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Proof of the Transversality for the Standard Map

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We consider the standard map. The stable and unstable manifolds of the saddle fixed point are proved to intersect transversely at the primary homoclinic point u for any parameter value. For the proof, we use the particular objects called the dominant axis (DA) and subdominant axis (SD), and symmetric periodic orbits that have orbital points on these axes. The periodic orbit named $1/q$-BE has the orbital point z_k at the intersection point of DA and SD. Let ξ_k be the slope of SD at z_k. Take a sequence of z_k accumulating at u as $k \to \infty$. We prove that the slope ξ_k monotonically decreases to the slope $\xi_u(u)$ of the unstable manifold at u (the monotone inclination property). Using Ushiki's theorem, the hyperbolic region (HR) is constructed. It is proved that the orbital point z_k in HR is a saddle point with reflection. Using the monotone inclination property and the properties of z_k in HR, the transversality at u for any value of a (> 0) is proved.

Key words: Standard Map, Stable and Unstable Manifolds, Transversality, Ushiki's Theorem, Dominant and Subdominant Axes

1. Introduction

The transverse intersection of the stable and unstable manifolds implies chaos. This is a famous result observed by Poincaré (1890, 1899, 1993). This transversality is not self-evident in the two-dimensional area-preserving maps. In this paper, we prove the transversality of the standard map T for the whole possible parameter range. Here, the map T is defined on the infinite cylinder (MacKay, 1993) as

$$T\ :\ y_{n+1} = y_n + f(x_n),\ x_{n+1} = x_n + y_{n+1}\ (\mathrm{Mod}\ 2\pi) \tag{1}$$

where $0 \le x < 2\pi$, $-\infty < y < +\infty$, and $f(x) = a\sin x$ $(a \ge 0)$. At $a > 0$, there exist two fixed points $P = (0, 0)$ and $Q = (\pi, 0)$ where P is a saddle fixed point and Q is an elliptic fixed point at $0 < a < 4$ and a saddle one with reflection at $a > 4$.

The transversality has already been proved for infinitesimally small $a > 0$ by Lazutkin *et al.* (1989). They used the complex analysis to determine the splitting angle of the stable and unstable manifolds at the primary homoclinic point and derived the exponentially small splitting. For $a > a_c = 4/3$, we have shown the transversality (Yamaguchi and Tanikawa, 2000). In our approach, $f(x)$ has been assumed to be a C^2-function. The slope and the curvature of the unstable manifold at P are used to determine a_c. The invariant curves including the stable and unstable manifolds do not satisfy the Lipschitz condition at $a > a_c$ (Mather, 1984). Therefore, the unstable manifold is not a graph. The possibility that the non-transverse intersection may appear at $a \le a_c$ is not removed. So, the purpose of the present paper is to prove the transversality without the restriction to the values of a.

Let us explain the behavior of the unstable and stable manifolds of P. There exist two branches of the unstable manifold separated by P. Let $W_u(P)$ be the branch going toward the upper-right direction. For convenience we cut the cylinder vertically at P. Then, we obtain an infinite rectangle extending to $y \to \pm\infty$. We obtain a saddle $P' = (2\pi, 0)$ as a copy of P (Fig. 1). There exist two branches of the stable manifold of P'. Let $W_s(P')$ be the branch coming from the upper-left direction as displayed in Fig. 1. In Fig. 1, it is easy to observe the transverse intersections of $W_u(P)$ and $W_s(P')$ at the primary homoclinic point u on the axis $x = \pi$ named $S_G(\pi)$ and at v on the axis $y = 2(x - \pi)$ named $S_H(\pi)$. The parameter value $a = 3/2$ of Fig. 1 is larger than a_c. So, we know that the transversality holds. Our purpose is to prove Theorem 1.1.

Theorem 1.1. Let $W_u(P)$ be the unstable manifold of P and $W_s(P')$ be the stable manifold of P'. These intersect at the primary homoclinic point u on the symmetry axis $S_G(\pi)$ $(x = \pi,\ y > 0)$. Let $\xi_u(u)$ be the slope of $W_u(P)$ at u and $\xi_s(u)$ be the slope of $W_s(P')$ at u. The relations $0 < \xi_u(u) < a/2 < \xi_s(u)$ hold for any $a > 0$.

In §2, several notations and the properties used in this paper are introduced. In §3, Theorem 1.1 is proved. In §4, our results are summarized.

2. Mathematical Tools

2.1 Periodic orbits and the symmetric axes

At $a > 0$, there are periodic orbits which move round cylinder. The existence of these orbits is guaranteed by the Poincaré-Birkhoff theorem (Poincaré, 1912; Birkhoff, 1913, 1927). Let p/q be an irreducible fraction. For every

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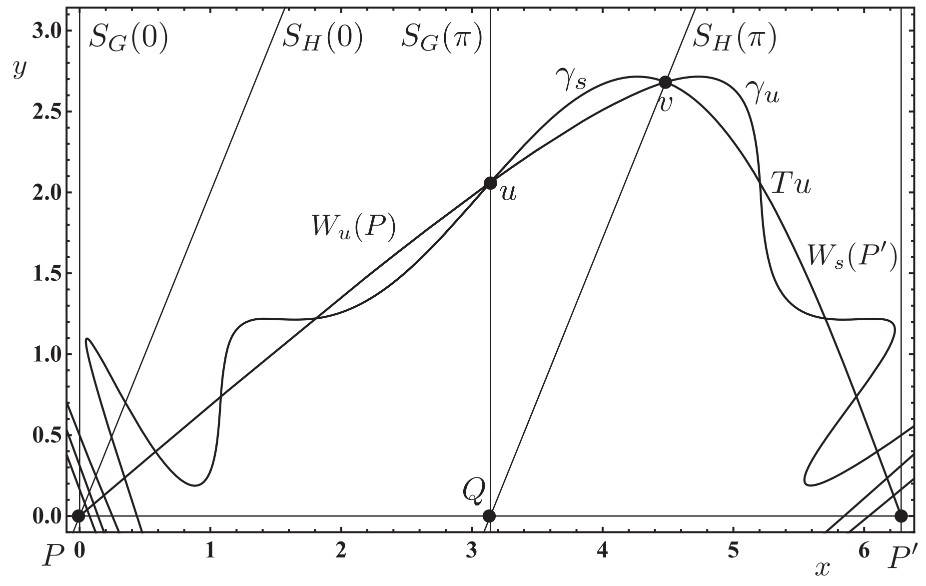

