Close Relationship between Ratios of "Golden Ratio Group" and Those of "Square-Root-of-2 Ratio Group"

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While the golden ratio ($1{:}\phi$) has been well known since the ancient Greece age and has already been studied in depth by many researchers, the square-root-of-2 ratio ($1{:}\sqrt{2}$) and other ratios of this group have gone largely unnoticed until now, despite the fact that the rhombic dodecahedron, one of the most important polyhedra, is based on the square-root-of-2 ratio. According to a popular theory, these ratios are independent of one another. Against it, however, the author found out that there exists close relationship between the ratios of the golden ratio group and those of the square-root-of-2 ratio group through careful study on three kinds of rhombic polyhedron.

Key words: Golden Ratio, Square-Root-of-2 Ratio, Rhombus, Rhombic Polyhedron, Dihedral

1. Introduction

Until now, it has been the established fact that the golden ratio is an independent ratio with no specific relationship with other special ratios like the square-root-of-2 ratio. Against it, however, the author puts forth the following proposition based on careful research into three kinds of rhombic polyhedron.

Proposition. *There exists a close relationship between the ratios of the golden ratio group and those of the square-root-of-2 ratio group. (Note: An explanation for these two groups is given in Sec. 2.)*

Good grounds for this proposition are shown in Sec. 5 and Sec. 6.

Section 5 deals with the agreement in the dihedral of solids transformed from three kinds of rhombic polyhedron, which provides evidence for this proposition. They are rhombic dodecahedron, rhombic triacontahedron, and rhombic enneacontahedron. A common property of these solids is that they can all be built up from the regular polyhedron. The rhombic triacontahedron and the rhombic enneacontahedron are totally different from each other in shape, face composition, etc. Nevertheless, transformations of these solids based on a simple specific principle amazingly result in exactly the same shape.

Section 2 indicates that there are several specific ratios of significance other than the golden ratio and the square-root-of-2 ratio and that they belong to either a golden ratio group or a square-root-of-2 ratio group. Except the "square-root-of-2 triplicate ratio", these ratios are all equal to the length ratio of one side of the regular polygon to a specific diagonal thereof. These ratios also represent the length ratios between two diagonals of specific rhombuses, and each of such rhombuses constitutes a specific group of rhombic polyhedron. These findings are explained in Sec. 3. The five kinds of regular polyhedron are classified into two groups; one is a square-root-of-2 ratio group and the other is a golden ratio group. Such classification is described in Sec. 4.

Section 6 reveals that plane-filling is realized by combination of the golden rhombus and the "square-root-of-2 square rhombus" and that another plane-filling becomes possible with a combination of the "square-root-of-2 rhombus" and the new rhombus based on a new ratio. The definition of these rhombuses is provided in Sec. 3.

2. Relationship with the Diagonal of the Regular Polygon

There exist several specific ratios of significance other than the golden ratio and the square-root-of-2 ratio. All these ratios are shown in Table 1 below. Except the square-root-of-2 triplicate ratio, they are all equal to the length ratio of one side of the regular polygon to a specific diagonal thereof. Yet, the golden ratio and the square-root-of-2 ratio are considered exceptional in that they represent the sole such ratio inherent in the regular pentagon and the regular tetragon, respectively. This is because there exists only one kind of diagonal in these polyhedra.

The ratio $1{:}\sqrt{2+\sqrt{3}}$ is derived from the dodecagon and, for this reason, it is tentatively named the "dodecagon ratio". According to a common view, the ratio $1{:}1+\sqrt{2}$ represents the silver ratio. Note that the term "silver ratio" sometimes means the ratio $1{:}\sqrt{2}$. But, in this paper this definition is not used. Of these ratios, the golden ratio, square-root-of-2 ratio, and silver ratio are self-ex-

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Table 1. Seven specific ratios of significance including the golden ratio and the square-root-of-2 ratio.

	Name	Ratio value	Remarks
Golden ratio group	Golden ratio	$1:(1+\sqrt{5})/2 = 1:\emptyset$	Length ratio of one side of the regular pentagon to its diagonal.
	Golden square ratio	$1:\emptyset^2 = 1:(3+\sqrt{5})/2$	Length ratio of one side of the regular decagon to its 2nd shorter diagonal.
Square-root-of-2 ratio group	Square-root-of-2 ratio	$1:\sqrt{2}$	Length ratio of one side of the square to its diagonal.
	Square-root-of-2 square ratio	$1:2$	Length ratio of one side of the regular hexagon to its longer diagonal.
	Dodecagon ratio	$1:\sqrt{2+\sqrt{3}}$	Length ratio of one side of the regular dodecagon to its shortest diagonal.
	Square-root-of-2 triplicate ratio	$1:2\sqrt{2}$	Length ratio of one-half edge of the regular octahedron to its space diagonal.
	Silver ratio	$1:1+\sqrt{2}$	Length ratio of one side of the regular octagon to its 2nd shorter diagonal.

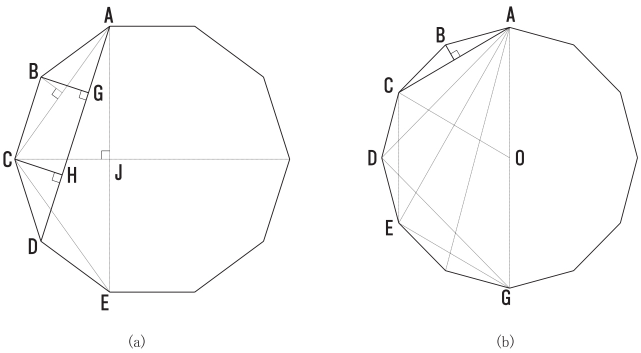


Fig. 1. (a) Regular decagon; (b) Regular dodecagon.

planatory. Therefore, explanations are given to the other ratios in Table 1.

The second ratio in Table 1 is expressed by AB:AD shown in Fig. 1(a), which shows a regular decagon. Its proof is given below.

Detailed calculation of the golden square ratio:

$\overrightarrow{AD} = \overrightarrow{AB}\cdot\cos\frac{\pi}{5} + \overrightarrow{GH} + \overrightarrow{CD}\cdot\cos\frac{\pi}{5}$. $\overrightarrow{AB} = \overrightarrow{BC} = \overrightarrow{CD} = \overrightarrow{GH}$.

$\overrightarrow{AD} = \overrightarrow{AB}\cdot\left(1 + 2\cdot\cos\frac{\pi}{5}\right)$. $\angle CAE = \frac{\pi}{5}$. $\overrightarrow{AJ} = \frac{\overrightarrow{AE}}{2}$.

$\cos\frac{\pi}{5} = \frac{\overrightarrow{AJ}}{\overrightarrow{AC}} = \frac{\overrightarrow{AE}}{2}\cdot\frac{1}{\overrightarrow{AC}} = \frac{\overrightarrow{AE}}{\overrightarrow{AC}}\cdot\frac{1}{2} = \frac{\emptyset}{2}$. $\overrightarrow{AD} = \overrightarrow{AB}\cdot(1+\emptyset)$. $\overrightarrow{AB}:\overrightarrow{AD} = 1:(1+\emptyset) = 1:\emptyset^2$

The foregoing calculation indicates that the golden square ratio is equal to the length ratio of one side of the regular decagon to its 2nd shorter diagonal.

Shown in Fig. 1(b) is a regular dodecagon, which also includes a regular tetragon (i.e. square) and a regular hexagon. In fact, the right isosceles triangle ADG is equivalent to one-half of a regular tetragon and the trapezoid ACEG is equivalent to one-half of a regular hexagon. In this figure, the triangle ACO is a regular triangle. Therefore, $\overrightarrow{AC} = \overrightarrow{AO} = \overrightarrow{AG}/2$. $\overrightarrow{AC}:\overrightarrow{AG} = 1:2$. This means that the "square-root-of-2 square ratio" is equal to the length ratio of one side of the regular hexagon to its longer diagonal.

