-4}(m) - \left(-\frac{1}{\Phi(m)}\right)^{n}\right\}\Phi^2(m)\\
&= \frac{1}{\sqrt{m^2+4}}\left\{\left(\Phi^2(m) - \frac{1}{\Phi^2(m)}\right)\Phi^{n-2}(m)\right.\\
&\qquad \left. +\Phi^{n-4}(m) - \left(-\frac{1}{\Phi(m)}\right)^{n}\right\}\Phi^2(m) \qquad (\because (5))\\
&= \frac{1}{\sqrt{m^2+4}}\left\{\Phi^n(m) - \Phi^{n-4}(m)\right.\\
&\qquad \left. +\Phi^{n-4}(m) - \left(-\frac{1}{\Phi(m)}\right)^{n}\right\}\Phi^2(m)\\
&= \frac{1}{\sqrt{m^2+4}}\left\{\Phi^n(m) - \left(-\frac{1}{\Phi(m)}\right)^n\right\}\Phi^2(m)\\
&= G_n(m)\Phi^2(m).
\end{aligned}$$

Thus, we arrive at the left-hand side of (7). [Q.E.D.]

Corollary

As n tends to infinity, the sides of the metallic right-triangles diverge. But because of the leading term in (4),

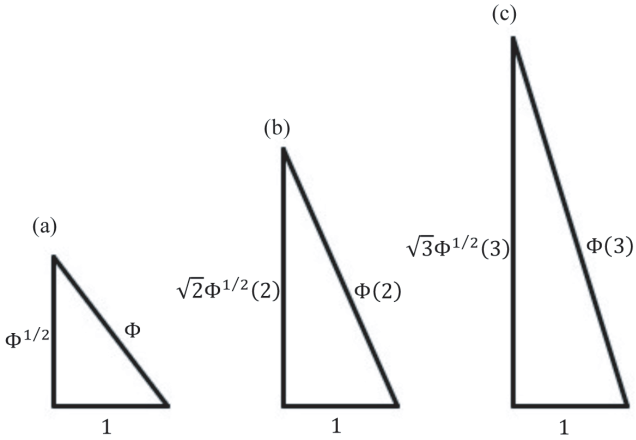

Fig. 2. Examples of the initial three metallic right-triangles: (a) the golden right-triangle (the Kepler triangle); (b) the silver or platinum right-triangle; (c) the bronze right-triangle. The metallic ratios are given in *Definition I* in the text.

the ratio amongst the sides converges such that

$$\begin{aligned}&\left(\sqrt{G_{n-2}(m)},\, \sqrt{m}\Phi^{n/2}(m),\, \sqrt{G_n(m)}\Phi(m)\right)\\ &\to \left(\frac{\Phi^{n/2}(m)}{(m^2+4)^{1/4}}\Phi^{-1}(m),\, \sqrt{m}\Phi^{n/2}(m),\, \frac{\Phi^{n/2}(m)}{(m^2+4)^{1/4}}\Phi(m)\right)\\ &= \left(\Phi^{-1}(m), m^{1/2}(m^2+4)^{1/4}, \Phi(m)\right)\frac{\Phi^{n/2}(m)}{(m^2+4)^{1/4}}.\end{aligned}$$

Therefore

(the short leg) : (the long leg) : (the hypotenuse)

$$\to \Phi^{-1}(m) : m^{1/2}(m^2+4)^{1/4} : \Phi(m), \qquad (8)$$

as n tends to infinity. The ratio satisfies the Pythagorean Theorem.

$$\Phi^2(m) = m\sqrt{m^2+4} + \frac{1}{\Phi^2(m)}, \qquad (9)$$

because of (5).

3. Examples

Figure 2 shows the initial three of the metallic right-triangles. Figure 2(a) is the golden right-triangle or the Kepler triangle. Figure 2(b) is *my* silver right-triangle. But this name is used to other triangle, so I may call it the platinum right-triangle. Figure 2(c) is the bronze right-triangle. These metallic ratios are given in *Definition I* above.

How many examples may we find embedded in the nature and artefacts?

4. Conclusion

What we get is the super-set of the Kepler triangle and its kin (Sugimoto, 2020). Those new triangles act as the trivium (three-way crossing) amongst the metallic means, the generalised Fibonacci sequences and the Pythagorean theorem.

Reference

Takeshi Sugimoto (2020) The Kepler Triangle and Its Kin, *FORMA*, **35**(1), 1–2.