Fig. 1. The unstable manifold $W_u(P)$ of the saddle fixed point $P = (0, 0)$ and the stable manifold $W_s(P')$ of the saddle fixed point $P' = (2\pi, 0)$ are displayed at $a = 1.5$. Here, $S_G(0)$, $S_H(0)$, $S_G(\pi)$ and $S_H(\pi)$ represent the symmetry axes. Two intersection points u and v of $W_u(P)$ and $W_s(P')$ are the primary homoclinic points. Two arcs $\gamma_u = [v, Tu]_{W_u(P)}$ and $\gamma_s = [u, v]_{W_s(P')}$ are also displayed.

rotation number p/q, a pair of saddle and elliptic orbits exist in the standard map. Since these orbits satisfy the order preservation, these are called the monotone periodic orbits or the Birkhoff periodic orbits (Yamaguchi and Tanikawa, 2007). Here, the order preservation means that for any pair of points in an orbit, the order of the x-coordinates of their iterates does not alter on the universal cover. We use the notation p/q-BE for the order preserving symmetric elliptic periodic orbit of rotation number p/q.

Through the linear stability analysis, the eigenvalues $\lambda_\pm$ of the linearized matrix M are determined. If these are complex conjugate, i.e., $\lambda_\pm = \alpha \pm i\beta$ ($|\lambda_\pm| = 1$), we call the periodic orbit the elliptic periodic orbit. If these satisfy the conditions $\lambda_- < -1 < \lambda_+ < 0$, the periodic orbit is a saddle periodic orbit with reflection. If these satisfy the conditions $0 < \lambda_- < 1 < \lambda_+$, the orbit is a saddle periodic orbit.

If a map is represented by the product of involutions, we say that the map has the reversibility in the meaning of Birkhoff (1927). The standard map T is reversible. Using the involutions G and H, we represent T as $T = H \circ G$ where $H \circ H = G \circ G = \mathrm{id}$ and $\det\nabla H = \det\nabla G = -1$. Here, we give the actions of G and H.

$$G\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y + f(x) \\ -x \text{ (Mod } 2\pi) \end{pmatrix}, \tag{2}$$

$$H\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y \\ y - x \text{ (Mod } 2\pi) \end{pmatrix}. \tag{3}$$

The set of fixed points of involution is called the symmetry axis. Let S_G be the symmetry axis of G and S_H be the symmetry axis of H. We give the representations for them on cylinder (see Fig. 1).

$$S_G(0) = \{(x, y) : x = 0\}, \quad S_G(\pi) = \{(x, y) : x = \pi\}, \tag{4}$$

$$S_H(0) = \{(x, y) : y = 2x\},$$
$$S_H(\pi) = \{(x, y) : y = 2(x - \pi)\}. \tag{5}$$

Definition 2.1 An orbit is symmetric if and only if it has a point on the symmetry axis.

Proposition 2.2 A periodic orbit is symmetric if and only if it has two of the points on the symmetry axis or axes.

By proposition 2.2, a $1/(2k+1)$-BE has one point z_0 on $S_H(0)$ and the other point z_k on $S_G(\pi)$.

2.2 Involutions for T^{2k+1}

Using the two involutions G and $T^{2k}H$ ($k \geq 0$), we express T^{2k+1} as

$$T^{2k+1} = T^{2k}H \circ G. \tag{6}$$

Let $S_{T^{2k}H}$ be the symmetry axis of $T^{2k}H$. Thus, T^{2k+1} has two symmetry axes $S_{T^{2k}H}$ and $S_G(\pi)$. Greene (1979) named $S_G(\pi)$ the dominant axis (DA). In this paper, we call $S_{T^{2k}H}$ the subdominant axis (SD). The initial point z_0 is on $S_H(0)$ and z_k on $S_G(\pi)$. We remark that z_k is the intersection point of DA and SD.

The representation of SD is $T^k S_H$. In fact,

$$T^{2k}H(T^k S_H) = T^{2k}T^{-k}HS_H = T^k S_H.$$

Let us operate $T^{-(2k+1)}$ to $S_{T^{2k}H}$.

$$\begin{aligned} T^{-(2k+1)}S_{T^{2k}H} &= G(HT^{-2k}S_{T^{2k}H}) \\ &= G(T^{2k}HS_{T^{2k}H}) = GS_{T^{2k}H}. \end{aligned} \tag{7}$$

The operation G to $S_{T^{2k}H}$ is equivalent with that of $T^{-(2k+1)}$ to $S_{T^{2k}H}$.

We use the following representation of involution G whose symmetry axis is $S_G(\pi)$.

$$G\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} y + f(x) \\ 2\pi - x \end{pmatrix}. \tag{8}$$

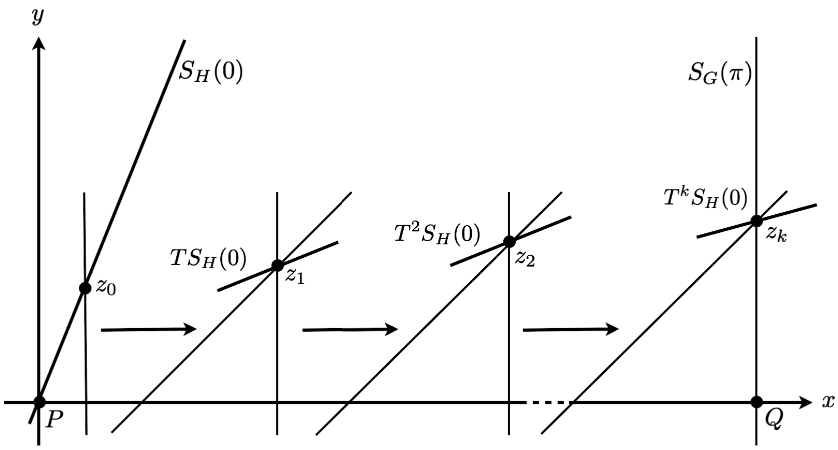

Fig. 2. The slopes of inclined thin lines are 1.