Detailed calculation of the "dodecagon ratio":

In Fig. 1(b), $\angle BAC = \frac{\pi}{12}$. $\overrightarrow{AC} = \overrightarrow{AB}\cdot\cos\left(\frac{\pi}{12}\right)\cdot 2$. $\frac{\overrightarrow{AC}}{\overrightarrow{AB}} = 2\cdot\cos\left(\frac{\pi}{12}\right)$.

Based on the half-angle formula of the trigonometrical function, $\cos\left(\frac{\pi}{12}\right) = \sqrt{\left(1+\cos\left(\frac{\pi}{6}\right)\right)/2}$.

$\frac{\overrightarrow{AC}}{\overrightarrow{AB}} = 2\cdot\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}} = 2\cdot\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = 2\cdot\sqrt{\frac{2+\sqrt{3}}{4}} = \sqrt{2+\sqrt{3}}$. $\overrightarrow{AB}:\overrightarrow{AC} = 1:\sqrt{2+\sqrt{3}}$

The above calculation indicates that the dodecagon ratio is equal to the length ratio of one side of the regular dodecagon to its shortest diagonal.

The last two ratios in Table 1 are derived by the use of Fig. 2 below, which is a regular octahedron. Shown in Fig. 2 below is a regular octahedron projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis. With the tetragon AHBC being a square-root-of-2 rhombus, $\overrightarrow{HO}:\overrightarrow{AO} = 1:\sqrt{2}$. Therefore, $\overrightarrow{HO}:\overrightarrow{AB} = 1:2\sqrt{2}$. This means that the square-root-of-2 triplicate ratio is equal to the length ratio of one-half edge of the regular octahedron to its space diagonal.

3. Relationship with the Diagonals of Rhombuses

Various ratios mentioned above also represent the length ratios between two diagonals of the following special rhombuses, as shown in Table 2.

Out of these angles, the acute angle of the square-root-of-2 rhombus (≈70°31′43.606″), obtuse angle of the square-root-of-2 rhombus (≈109°28′16.394″), obtuse angle of the golden rhombus (≈116°33′54.184″), obtuse angle of the golden square rhombus (≈138°11′22.866″), and obtuse angle of the square-root-of-2 triplicate rhombus (≈141°03′27.212″) are respectively equal to the dihedral angles (those of two neighboring faces) of the regular tetrahedron, regular octahedron, regular dodecahedron, regular icosahedron, and triangular bipyramid. The triangular bipyramid is a type of hexahedron, being the first in the infinite set of face-transitive bipyramids. It is the dual of the triangular prism with 6 isosceles triangle faces. Particularly, the obtuse angle of the square-root-of-2

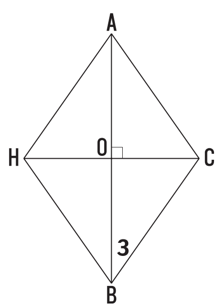


Fig. 2. Regular octahedron projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis.

Table 2. Length ratios between two diagonals of special rhombuses.

	Name	Length ratio of two diagonals	Acute angle	Obtuse angle
Golden ratio group	Golden rhombus	$1:(1+\sqrt{5})/2 = 1:\emptyset$	63°26′05.816"	116°33′54.184"
	Golden square rhombus	$1:\emptyset^2 = 1:(3+\sqrt{5})/2$	41°48′37.134"	138°11′22.866"
Square-root-of-2 ratio group	Square-root-of-2 rhombus	$1:\sqrt{2}$	70°31′43.606"	109°28′16.394"
	Square-root-of-2 square rhombus	$1:2$	53°07′48.368"	126°52′11.632"
	Square-root-of-2 triplicate rhombus	$1:2\sqrt{2}$	38°56′32.788"	141°03′27.212"
	Dodecagon ratio rhombus	$1:\sqrt{2+\sqrt{3}}$	54°44′08.197"	125°15′51.803"

Table 3. Special rhombuses constituting a specific rhombic polyhedron.

	Rhombic polyhedron	Regular polyhedron fundamental to the formation of rhombic polyhedron	Rhombus constituting the surface
Golden ratio group	Rhombic dodecahedron of the 2nd kind	Regular icosahedron	Golden rhombus
	Rhombic icosahedron		
	Rhombic triacontahedron		
	Rhombic enneacontahedron	Regular dodecahedron	Square-root-of-2 rhombus (60 faces), Golden square rhombus (30 faces)
Square-root-of-2 ratio group	Rhombic dodecahedron	Regular tetrahedron, Regular hexahedron	Square-root-of-2 rhombus

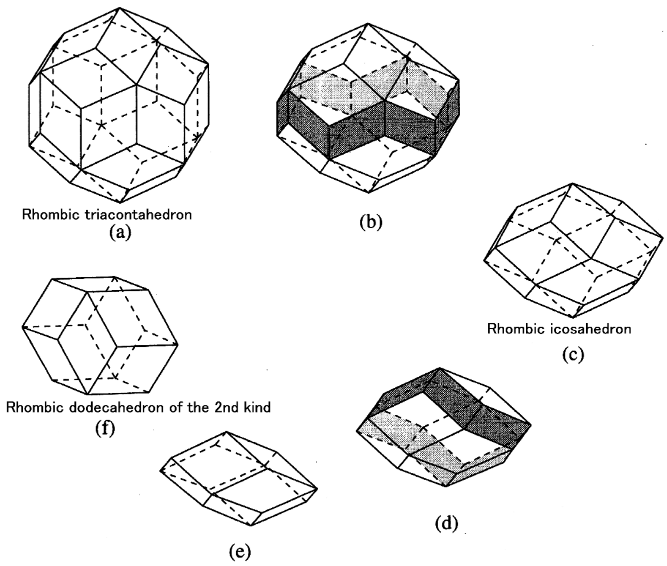


Fig. 3. How to make the rhombic icosahedron and the rhombic dodecahedron of the 2nd kind from the rhombic triacontahedron.

rhombus (≈109°28′16.394″) is called the regular tetrahedron angle (called Maraldi’s angle, also) in the field of crystallography.

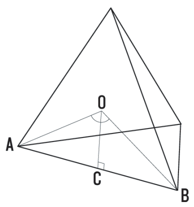
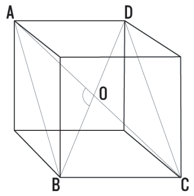
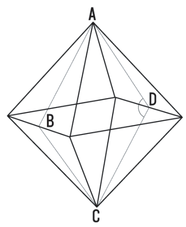
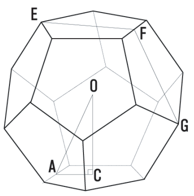
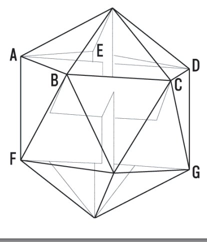
Here, it is to be noted that twice the acute angle of the golden rhombus (≈63°26′05.816″) is equal to the obtuse angle of the “square-root-of-2 square rhombus” (≈126°52′11.632″) and that twice the acute angle of the square-root-of-2 rhombus (≈70°31′43.606″) is equal to the obtuse angle of the “square-root-of-2 triplicate rhombus” (≈141°03′27.212″) and that twice the acute angle of the “dodecagon ratio rhombus” (≈54°44′08.197″) is equal to the obtuse angle of the square-root-of-2 rhombus (≈109°28′08.197″).

Furthermore, these rhombuses constitute a specific rhombic polyhedron as listed in Table 3 below. This fact is studied in [1].

Of these poyhedra, the rhombic dodecahedron of the 2nd kind and the rhombic icosahedron are created from the rhombic triacontahedron. This process is illustrated in [3]. For reference purpose, this illustration is excerpted and shown in Fig. 3 below. This means that the rhombic dodecahedron of the 2nd kind, rhombic icosahedron and rhombic triacontahedron all fall under the same category.