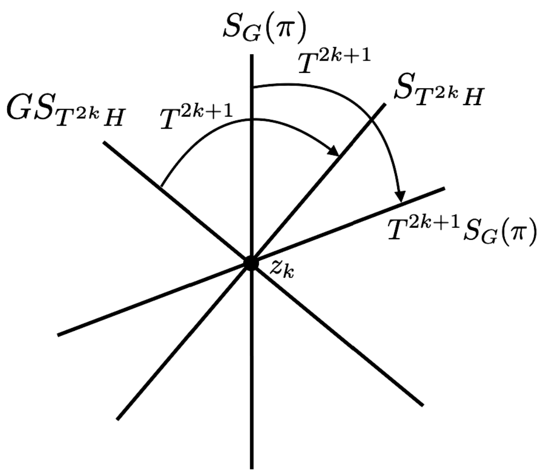

Fig. 3. The point $z_k \in S_G(\pi)$ is the orbital point of $1/(2k+1)$-BE. The configuration among the dominant axis $S_G(\pi)$, the subdominant axis $S_{T^{2k}H}$, the image $T^{2k+1}S_G(\pi)$ and the image $GS_{T^{2k}H}$ in the vicinity of z_k on cylinder.

Suppose that the curve represented by $y = F(x)$ passes through $z = (x, y) \in S_G(\pi)$. The image of $y = F(x)$ with respect to involution G is

$$y = F(2\pi - x) - f(x) \equiv F_G(x). \tag{9}$$

At $z = (x, y)$, let $\xi(z) = dF(x)/dx$ be the slope of $F(x)$ and $\xi_G(z) = dF_G(x)/dx$ be the slope of $F_G(x)$ at z. We obtain the relation:

$$\xi(z) + \xi_G(z) = a. \tag{10}$$

Differentiating T with respect to x_n, we obtain the map to determine the relation between the slope $\xi_n = dy_n/dx_n$ and $\xi_{n+1} = dy_{n+1}/dx_{n+1}$.

$$\xi_{n+1} = \frac{(\xi_n + f'(x_n))}{(\xi_n + f'(x_n) + 1)} \tag{11}$$

where $f'(x_n) = a\cos x_n$. We note the following relations.

$$\frac{dy_{n+1}}{dx_n} = \frac{dy_{n+1}}{dx_{n+1}}\frac{dx_{n+1}}{dx_n} = \xi_{n+1}(\xi_n + f'(x_n) + 1).$$

Differentiating Eq. (11) with respect to x_n, we obtain the map for the second derivative $\eta_n = d\xi_n/dx_n$.

$$\eta_{n+1} = \frac{(\eta_n - f(x_n))}{(\xi_n + f'(x_n) + 1)^3}. \tag{12}$$

Here $f''(x_n) = -f(x_n) = -a\sin x_n$ is used.

2.3 Properties of the subdominant axis

We remark that $1/(2k+1)$-BE has one orbital point z_k which is the intersection point of DA and SD. Let us discuss the relative disposition of the dominant and subdominant axes in the vicinity of z_k. The discussion is done on the universal cover: $-\infty < x < +\infty, -\infty < y < +\infty$ restricted to $0 \le y \le 2\pi$. The image $T^kS_H(0)$ is a curve connecting $(0, 0)$ and $((2k+1)\pi, 2\pi)$, and passes through $z_k \in S_G(\pi)$. The image $T^kS_H(0)$ is the graph of a monotone increasing function in $0 \le x \le \pi$ as is easily shown by the use of T. Thus, $T^kS_H(0)$ is the graph in the vicinity of z_k. For example, the vertical line passing through z_0 leans to the right direction under the operation of T and the slope of image is one. This movement is displayed in Fig. 2.

From the relations $T^{2k+1}S_G(\pi) = (T^{2k}H)(GS_G(\pi)) = (T^{2k}H)S_G(\pi)$, we obtain that the subdominant axis $S_{T^{2k}H}$ is sandwiched by $S_G(\pi)$ and its image $T^{2k+1}S_G(\pi)$ (Fig. 3).

Next, we give the relation of the subdominant axis and the period-doubling bifurcation.

Proposition 2.3. Let $z_k \in S_G(\pi)$ be the orbital point of $1/(2k+1)$-BE. Let $\xi(z_k)$ and $\xi_G(z_k)$ respectively be the slopes of the subdominant axis $T^kS_H(0)$ and its image $GT^kS_H(0)$ at z_k. Then, there exists a critical parameter value $a_c(1/(2k+1))$ of the period-doubling bifurcation such that (i)–(iii) hold.
(i) $\xi_G(z_k) < a/2 < \xi(z_k)$ for $0 < a < a_c(1/(2k+1))$, and z_k is an elliptic point;
(ii) $\xi_G(z_k) = a/2 = \xi(z_k)$ at $a = a_c(1/(2k+1))$; and
(iii) $\xi_G(z_k) > a/2 > \xi(z_k)$ for $a > a_c(1/(2k+1))$, and z_k is a saddle point with reflection.

Proof. We restrict $S_H(0)$ to $0 \le x \le \pi$ and call it with the same name. Then, its left and right end points are $P = (0, 0)$ and $R_0 = (\pi, 2\pi)$. The left and right points of $T^kS_H(0)$ are P and $R_k = ((2k+1)\pi, 2\pi)$. Let z_k

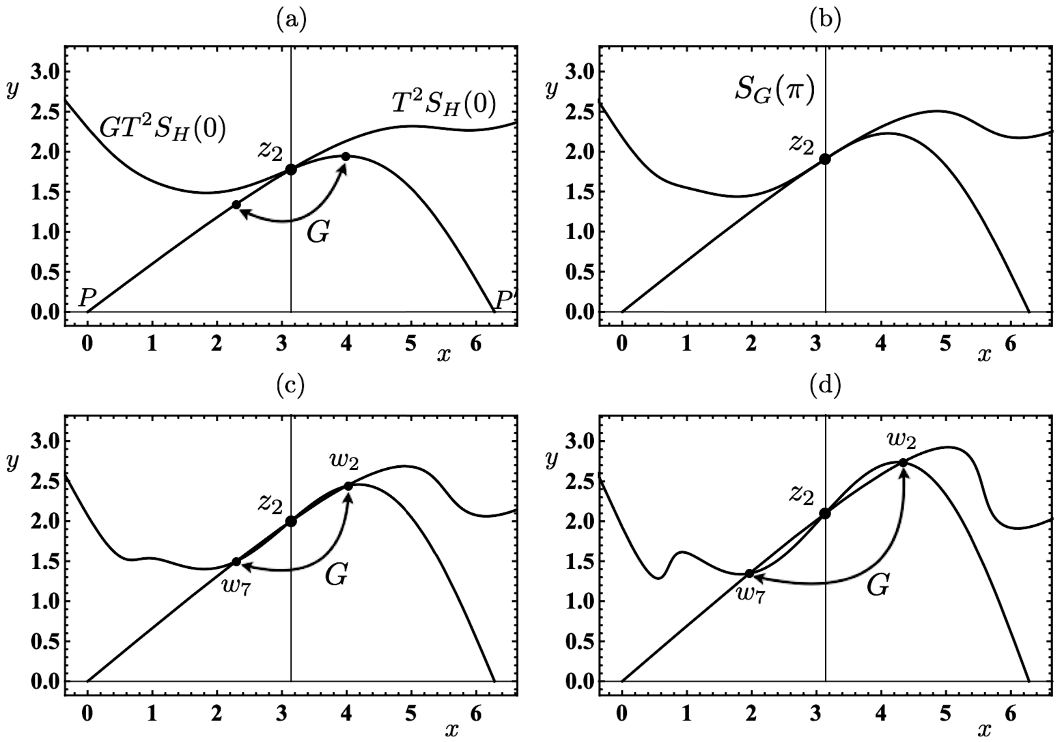

Fig. 4. (a) Before the period-doubling bifurcation of 1/5-BE. $a = 0.8$. (b) Bifurcation point. $a = a_c(1/5) = 1.047938$. (c) After the bifurcation. $a = 1.25$. (d) After the bifurcation. $a = 1.5$.

be the intersection point of $T^kS_H(0)$ and $S_G(\pi)$. We note that $GP = P' = (2\pi, 0)$ and $GR_k = (-(2k-1)\pi, 2\pi)$. The image $GT^kS_H(0)$ is the curve connecting GR_k and P', and passes through $z_k = (\pi, y_k)$. The slope of $S_H(0)$ is 2 which is of course smaller than that of the vertical line containing z_0. The image of the vertical line containing z_m $(0 \le m < k)$ under T has slope 1. Then the slope of $T^{m+1}S_H(0)$ at z_{m+1} is less than 1. From this fact, we have the relation $\xi(z_k) < 1$ (see Fig. 2).

From Eq. (10), we have $\xi_G(z_k) = a - \xi(z_k)$. At $a = 0$, we have $\xi(z_k) = 2/(2k+1) > 0$ and $\xi_G(z_k) = -\xi(z_k) < 0$. For $a \ge 2$, we have $\xi_G(z_k) = a - \xi(z_k) > 1 > \xi(z_k)$. From this fact, there exists a value of a (> 0) such that the relation $\xi_G(z_k) = \xi(z_k)$ holds. Let this critical value be $a^*(1/(2k+1))$.

Suppose the situation that $\xi_G(z_k) > \xi(z_k)$ holds. GR_k and P', respectively, are in the upper and lower region of $\{(x, y) : -(2k-1)\pi \le x \le (2k+1)\pi, 0 \le y \le 2\pi\}$ divided by $T^kS_H(0)$. Thus, of course, $GT^kS_H(0)$ connecting P' and GR_k intersects $T^kS_H(0)$ at least at one point. The orbital point z_k is one of the intersection points. Due to $\xi_G(z_k) > \xi(z_k)$, the arc of $GT^kS_H(0)$ connecting z_k and P' intersects $T^kS_H(0)$ in the region satisfying $x > \pi$. Let the intersection point be w. The arc of $GT^kS_H(0)$ connecting z_k and GR_k also intersects $T^kS_H(0)$ in the region satisfying $x < \pi$. Let the intersection point be w^*. We remark that the relation $w^* = Gw$ holds and the relations $T^{(2k+1)}w = w^*$ and $T^{(2k+1)}w^* = w$ hold. Therefore, w and w^* are the points of the periodic orbit with period $2(2k+1)$. This orbit does not exist if $\xi_G(k) < \xi(k)$ when parameter a is small. Consequently, this orbit is born at some value of a when $\xi_G(k) = \xi(k)$ through period-doubling bifurcation. We denote the value of a by $a_c(1/(2k+1)) = a^*(1/(2k+1))$.

The above bifurcation process implies that point z_k is an elliptic point before the bifurcation, and is a saddle point with reflection after the bifurcation. Thus, Proposition is proved.

We introduce the example to understand Proposition 2.3. The bifurcation process of 1/5-BE are depicted in Fig. 4 where SD $(T^2S_H(0))$ and its image $GT^2S_H(0)$ are also shown. From z_2, the daughter periodic points w_2 and w_7 are born. The relation $w_2 = Gw_7$ holds. Let the slope of SD at z_2 be $\xi(z_2)$ and that of the image be $\xi_G(z_2)$. We can confirm the relations : $\xi_G(z_2) < a/2 < \xi(z_2)$ in Fig. 4(a) before the period-doubling bifurcation, $\xi_G(z_2) = a/2 = \xi(z_2)$ in Fig. 4(b) at the bifurcation point and $\xi_G(z_2) > a/2 > \xi(z_2)$ in Figs. 4(c) and (d) after the bifurcation.