Figure 3 illustrates the process of making the rhombic icosahedron and the rhombic dodecahedron of the 2nd kind from the rhombic triacontahedron, (a) → (b) → (c) → (d) → (e) → (f). The rhombic icosahedron is created by pulling out 10 rhombuses, i.e. shadowed portion of (b), from the triacontahedron and by combining the upper portion with the lower portion. Further, the rhombic dodecahedron of the 2nd kind is formed by pulling out 8 rhombuses, i.e. shadowed portion of (d), from the icosahedron.

Table 4. Five kinds of regular polyhedron classified into two groups.

	Regular polyhedron	Underlying ratio: Explanation	Underlying ratio: Figure	Appearing angle or underlying angle
	Regular Tetrahedron	Isosceles triangle AOB is formed by connecting the median point O and two vertices A and B. C is the center of $\overrightarrow{AB}$. Then, $\overrightarrow{CO}:\overrightarrow{AC} = 1:\sqrt{2}$.		∠AOB ≈ 109°28′16"394 This angle is equal to the obtuse angle of the square-root-of-2 rhombus. The dihedral of the regular tetrahedron is approx. equal to 70°31′43"606. This angle is identical with the acute angle of the square-root-of-2 rhombus.
Square-root-of-2 ratio group	Regular hexahedron (cube)	$\overrightarrow{AB}$ and $\overrightarrow{DC}$ represent two sides facing each other across the median point O. The rectangle ABCD is formed by joining A to D, and, B to C. Then, $\overrightarrow{AD}:\overrightarrow{AB} = 1:\sqrt{2}$.		O represents the vertex of two diagonals of rectangle ABCD. ∠AOB ≈ 109°28′16"394. This angle is identical with the obtuse angle of the square-root-of-2 rhombus.
	Regular octahedron	B and D represent the centers of two sides facing each other across the median point of the solid. $\overrightarrow{BD}$ and $\overrightarrow{AC}$ represent two diagonals of the tetragon ABCD which is a rhombus. Then, $\overrightarrow{BD}:\overrightarrow{AC} = 1:\sqrt{2}$.		The dihedral ADC is equal to 109°28′16.394" This angle is identical with the obtuse angle of the square-root-of-2 rhombus.
Golden ratio group	Regular dodecahedro n	C represents the center of a regular pentagon. A is the center of its side. O represents the median point of the solid. The right triangle ACO is formed by drawing a perpendicular line from O to C. $\overrightarrow{AC}:\overrightarrow{CO} = 1:$ Ø		The dihedral EFG is equal to 116°33′54.184". This angle is identical with the obtuse angle of the golden rhombus.
	Regular icosahedron	Twelve vertices of the solid also serve as vertices of 3 rectangles, such as AFGD, crossing each other at right angles at the median point of the solid. These 3 rectangles are all congruent. The pentagon ABCDE is a regular pentagon. $\overrightarrow{AC} = \overrightarrow{AD}$, $\overrightarrow{AB} = \overrightarrow{AF}$. $\overrightarrow{AB}:\overrightarrow{AC} = 1:$ Ø . Therefore, $\overrightarrow{AF}:\overrightarrow{AD} = 1:$ Ø.		The dihedral of the solid is equal to 138°11′22"866. This angle is identical with the obtuse angle of the golden square rhombus.

4. Relationship with the Regular Polyhedron

Five kinds of regular polyhedron are classified into two groups: (1) square-root-of-2 ratio group; (2) golden ratio group, as described in Table 4.

5. Agreement in the Dihedral of Solids Transformed from Rhombic Polyhedra

This section deals with the "Golden Transformation" and the "Square-Root-of-2 Transformation". The former represents a process of replacing "square-root-of-2 rhombuses" with "golden rhombuses", subsequently partitioning each of them into 2 congruent triangles. The latter process comes in two types. One is to replace golden rhombuses with square-root-of-2 rhombuses, subsequently partitioning them into two congruent triangles, whereas the other is to replace "golden square rhombuses" with "square-root-of-2 triplicate rhombuses", subsequently partitioning them into two congruent triangles. These processes result in the creation of a new polyhedron. It is to be noted, however, that, such partitioning is really required to have it closed.

Three kinds of polyhedron, i.e. rhombic dodecahedron, rhombic triacontahedron, and rhombic enneacontahedron, undergo these processes. They are all built up from a specific regular polyhedron. This fact is studied in [1]. It is to be pointed out here that such regular polyhedron is classified into either the square-root-of-2 ratio group or the golden ratio group as shown in Table 4. Accordingly, the polyhedron of the square-root-of-2 ratio group is supposed to undergo the Golden Transformation and the one of the golden ratio group is supposed to undergo the Square-Root-of-2 Transformation.

5.1 Solids Transformed from the Rhombic Dodecahedron

The rhombic dodecahedron is composed of 12 square-root-of-2 rhombuses and is the dual solid of cuboctahedron, one of quasi-regular polyhedra. The rhombic dodecahedron has two kinds of cross section. One is the square and the other is the regular hexagon. Shown in Fig. 4 left is the latter.

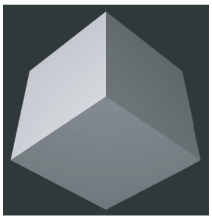


Fig. 4. Rhombic dodecahedron.

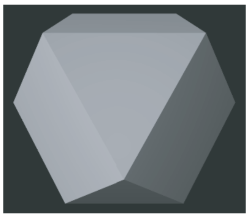


Fig. 5. Cuboctahedron.

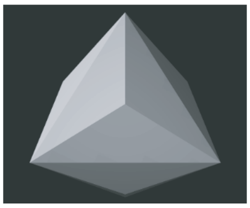
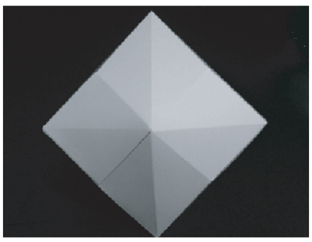


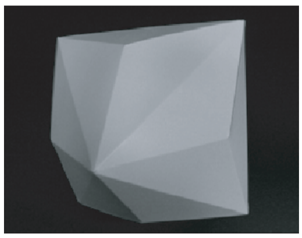
Fig. 6. Triakis octahedron.

According to Watanabe and Betsumiya [1], it is built up from either the regular tetrahedron or regular hexahedron (cube) that belong to the square-root-of-2 ratio group. For this reason, it is supposed to undergo the Golden Transformation.

Each face of the rhombic dodecahedron is replaced with two isosceles triangles arising from the partition of golden rhombus mountain-folded by the longer diagonal. Then, a convex polyhedron composed of 24 congruent isosceles triangles is created. In terms of the number of faces, edges, vertices, the face type, and the vertices by type, its shape is basically the same as the triakis octahedron. For this reason, this new solid is tentatively named “Triakis Octahedron of the 2nd kind”. It can be seen as a regular octahedron with a regular triangular pyramid covering each face. There are a couple of differences. One difference is in the internal angle of the isosceles triangle and the other is in the dihedral. In the case of the triakis octahedron, the obtuse angle is equal to $\cos^{-1}(1/4 - \sqrt{2}/2) \approx 117.12'2.06''$ and the acute angle is equal to $\cos^{-1}(1/2 + \sqrt{2}/4) \approx 31.23'58.97''$. Its dihedral is equal to $\cos^{-1}\{-(3+8\sqrt{2})/17\} \approx 147.21'0.36''$. In the case of the new polyhedron, on the other hand, the obtuse angle is equal to $\tan^{-1}\{(1+\sqrt{5})/2\} \approx 116.33'54.184''$ and the acute angle is $\tan^{-1}\{2/(1+\sqrt{5})\} \approx 31.43'2.91''$. Refer to Figs. 7(a) and (b). Its dihedral is calculated below. Provided in Figs. 8(a) and (b) are illustrations of this pyramid.

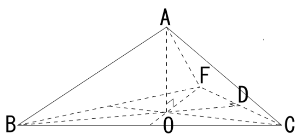


(a)

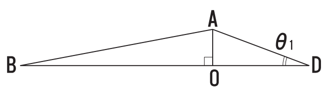


(b)

Fig. 7. Triakis octahedron of the 2nd kind: (a) top view; (b) diagrammatic perspective view.



(a)



(b)

Fig. 8. (a) Perspective illustration of the pyramid; (b) Pyramid projected to a plane perpendicular to $\overrightarrow{CF}$.

Detailed calculation: [Assumption: $\overrightarrow{\mathrm{BC}} = 1 + \sqrt{5}$.]

$$\overrightarrow{\mathrm{AB}} = \sqrt{\frac{5+\sqrt{5}}{2}}. \quad \overrightarrow{\mathrm{BD}} = \frac{1+\sqrt{5}}{2} \cdot \sqrt{3}. \quad \overrightarrow{\mathrm{BO}} = \frac{2}{3} \cdot \overrightarrow{\mathrm{BD}} = \frac{2}{3} \cdot \frac{1+\sqrt{5}}{2} \cdot \sqrt{3} = \frac{1+\sqrt{5}}{\sqrt{3}}.$$

$$\overrightarrow{\mathrm{AO}}^2 = \frac{5+\sqrt{5}}{2} - \frac{(1+\sqrt{5})^2}{3} = \frac{15+3\sqrt{5}-(1+2\sqrt{5}+5)\cdot 2}{6} = \frac{3-\sqrt{5}}{6}. \quad \overrightarrow{\mathrm{AO}} = \sqrt{\frac{3-\sqrt{5}}{6}}.$$

$$\overrightarrow{\mathrm{DO}} = \frac{1}{3} \cdot \overrightarrow{\mathrm{BD}} = \frac{(1+\sqrt{5}) \cdot \sqrt{3}}{6}.$$

$$\tan\theta_1 = \frac{\overrightarrow{\mathrm{AO}}}{\overrightarrow{\mathrm{DO}}} = \frac{\sqrt{\frac{3-\sqrt{5}}{6}}}{\frac{1+\sqrt{5}}{6}\sqrt{3}} = \frac{\sqrt{3-\sqrt{5}}\cdot 6}{\sqrt{3}\cdot(1+\sqrt{5})\cdot\sqrt{2}\cdot\sqrt{3}} = \frac{6\cdot\sqrt{3-\sqrt{5}}}{3\cdot\sqrt{2}\cdot(1+\sqrt{5})} = \frac{\sqrt{2}\cdot\sqrt{3-\sqrt{5}}}{1+\sqrt{5}} = \frac{\sqrt{2}\cdot\sqrt{3-\sqrt{5}}}{1-5}(1-\sqrt{5}) = \frac{\sqrt{6-2\cdot\sqrt{5}}}{4}\cdot(\sqrt{5}-1).$$

$\theta_1 \approx 20°54'18.567''$.

It is to be noted here that this angle θ_1 represents the dihedral of the pyramid against its base, which is just equal to one-half of the acute angle of the golden square rhombus.

Figure 9(a) provides a view of this polyhedron (with 24 faces) projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis. The dihedral of the regular octahedron δ_1 is equal to approximately $109°28'16.394''$. Therefore, the new dihedral $\angle$ECD is calculated by adding $2\theta_1$ to $109°28'16.394''$, i.e.

$$\angle \mathrm{ECD} = 109°28'16.394'' + (20°54'18.567'')\cdot 2 = 151°16'53.528''.$$

The above-mentioned polyhedron has a convex surface. If the triangle pyramids are replaced by concave ones, shown in Fig. 9(b), a new nonconvex polyhedron is created. It is tentatively named "Triakis Octahedron of the 3rd kind". The angle δ_1 in this case is expressed as

$$\delta_1 = 109°28'16.394'' - (20°54'18.567'')\cdot 2 = 67°39'39.260''.$$

5.2 Solids Transformed from the Rhombic Triacontahedron

The rhombic triacontahedron is composed of 30 golden rhombuses and is the dual solid of icosidodecahedron, one of quasi-regular polyhedra. According to Watanabe and Betsumiya [1], it is built up from the regular icosahedron that belongs to the golden ratio group; accordingly, it is supposed to undergo the Square-Root-of-2 Transformation.

Replacing all golden rhombuses of this polyhedron with square-root-of-2 rhombuses partitioned into two congruent triangles results in the creation of a new polyhedron with 60 faces. Solids thus transformed come in two types. One has a shape of the regular icosahedron with a regular triangular pyramid stuck on each of its face (Fig. 12), and the other has a shape of the regular dodecahedron with a pentagonal pyramid stuck on each of its face (Fig. 19).

5.2.1 Icosahedron with Regular Triangular Pyramids Stuck on Its Faces

Each face of the rhombic triacontahedron is replaced with two isosceles triangles arising from the partition of the square-root-of-2 rhombus valley-folded along the longer diagonal. Then, a nonconvex polyhedron composed of 60 congruent isosceles triangles is created. In terms of the number of faces, edges, vertices, the face type, and the vertices by type, its shape is basically the same as the triakis icosahedron shown in Fig. 11. For this reason, this new solid is tentatively named "Triakis Icosahedron of the 2nd kind". It can be seen as an icosahedron with a regular triangular pyramid covering each face.

There are a couple of differences. One difference is in the internal angle of the isosceles triangle and the other is in the dihedral. In the case of the triakis icosahedron, the obtuse angle is equal to $\cos^{-1}(-3\phi/10) \approx 119.02'21.66''$ and the acute angle is equal to $\cos^{-1}\{(\phi+7)/10\} \approx 30.28'49.17''$. Its dihedral is equal to $\cos^{-1}\{-(3+8\sqrt{2})/17\} \approx 160.36'45.19''$. In the case of the new polyhedron, on the other hand, the obtuse angle is equal to approximately $109°28'16.394''$ and the acute angle is equal to approximately $35°15'51.803''$. Refer to Figs. 12(a) and (b). Its dihedral is calculated below.

Provided in Fig. 13(a) is a perspective illustration of this pyramid. Three sides of its base make up a regular triangle and each of its lateral faces is equivalent to one-half of the square-root-of-2 rhombus partitioned into two congruent triangles by the longer diagonal.

Detailed calculation: [Assumption: $\overrightarrow{\mathrm{BC}} = 2\sqrt{2}$]

$$\overrightarrow{\mathrm{AB}} = \sqrt{3}. \quad \overrightarrow{\mathrm{BD}} = 2\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \sqrt{6}. \quad \overrightarrow{\mathrm{BO}} = \frac{2}{3} \cdot \overrightarrow{\mathrm{BD}} = \frac{2}{3} \cdot \sqrt{6}.$$

$$\overrightarrow{\mathrm{AO}}^2 = \left(\sqrt{3}\right)^2 - \left(\frac{2}{3}\cdot\sqrt{6}\right)^2 = 3 - \frac{24}{9} = \frac{9-8}{3} = \frac{1}{3}. \quad \overrightarrow{\mathrm{AO}} = \sqrt{\frac{1}{3}}.$$

Given in Fig. 13(b) is a view of the pyramid projected to a plane perpendicular to $\overrightarrow{\mathrm{CF}}$. The angle θ_2 is the dihedral of the pyramid against its base.

Detailed calculation:

$$\overrightarrow{\mathrm{DO}} = \frac{1}{3} \cdot \overrightarrow{\mathrm{BD}} = \frac{\sqrt{6}}{3} \quad \tan\theta_2 = \frac{\overrightarrow{\mathrm{AO}}}{\overrightarrow{\mathrm{DO}}} = \frac{\sqrt{\frac{1}{3}}}{\frac{\sqrt{6}}{3}} = \frac{3}{\sqrt{3}\cdot\sqrt{6}} = \frac{3}{\sqrt{18}} = \frac{3}{3\cdot\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

$\theta_2 \approx 35°15'51.803''$.