3. Proof of Theorem 1.1

3.1 Preparations

Let $z = (x, y)$ be any point in region $\mathcal{D} = \{(x, y) : 0 < x < \pi$ and $y > 0\}$, and $Tz = (x', y')$ be its image. Let $\pi_x(z)$ be the x-coordinate of z and $\pi_y(z)$ be the y-coordinate. Let $[A, B]_{\mathcal{C}}$ be a closed arc on a curve or a one dimensional manifold $\mathcal{C}$ where $A, B \in \mathcal{C}$. Thus, for example, $[A, B]_{W_u(P)}$ is a closed arc on the unstable manifold $W_u(P)$. We similarly define an arc of $W_s(P')$.

From the fact $f(x) > 0$ for $z = (x, y) \in \mathcal{D}$, the relation $y' = y + f(x) > y$ is derived. From the relation $x' = x + y'$, we obtain the relation $x' > x$. As a result, the image Tz locates to the upper-right of z. Thus, we obtain Proposition 3.1.

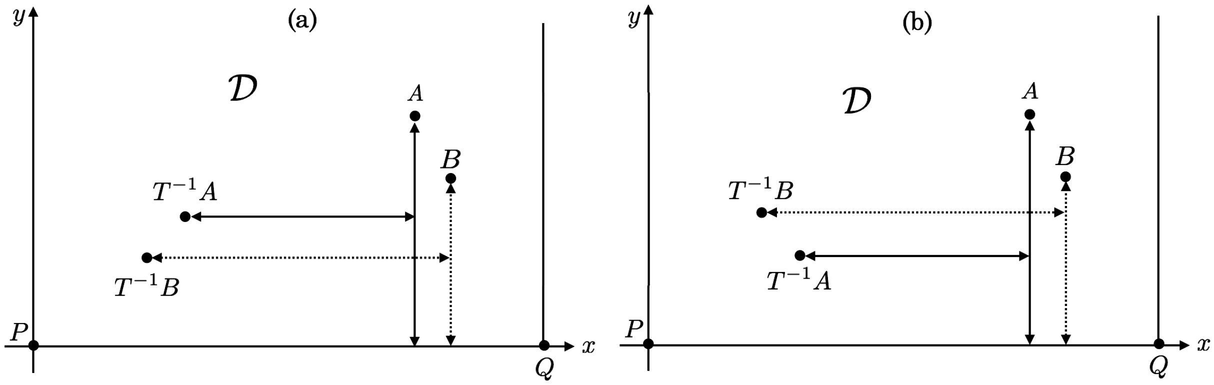

Fig. 5. Proof of the order preservation of left-right.

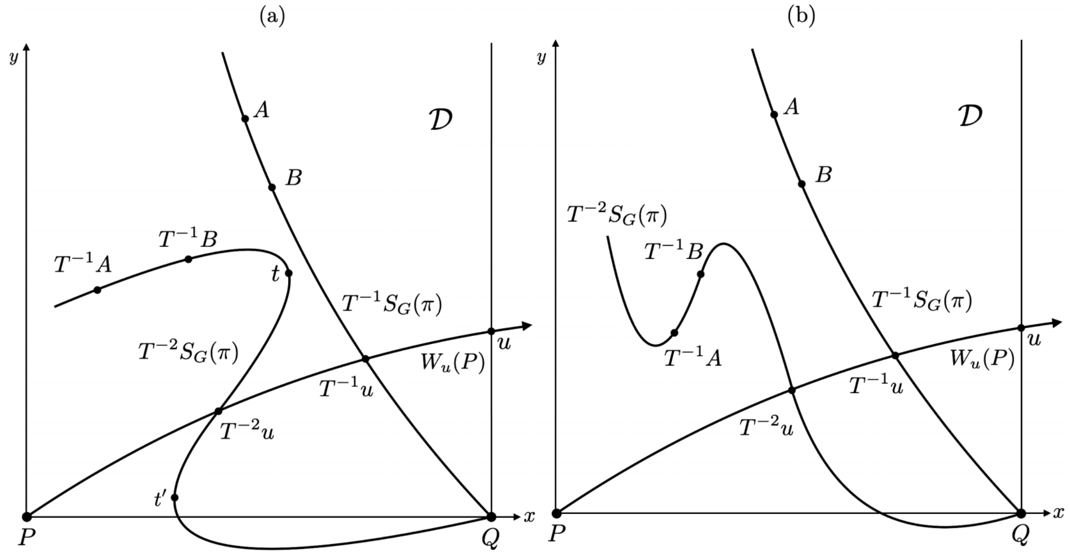

Fig. 6. Two situations satisfying the condition that the relation $\pi_y(T^{-1}A) < \pi_y(T^{-1}B)$ holds.

Proposition 3.1. If z is in $\mathcal{D} = \{(x, y) : 0 < x < \pi \text{ and } y > 0\}$, Tz is to the upper-right of z. If z is on $S_G(\pi)$, then $Tz = (\pi + \pi_y(z), \pi_y(z))$.

Let z be the point of $W_u(P)$, and be close to P along $W_u(P)$. From Property 3.1, $T^k z$ for some $k > 0$ will be in the region $x > \pi$ and $y > 0$. This implies the existence of the first intersection point u of $W_u(P)$ with $S_G(\pi)$. Let us introduce Proposition 3.2 (Yamaguchi and Tanikawa, 2000).

Proposition 3.2. The slope of $\Gamma_u = [P, u]_{W_u(P)}$ is positive and its curvature is negative.

Here we define two arcs γ_u and γ_s (see Fig. 1).

Definition 3.3.

$$\gamma_u = [v, Tu]_{W_u(P)}, \tag{13}$$

$$\gamma_s = [u, v]_{W_s(P')}. \tag{14}$$

where $\gamma_s = H\gamma_u$.

Proposition 3.4. Suppose that A and B locate in $\mathcal{D}$ satisfying the conditions $\pi_x(A) < \pi_x(B)$ and $\pi_y(A) > \pi_y(B)$. The relation $\pi_x(T^{-1}A) < \pi_x(T^{-1}B)$ holds.

Proof. Assume that the relation $\pi_x(T^{-1}A) > \pi_x(T^{-1}B)$ holds. This configuration is depicted in Fig. 5. In Fig. 5(a), the situation satisfying the condition $\pi_y(T^{-1}A) > \pi_y(T^{-1}B)$ is shown. The length of solid vertical arrow is equal to that of solid horizontal one. While the length of dotted vertical arrow is less than that of dotted horizontal one. This contradicts the property of T. The same contradiction is derived for the situation shown in Fig. 5(b). For the situation satisfying the condition $\pi_x(T^{-1}A) = \pi_x(T^{-1}B)$, using the reason mentioned above, the contradiction is derived.

Proposition 3.5. Suppose that A and B locate on $T^{-n}S_G(\pi) \in \mathcal{D}$ for $n \geq 1$ where the relation $\pi_x(A) < \pi_x(B)$ holds. Two images $T^{-1}A$ and $T^{-1}B$ locate on $T^{-n-1}S_G(\pi) \in \mathcal{D}$. Thus, (i)–(iii) hold.
(i) $\pi_y(A) > \pi_y(B)$.
(ii) $\pi_x(T^{-1}A) < \pi_x(T^{-1}B)$ (Order preservation of left-right).
(iii) $\pi_y(T^{-1}A) > \pi_y(T^{-1}B)$.

Proof of (i). First, we consider the case with $n = 1$. The image $T^{-1}S_G(\pi) \in \mathcal{D}$ is represented as

$$y = \pi - x - f(x)\ (> 0). \tag{15}$$

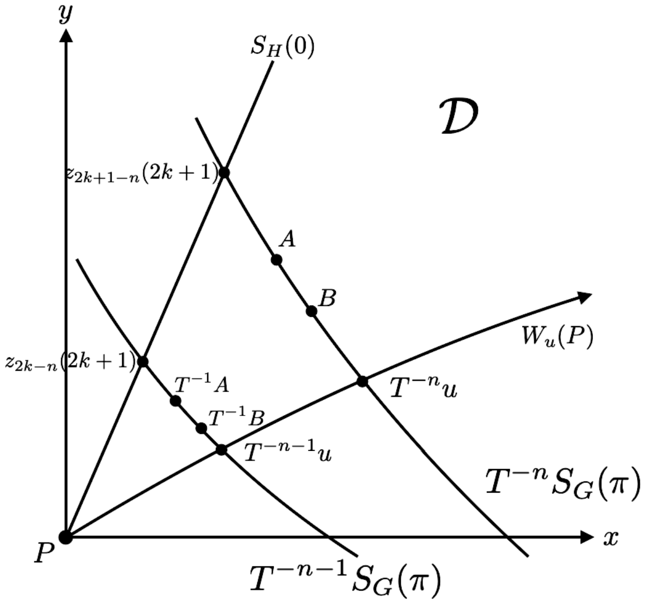

Fig. 7. The appropriate configuration.

The slope $dy/dx = -1 - f'(x)$ is negative for $T^{-1}S_G(\pi) \in \mathcal{D}$. Thus, (i) for $n = 1$ is proved. We remark that the second derivative $d^2y/dx^2 = -f''(x) = f(x)$ is positive. The curvature of $T^{-1}S_G(\pi) \in \mathcal{D}$ is positive.

Proof of (ii). From Proposition 3.4, (ii) is derived.

Proof of (iii). In order to prove (iii), we prove that the slope of $T^{-2}S_G(\pi) \in \mathcal{D}$ is negative. We derive the contradiction if the relation $\pi_y(T^{-1}A) < \pi_y(T^{-1}B)$ holds. From the assumption, there exists a portion of $T^{-2}S_G(\pi) \in \mathcal{D}$ satisfying the condition that the slope is positive. This situation is displayed in Fig. 6.

This situation displayed in Fig. 6(a) includes two contradictions. The first fact is the existence of two turning points t and t'. The second one is the existence of portion whose curvature is negative. For example, the curvature of $[T^{-1}A, t]_{T^{-2}S_G(\pi)}$ is negative. In Fig. 6(b), the turning point does not exist but there exists the portion whose curvature is negative (see $[T^{-1}B, T^{-2}u]_{T^{-2}S_G(\pi)}$).

We have the relation.

$$\xi(Tt) = \frac{(\xi(t) + f'(x_t))}{(\xi(t) + f'(x_t) + 1)}. \tag{16}$$

Suppose that $\xi(t)$ diverges and the image Tt locates on $T^{-1}S_G(\pi) \in \mathcal{D}$. Thus, we have that the slope $\xi(Tt) = 1$ is positive. This contradicts the fact that the slope of $T^{-1}S_G(\pi) \in \mathcal{D}$ is negative.

In $T^{-1}S_G(\pi) \in \mathcal{D}$, there is no maximum point at which the slope is zero and is no turning point at which the slope diverges. Combining these facts and $\xi(Tt) < 0$, we obtain that the denominator of Eq.(16) is positive and the numerator is negative.

Next, we use the following relation:

$$\eta(Tt) = \frac{(\eta(t) - f(x_t))}{(\xi(t) + f'(x_t) + 1)^3}. \tag{17}$$

Since the denominator is positive, and $\eta(Tt)$ and $f(x_t)$ are positive, we have the relation $\eta(t) > f(x_t) > 0$. This implies that the curvature of $T^{-2}S_G(\pi) \in \mathcal{D}$ is positive.

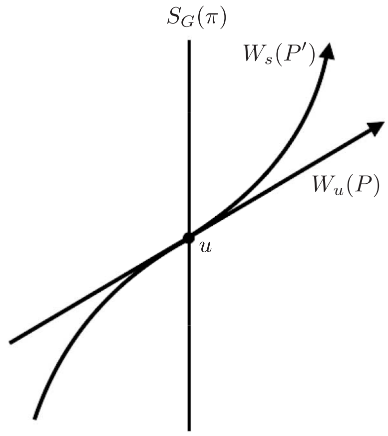

Fig. 8. Non-transverse intersection of $W_u(P)$ and $W_s(P')$ at u.