Each lateral face of this regular triangular pyramid is equivalent to one-half of the square-root-of-2 rhombus based on dual-partitioning by the longer diagonal. It is to be noted here that this dihedral θ_2 is just equal to one-half of the acute angle of the square-root-of-2 rhombus.

Provided in Fig. 14(a) is a view of this polyhedron (with 60 faces) projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis. The dihedral of the regular icosahedron δ_2 is approximately equal to $138°11'22.866''$. Therefore, the dihedral created by the longer diagonal of the square-root-of-2 rhombus is calculated by adding $138°11'22.866''$ to the aforementioned dihedral $35°15'51.803''$ multiplied by 2.

$$138°11'22.866'' + (35°15'51.803'')\cdot 2 = 208°43'06.472'' = 360° - 151°16'53.528''.$$

This means that the convex portion of the Triakis Octahedron of the 2nd kind with the dihedral being approx. $151°16'53.528''$ tightly fits into the concave portion of the Triakis Icosahedron of the 2nd kind with the dihedral

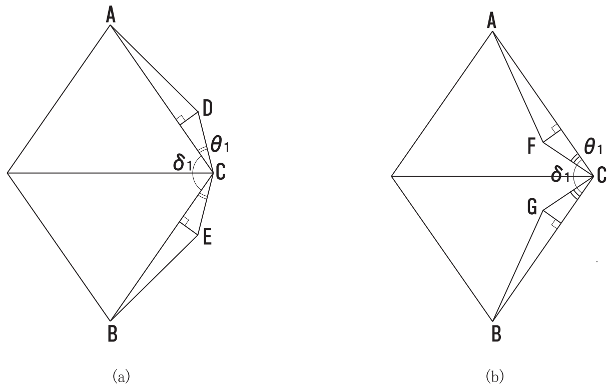


Fig. 9. (a) Triakis Octahedron of the 2nd kind; (b) Triakis Octahedron of the 3rd kind projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis.

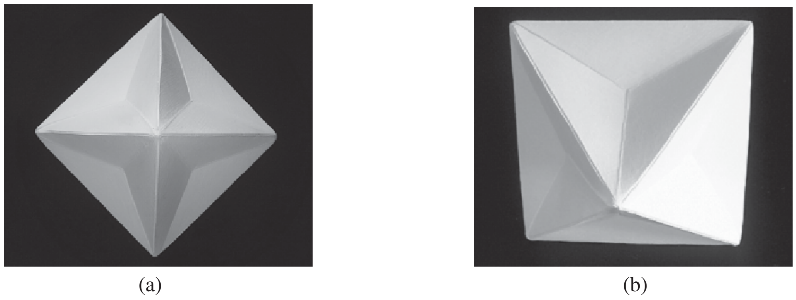


Fig. 10. Triakis Octahedron of the 3rd kind: (a) top view; (b) diagrammatic perspective view.

being approx. 208°43′06.472″, achieving local space-filling. (Refer to Fig. 15.)

In the case of the “Triakis Icosahedron of the 2nd kind”, square-root-of-2 rhombuses are valley-folded along the longer diagonal. If they are mountain-folded, a stellated nonconvex polyhedron is created. It is very similar to the great dodecahedron, one of the Kepler-Poinsot polyhedra, which is a nonconvex regular polyhedron. It can be seen as a “Triakis Icosahedron of the 2nd kind” with the vertices directed inward and, accordingly, it is tentatively named “Triakis Icosahedron of the 3rd kind”. Refer to Figs. 16(a) and (b). Only differences between the great dodecahedron and the Triakis Icosahedron of the 3rd kind lie in the angles of the isosceles triangle that constitutes the polyhedron. Its vertex angle and basic angle are respectively 108° and 36° in the case of the great dodecahedron, whereas they are approximately equal to 109°28′16.394″ and 35°15′51.803″, respectively, in the case of the “Triakis Icosahedron of the 3rd kind”.

Figure 14(b) represents a view of “Triakis Icosahedron of the 3rd kind” projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis. Accordingly, the dihedral created by the longer diagonal of the square-root-of-2 rhombus is figured out by subtracting the aforementioned dihedral multiplied by 2 from 138°11′22.866″.

$$138°11'22.866'' - (35°15'51.803'')\cdot 2 = 67°39'39.260''.$$

This dihedral exactly coincides with the dihedral of the Triakis Octahedron of the 3rd kind.

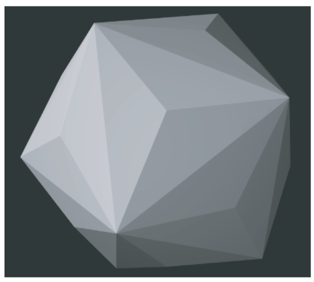
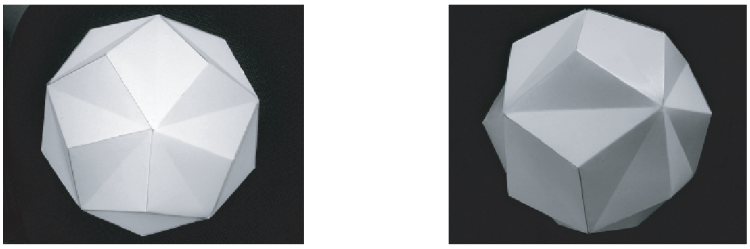


Fig. 11. Triakis icosahedron.

5.2.2 Dodecahedron with Pentagonal Pyramids Stuck on Its Faces

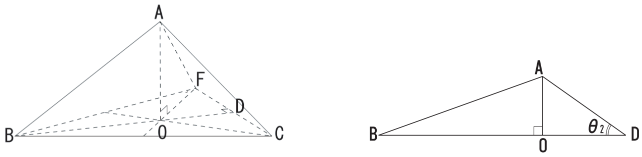
Each face of the rhombic triacontahedron shown in Fig. 17 is replaced with two isosceles triangles arising from the partition of the square-root-of-2 rhombus mountain-folded by the shorter diagonal. Then, a convex polyhedron composed of 60 congruent isosceles triangles is created. In terms of the number of faces, edges, vertices, the face type, and the vertices by type, its shape is basically the same as the pentakis dodecahedron shown in Fig. 18. For this reason, this new solid is tentatively named “Pentakis Dodecahedron of the 2nd kind”. It can be seen as a dodecahedron with a pentagonal pyramid covering each face.

There are a couple of differences. One is the internal angle of the isosceles triangle, which makes another difference in the dihedral. In the case of the pentakis do-



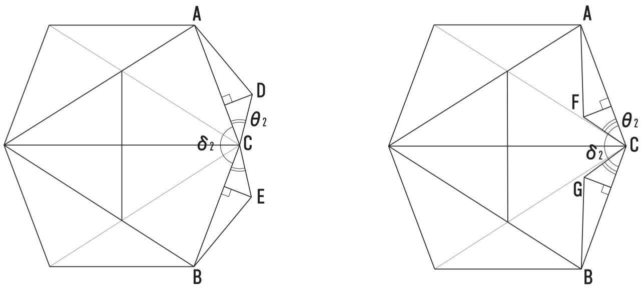
(a) (b)

Fig. 12. Triakis Icosahedron of the 2nd kind: (a) top view; (b) diagrammatic perspective view.



(a) (b)

Fig. 13. (a) Perspective illustration of the pyramid; (b) Pyramid projected to a plane perpendicular to $\overrightarrow{\mathrm{CF}}$.



(a) (b)

Fig. 14. (a) Triakis Icosahedron of the 2nd kind; (b) Triakis Icosahedron of the 3rd kind projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis.

decahedron, the obtuse angle is equal to $\sin^{-1}(\sqrt{58+18\sqrt{5}}/12) \approx 55°41'26.3''$ and the acute angle is equal to $\cos^{-1}(\sqrt{58+18\sqrt{5}}/12)\cdot 2 \approx 68°36'67.4''$. Its dihedral is equal to $\cos^{-1}\{-(80 + 9\sqrt{5})/109\} \approx 156°43'6.79''$. In the case of the new polyhedron, on the other hand, the obtuse angle is equal to approximately $54°44'08.197''$ and the acute angle is equal to approximately $70°31'43.606''$. Refer to Figs. 19(a) and (b). Its dihedral is calculated upper right.

Provided in Fig. 20(a) is a perspective illustration of the pentagonal pyramid. Five sides of its base make up a regular pentagon and each of its lateral faces is equivalent to one-half of the square-root-of-2 rhombus partitioned into two congruent triangles by the shorter diagonal. Figure 20(b) illustrates the pentagonal pyramid projected to a plane perpendicular to $\overrightarrow{\mathrm{CD}}$.

Detailed calculation: [Assumption: $\overrightarrow{\mathrm{CD}} = 2$; $\angle \mathrm{FHG} = \theta_3$]

$\overrightarrow{\mathrm{AF}} = \sqrt{3}$. $\overrightarrow{\mathrm{FH}} = \sqrt{2}$.

$\overrightarrow{\mathrm{AH}} = 2\emptyset \cdot \sin\frac{2\pi}{5}$. $\overrightarrow{\mathrm{AG}} \cdot \cos\frac{3\pi}{10} = 1$. $\overrightarrow{\mathrm{AG}} = \frac{1}{\cos\frac{3\pi}{10}}$.

$\overrightarrow{\mathrm{GH}} = 2\emptyset \cdot \sin\frac{2\pi}{5} - \frac{1}{\cos\frac{3\pi}{10}}$. $\overrightarrow{\mathrm{FH}} \cdot \cos\theta_3 = \overrightarrow{\mathrm{GH}} = 2\emptyset \cdot \sin\frac{2\pi}{5} - \frac{1}{\cos\frac{3\pi}{10}}$.

$$\cos\theta_3 = \left(2\emptyset \cdot \sin\frac{2\pi}{5} - \frac{1}{\cos\frac{3\pi}{10}}\right) \cdot \frac{1}{\overrightarrow{\mathrm{FH}}} = \left(2\emptyset \cdot \sin\frac{2\pi}{5} - \frac{1}{\cos\frac{3\pi}{10}}\right) \cdot \frac{1}{\sqrt{2}}.$$

$\theta_3 \approx 13°16'57.092''$.

This angle θ_3 is the dihedral θ_3 ($\approx 13°16'57.092''$) of the pentagonal pyramid against its base. It is to be noted here that this dihedral is just equal to one-fourth of the acute angle of the square-root-of-2 square rhombus ($\approx 53°07'48.368''$).

Figure 20(a) represents a view of "Pentakis Dodecahedron of the 2nd kind" projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis. The dihedral of the regular dodecahedron δ_3 is approximately

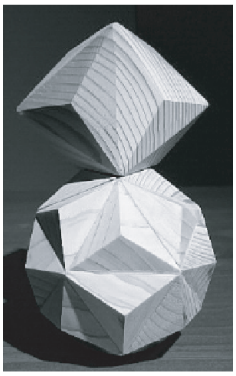
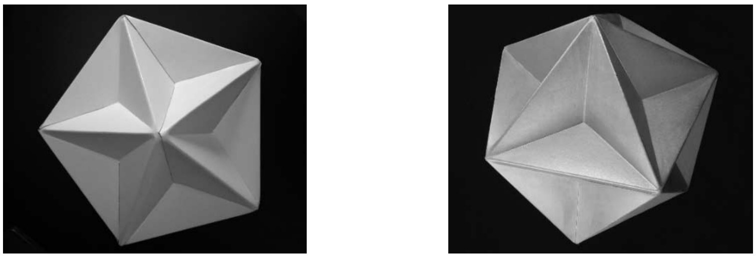


Fig. 15. Triakis Octahedron of the 2nd kind (upper) and Triakis Icosahedron of the 2nd kind (lower) tightly fit into each other.



(a) (b)

Fig. 16. Triakis Icosahedron of the 3rd kind: (a) top view; (b) diagrammatic perspective view.

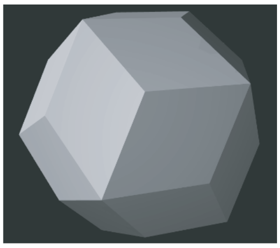


Fig. 17. Rhombic triacontahedron.

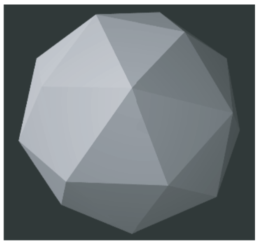


Fig. 18. Pentakis dodecahedron.

equal to 116°33′54.184″. Accordingly, the dihedral created by the shorter diagonal of the square-root-of-2 rhombus is figured out by adding 116°33′54.184″ to the aforementioned dihedral 13°16′57.092″ multiplied by 2.

$$\angle DCE = 116°33'54.184'' + (13°16'57.092'')\cdot 2 = 143°07'48.368''.$$

In the case of the “Pentakis Dodecahedron of the 2nd kind”, square-root-of-2 rhombuses are mountain-folded by the short diagonal. If they are valley-folded, a new nonconvex polyhedron with the vertices directed inward is created. It is tentatively named “Pentakis Dodecahedron of the 3rd kind” as shown in Figs. 22(a) and (b).

Figure 21(b) provides a view of the “Pentakis Dodecahedron of the 3rd kind” projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis. The dihedral created by the shorter diagonal of the square-root-of-2 rhombus is given as follows

$$\angle FCG = 116°33'54.184'' - (13°16'57.092'')\cdot 2 = 90°.$$

5.3 Solids Transformed from Rhombic Enneacontahedron

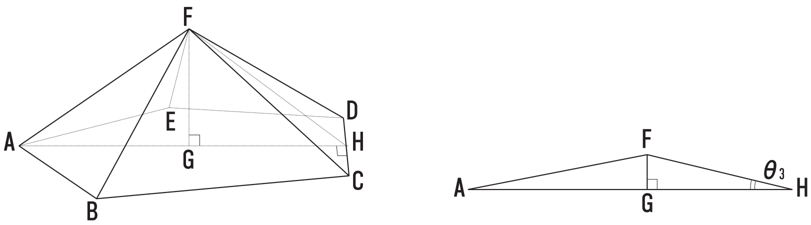
The rhombic enneacontahedron is composed of 60 square-root-of-2 rhombuses and 30 golden square rhombuses (90 faces in total). According to Watanabe and Betsumiya [1], it is built up from the regular dodecahedron that belongs to the golden ratio group. Accordingly, it is supposed to undergo the Square-Root-of-2 Transformation. In this case, only golden square rhombuses are all replaced with the square-root-of-2 triplicate rhombuses. The solid thus transformed come in two types. One has a shape of the regular icosahedron with a regular triangular pyramid stuck on each of its face; the other has a shape of the regular dodecahedron with a pentagonal pyramid stuck on each of its face.

5.3.1 Regular Icosahedron with Regular Triangular Pyramids Stuck on Its Faces

Sixty square-root-of-2 rhombuses are partitioned into

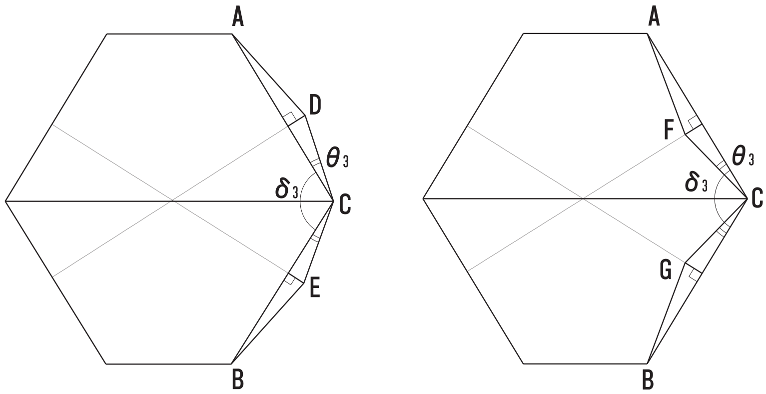
(a) (b)

Fig. 19. Pentakis Dodecahedron of the 2nd kind: (a) top view; (b) diagrammatic perspective view.