The situation satisfying the relation $\pi_y(T^{-1}A) < \pi_y(T^{-1}B)$ has a contradiction. The same contradiction is derived for the case satisfying the relation$\pi_y(T^{-1}A) = \pi_y(T^{-1}B)$. Thus, (iii) is proved.

Repeating the above mentioned procedure for $n = 2, 3, \cdots$, Proposition 3.5 is proved.

The appropriate configuration for A, B, $T^{-1}A$ and $T^{-1}B$ is illustrated in Fig. 7.

Proposition 3.6. Let Γ_k be $[P, z_k(2k+1)]_{T^kS_H(0)}$ $(k \geq 1)$. In the limit $k \to \infty$, Γ_k accumulates at Γ_u.

Proof. Suppose the contrary that points $z_k(2k+1)$ accumulate at u^* which is above u on $S_G(\pi)$ and Γ_k accumulates Arc$[P, u^*]$ which is the invariant curve passing through P. By linear stability analysis, it is proved that there exist the stable manifold $W_s(P)$ and the unstable manifold $W_u(P)$. There is no invariant curve passing through P except these manifolds. The existence of Arc$[P, u^*]$ contradicts this fact. Thus, Arc$[P, u^*]$ is Γ_u.

3.2 Monotone inclination property

Let us consider $1/q_k$-BE where $q_k = 2k + 1$ $(k \geq 1)$ is the period. Let $S_H^+(0)$ be the part of the symmetry axis $y = 2x$ with $y > 0$. Let $z_0(q_k) \in S_H^+(0)$ and $z_k(q_k) \in S_G(\pi)$ be the orbital point of $1/q_k$-BE. We remark that the orbital points $z_j(q_k)$ $(1 \leq j \leq k-1)$ locate in the region sandwiched by $S_H(0)$ and $S_G(\pi)$ with $y > 0$. For simplicity, the abbreviation $\xi_j(q_k)$ for the slope $\xi(z_j(q_k))$ at $z_j(q_k)$ $(0 \leq j \leq k)$ will be used in the following discussion.

Proposition 3.7. For $q_k = 2k + 1$ $(k \geq 1)$, the following relations hold.

$$\xi_k(q_k) > \xi_{k+1}(q_{k+1}), \tag{18}$$

$$\lim_{k \to \infty} \xi_k(q_k) = \xi_u(u). \tag{19}$$

Proof of Eq. (18).

We are going to prove the inequality by induction. The relations $\pi_y(z_1(3)) > \pi_y(z_2(5)) > \pi_y(z_3(7)) > \cdots > \pi_y(z_k(2k+1)) > \cdots$ hold on $S_G(\pi)$ by Proposition 3.6.

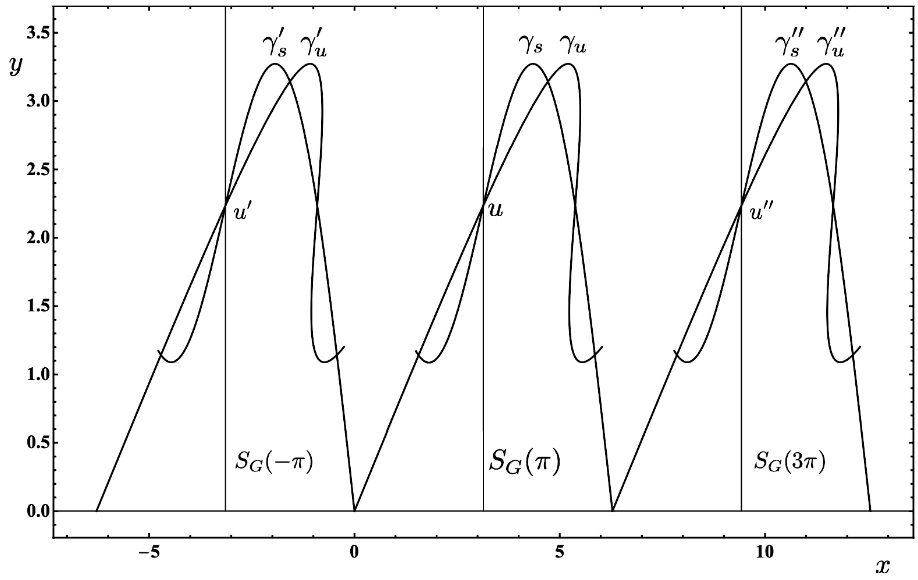

Fig. 9. Two arcs γ_u' and γ_s' exist in the region $-\pi < x < 0$, γ_u and γ_s in the region $\pi < x < 2\pi$ and γ_u'' and γ_s'' in the region $3\pi < x < 4\pi$ on the universal cover.

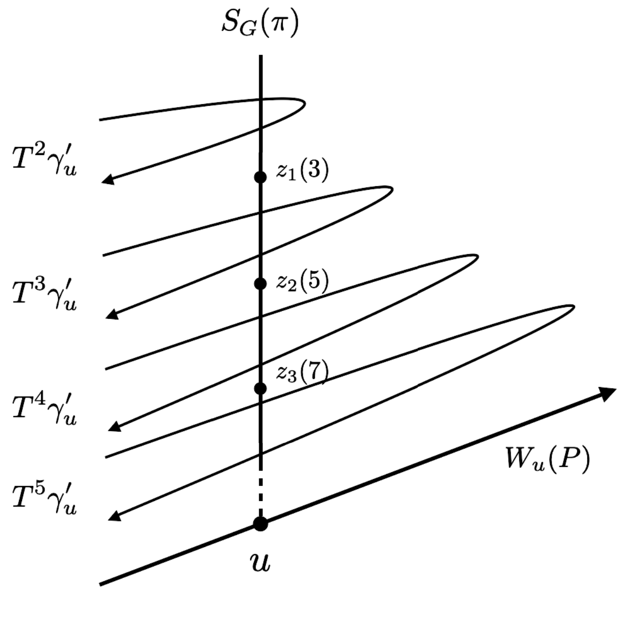

Fig. 10. Configuration of orbital points $z_k(2k+1)$ $(k = 1, 2, 3)$, $T^{k+1}\gamma_u'$ $(k = 1, 2, 3, 4)$ and u for a large $a > 0$ on the universal cover.