(a) (b)

Fig. 20. (a) Perspective illustration of the pentagonal pyramid; (b) Pentagonal pyramid projected to a plane perpendicular to $\overrightarrow{CD}$.



(a) (b)

Fig. 21. (a) Pentakis Dodecahedron of the 2nd kind; (b) Pentakis Dodecahedron of the 3rd kind projected to a XY-plane with the Z-axis serving as 2-fold rotational symmetry axis.

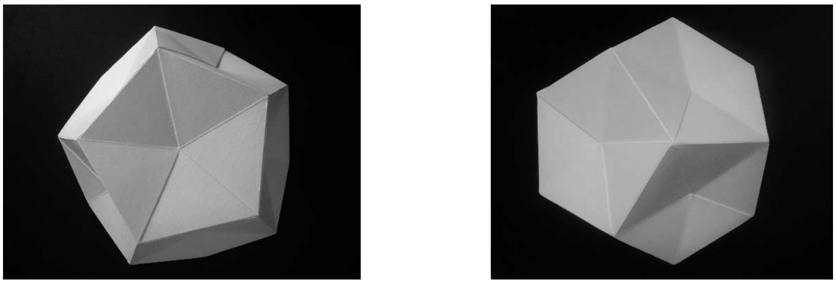
two by the longer diagonal mountain-folded and 30 square-root-of-2 triplicate rhombuses are partitioned into two by the shorter diagonal valley-folded. Then, a nonconvex polyhedron composed of 120 obtuse isosceles triangles and 60 acute isosceles triangles is created. It is shown in Fig. 23 below. Figure 24(a) is a perspective illustration of this triangular pyramid. The base of this pyramid BCD is a regular triangle and its three lateral faces ABC, ABD, and ACD are all congruent isosceles triangles. The pyramid's oblique edges AB, AC, AD are all equal to the longer diagonal of the square-root-of-2 rhombus.

Figure 24(b) shows the isosceles triangle ABC. The angle BAC consists of two halves of acute angle of the square-root-of-2 rhombus and one acute angle of the square-root-of-2 triplicate rhombus. Accordingly,

$$\angle \mathrm{BAC} \approx 35°15'51.803'' + 35°15'51.803'' + 38°56'32.788'' = 109°28'16.394''.$$

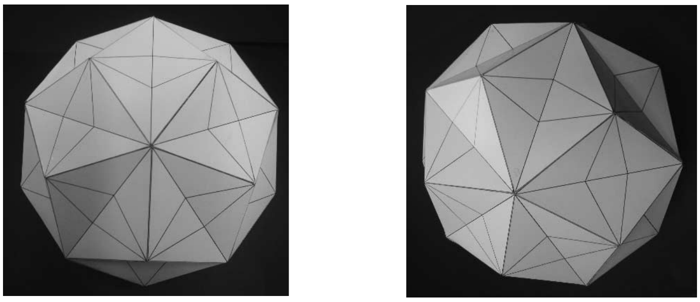
It is equal to the obtuse angle of the square-root-of-2 rhombus. This means that the isosceles triangle ABC has the same shape as one-half of the-square-root-of 2 rhombus partitioned by the longer diagonal. Therefore, the pyramid's dihedral against its base θ_3 is the same as the one of the aforementioned Triakis Icosahedron of the 2nd kind. θ_3 = 35°15′51.803″. The dihedral of this polyhedron is calculated in the same way as the Triakis Icosahedron of the 2nd kind.

$$138°11'22.866'' + (35°15'51.803'')\cdot 2 = 208°43'06.472'' = 360° - 151°16'53.528''.$$



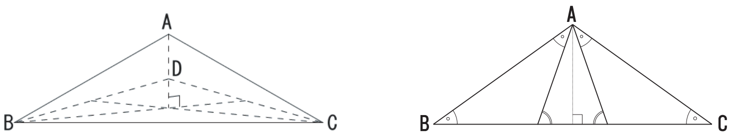
(a) (b)

Fig. 22. Pentakis Dodecahedron of the 3rd kind: (a) top view; (b) diagrammatic perspective view.



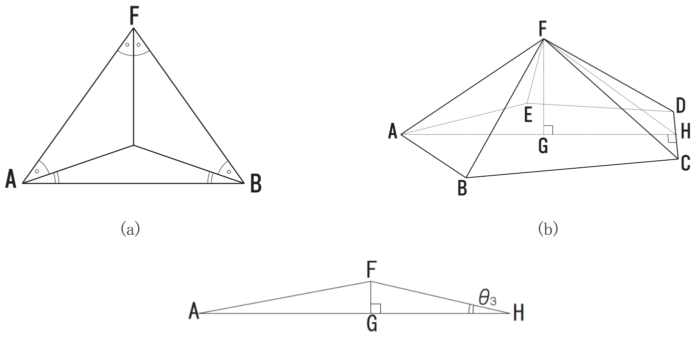
(a) (b)

Fig. 23. Nonconvex polyhedron transformed from the rhombic enneacontahedron: (a) top view; (b) diagrammatic perspective view.



(a) (b)

Fig. 24. (a) Perspective illustration of the triangular pyramid; (b) Isosceles triangle ABC.



(c)

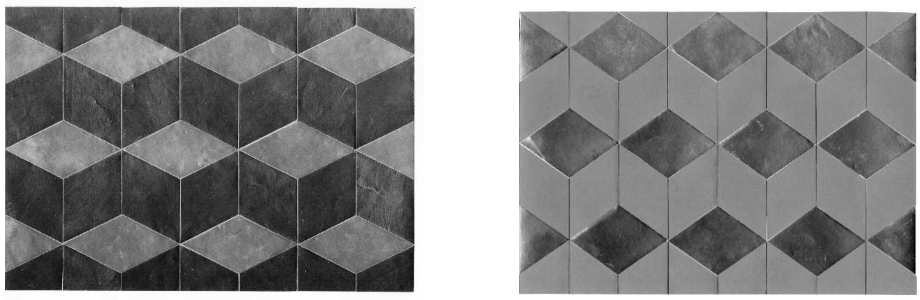
Fig. 25. (a) Lateral face of the pentagonal pyramid; (b) Perspective illustration of the pentagonal pyramid; (c) Pentagonal pyramid projected to a plane perpendicular to $\overrightarrow{CD}$.

This angle precisely agrees with the dihedral of the above-mentioned “Triakis Octahedron of the 2nd kind” and “Triakis Icosahedron of the 2nd kind”.

In the case of this polyhedron, all the-square-root-of-2 rhombuses are mountain-folded by the longer diagonal and all the square-root-of-2 triplicate rhombuses are valley-folded along the shorter diagonal. If all the square-root-of-2 rhombuses are valley-folded along the longer

Table 5. Two sets of dihedral of the solids being in perfect agreement.

	Angles in dihedral structure	Rhombic polyhedron transformed	Transformation process
1st set	• 151°16′53″528 • 67°39′39″260	Rhombic dodecahedron	Golden Transformation: replace the square-root-of-2 rhombuses with the golden rhombuses and divide them into two by the longer diagonal.
		Rhombic triacontahedron	Square-Root-of-2 Transformation: replace the golden rhombuses with the square-root-of-2 rhombuses and divide them into two by the longer diagonal.
		Rhombic enneacontahedron	Square-Root-of-2 Transformation: [1st step] replace the golden square rhombuses with the square-root-of-2 triplicate rhombuses and divide them into two by the shorter diagonal; [2nd step] divide the square-root-of-2 rhombuses into two by the longer diagonal.
2nd set	• 143°07′48″368 • 90°	Rhombic triacontahedron	Square-Root-of-2 Transformation: replace the golden rhombuses with the square-root-of-2 rhombuses and divide them into two by the shorter diagonal.
		Rhombic enneacontahedron	Square-Root-of-2 Transformation: [1st step] replace the golden square rhombuses with the square-root-of-2 triplicate rhombuses and divide them into two by the longer diagonal; [2nd step] divide the square-root-of-2 rhombuses into two by the longer diagonal.



(a) (b)

Fig. 26. (a) Plane filling by the golden rhombus & the square-root-of-2 square rhombus; (b) Plane filling by the square-root-of-2 rhombus & the dodecagon ratio rhombus.

diagonal and if all the square-root-of-2 triplicate rhombuses are mountain-folded along the shorter diagonal, a stellated nonconvex polyhedron is created. It has a shape being the same as "Triakis Icosahedron of the 3rd kind (see Fig. 16)", which is very much similar to the great dodecahedron. Therefore, the pyramid's dihedral against its base θ_3 is also the same, i.e. $\theta_3 \approx 35°15'51.803''$. The dihedral of this polyhedron is calculated in the same way as the "Triakis Icosahedron of the 3rd kind".

$$138°11'22.866'' - (35°15'51.803'')\cdot 2 = 67°39'39.260''.$$

This angle exactly agrees with the dihedral of the above-mentioned "Triakis Octahedron of the 3rd kind" and "Triakis Icosahedron of the 3rd kind".

5.3.2 Regular Dodecahedron with Pentagonal Pyramids Stuck on Its Faces

Sixty square-root-of-2 rhombuses and 30 square-root-of-2 triplicate rhombuses are both partitioned into two by the longer diagonal and mountain-folded. Then, a convex polyhedron composed of 180 obtuse isosceles triangles is created. It is a polyhedron with a pentagonal pyramid stuck on each face of the regular dodecahedron and with the vertices all directed outward.

Shown in Fig. 25(b) is a perspective illustration of this pentagonal pyramid.

The base ABCDE is a regular pentagon and its five lateral faces ABF, BCF, CDF, DEF, AEF are all congruent isosceles triangles. The triangle's oblique edges and base edges are respectively equal to the side and the shorter diagonal of the square-root-of-2 rhombus. Figure 25(c) illustrates the pentagonal pyramid projected to a plane perpendicular to $\overrightarrow{CD}$.

Figure 25(a) indicates a lateral face of the pentagonal pyramid ABF. Its vertex angle AFB is composed of two halves of the square-root-of-2 rhombus; its base angle FAB and FBA both consist of one-half of the acute angle of the square-root-of-2 rhombus and one-half of the acute angle of the square-root-of-2 triplicate rhombus.

$$\angle AFB = 35°15'51.803'' + 35°15'51.803'' = 70°31'43.606''.$$

$$\angle FAB = \angle FBA = 35°15'51.803'' + 19°28'16.394'' = 54°44'08.197''.$$

This means that the triangle ABF is equivalent to one-half of the square-root-of-2 rhombus partitioned by the shorter diagonal. Accordingly, the dihedral of the pentagonal pyramid against its base is the same as the one of the pentagonal pyramid of the Pentakis Dodecahedron of the 2nd kind. It means that the dihedral θ_3 is approximately equal to 13°16′57.092″. The dihedral of this polyhedron, therefore, is figured out in the same way as Pentakis Dodecahedron of the 2nd kind.

$$116°33'54.184'' + (13°16'57.092'')\cdot 2 = 143°07'48.368''.$$

This angle precisely agrees with the dihedral of the above-mentioned Pentakis Dodecahedron of the 2nd kind.

In the case of this polyhedron, all the-square-root-of-2 rhombuses and all the square-root-of-2 triplicate rhombuses are partitioned into two congruent triangles by the longer diagonal mountain-folded. If all the square-root-of-2 rhombuses are valley-folded while all the square-root-of-2 triplicate rhombuses are mountain-folded, a nonconvex polyhedron consisting of 180 obtuse isosceles triangles is created. It is a polyhedron with a pentagonal pyramid stuck on each face of the regular dodecahedron and with the vertices all directed inward. The dihedral of the pentagonal pyramid against its base is the same as the above-mentioned “Pentakis Dodecahedron of the 3rd kind”. That is, the dihedral θ_3 is approx. equal to 13°16′57.092″. The dihedral of this polyhedron, therefore, is figured out in the same way as Pentakis Dodecahedron of the 3rd kind.

$$116°33'54.184'' - (13°16'57.092'')\cdot 2 = 90°.$$

This angle also agrees precisely with the dihedral of “Pentakis Dodecahedron of the 3rd kind”.

5.4 Two Sets of Dihedral Being in Perfect Agreement

It has become apparent that there exist two sets of dihedral of the solids being in perfect agreement. These solids are created through transformation from three kinds of rhombic polyhedron. Such dihedrals are listed in Table 5. The foregoing discussion clearly indicates that there exists a close relationship between the ratios of the golden ratio group and those of the square-root-of-2 ratio group, specifically between the golden ratio and the square-root-of-2 ratio, and between the golden square ratio and the square-root-of-2 triplicate ratio.

6. Plane Filling

There also exists a close relationship between the golden ratio and the square-root-of-2 square ratio and between the square-root-of-2 ratio and the dodecagon ratio. These ratios are linked by the following formulae:

• [Obtuse angle of the golden rhombus 116°33′54.184″ × 2]
+ [Obtuse angle of the square-root-of-2 square rhombus 126°52′11.635″] = 360°

• [Obtuse angle of the square-root-of-2 rhombus 109°28′16.394″]
+ [Obtuse angle of the dodecagon ratio rhombus 125°15′51.803″ × 2] = 360°

These formulae indicate that a combination of two kinds of rhombus enables two separate sets of plane filling. One is the combination of the golden rhombus and the square-root-of-2 square rhombus and the other is the combination of the square-root-of-2 rhombus and the dodecagon ratio rhombus. Refer to Figs. 26(a) and (b).

7. Summary

The fruits of the above-mentioned study are summarized below.

• Until now, it has been the established theory that the golden ratio is an independent ratio with no specific relationship with other special ratios like the square-root-of-2 ratio. It has been found, however, that dihedrals of solids transformed from three kinds of rhombic polyhedron are in perfect agreement and that there exist two sets of such dihedrals. They are the rhombic dodecahedron, rhombic triacontahedron, and rhombic enneacontahedron. The processes used for such transformation are: (1) golden transformation and (2) square-root-of-2 transformation, which are both very simple.

• The rhombic triacontahedron, and the rhombic enneacontahedron are totally different from each other in shape, face composition, etc. Nevertheless, the square-root-of-2 transformation of these solids has amazingly resulted in exactly the same shape.

• Plane-filling is realized by the combination of the golden rhombus and the square-root-of-2 square rhombus. Another plane-filling becomes possible with the combination of the square-root-of-2 rhombus and the dodecagon ratio rhombus.

• The foregoing discussion clearly indicates that there exists a close relationship between the ratios of the golden ratio group and those of the square-root-of-2 ratio group, specifically between the golden ratio and the square-root-of-2 ratio and between the golden square ratio and the square-root-of-2 triplicate ratio, and between the golden ratio and the square-root-of-2 square ratio.

• There are several specific ratios of significance other than the golden ratio and the square-root-of-2 ratio, which however belong to either a golden ratio group or a square-root-of-2 ratio group.

• Except the square-root-of-2 triplicate ratio, these ratios are all equal to the length ratio of one side of the regular polygon to a specific diagonal thereof. Only the square-root-of-2 triplicate ratio is equal to the length ratio of one-half side of the regular octahedron to its space diagonal.

• They also represent the length ratios between two diagonals of specific rhombuses, and such rhombuses constitute specific rhombic polyhedron.

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