First, we discuss the slopes of the images of symmetry axis. Take, for $q_k = 2k+1$, two points $z_0(q_k)$ and $z_0(q_{k+1})$ on $S_H(0)$. Here, these points are the points of $1/q_k$-BE and $1/q_{k+1}$-BE. Trivially, $\xi_0(q_k) = \xi_0(q_{k+1}) = 2$. The slope $\xi_1(q_{k+1})$ is determined by the relation (see Eq. (11)).

$$\xi_1(q_{k+1}) = \frac{\xi_0(q_{k+1}) + f'(x_0)}{\xi_0(q_{k+1}) + f'(x_0) + 1} = \frac{2 + f'(x_0)}{3 + f'(x_0)}. \quad (20)$$

where $z_0(q_k) = (x_0(q_k), y_0(q_k))$, and $x_0 = x_0(q_{k+1})$. $f'(x) \equiv df(x)/dx$ is the slope of the graph of function $f(x)$ at x.

The point $(\pi/3, 2\pi/3)$ on $S_H(0)$ is mapped by T to the point $(\pi + \sqrt{3}a/2, 2\pi/3 + \sqrt{3}a/2)$ out of $\mathcal{D}$ with $x > \pi$, whereas points $z_j(q_k)$ $(0 \le j \le k-1)$ stay in $\mathcal{D}$. Thus, we have $0 < x_0 < \pi/3$ because the initial point $z_0(q_k) \in S_H(0)$ is below $(\pi/3, 2\pi/3)$. This implies that the numerator $2 + a\cos x_0$ and the denominator $3 + a\cos x_0$ are positive, hence $0 < \xi_1(q_{k+1}) < 1$. From this relation, we obtain the relation $\xi_0(q_k) = 2 > \xi_1(q_{k+1}) > 0$.

Next, suppose that the relation $\xi_{j-1}(q_k) > \xi_j(q_{k+1})$ holds. The slopes $\xi_j(q_k)$ and $\xi_{j+1}(q_{k+1})$ are derived as follows.

$$\xi_j(q_k) = \frac{\xi_{j-1}(q_k) + f'(x_{j-1})}{\xi_{j-1}(q_k) + f'(x_{j-1}) + 1} = \frac{B}{A}, \quad (21)$$

$$\xi_{j+1}(q_{k+1}) = \frac{\xi_j(q_{k+1}) + f'(x_j)}{\xi_j(q_{k+1}) + f'(x_j) + 1} = \frac{D}{C}. \quad (22)$$

where $x_{j-1} = x_{j-1}(q_k)$ and where $x_j = x_j(q_{k+1})$.

From Eq. (21), $T^k S_H(0)$ is vertical at $x_j(q_k)$ if $A = 0$, while $T^k S_H(0)$ is horizontal at $x_j(q_k)$ if $B = 0$. Therefore, B is positive for sufficiently large $k > 0$ because the slope $x_j(q_k)$ becomes close to the slope of $W_u(P)$ which is positive. Hence we have $A > 0$. In a similar manner, we derive that C and D are positive.

We calculate the difference

$$\xi_j(q_k) - \xi_{j+1}(q_{k+1}) = \frac{BC - AD}{AC} \quad (23)$$

where $AC > 0$ and

$$BC - AD = (\xi_{j-1}(q_k) - \xi_j(q_{k+1})) + (f'(x_{j-1}) - f'(x_j)). \quad (24)$$

From the induction hypothesis, the first term in Eq. (24) is positive. The second term is positive because the order preservation of left-right (Proposition 3.5(ii)) gives $x_{j-1} < x_j$ in $\mathcal{D}$ and hence $f'(x_{j-1}) > f'(x_j)$. As a result, $\xi_j(q_k) > \xi_{j+1}(q_{k+1})$ is proved. Here, we let $j = k$ and obtain Eq. (18).

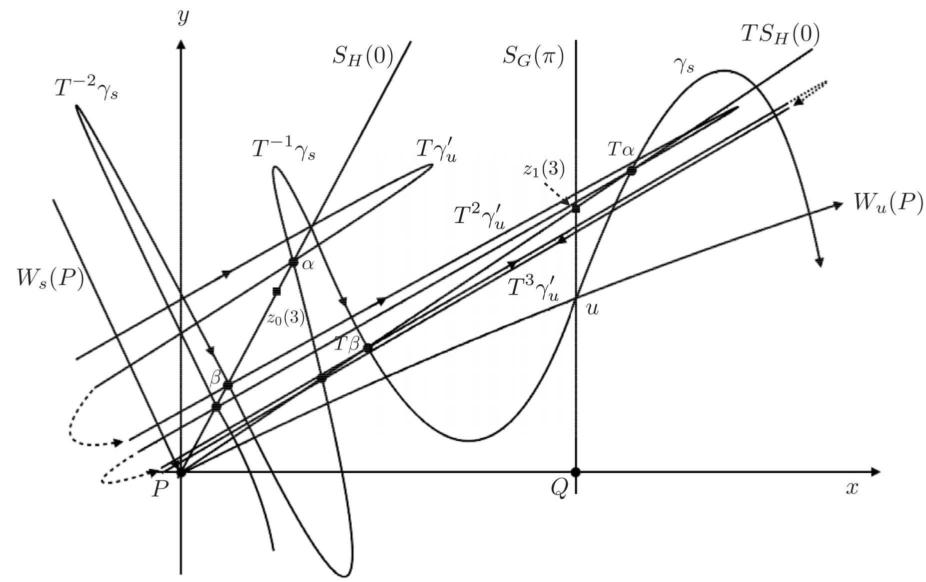

Fig. 11. Relations among $T^2\gamma_u'$, $T^3\gamma_u'$ and $z_1(3) \in S_G(\pi)$ on the universal cover.

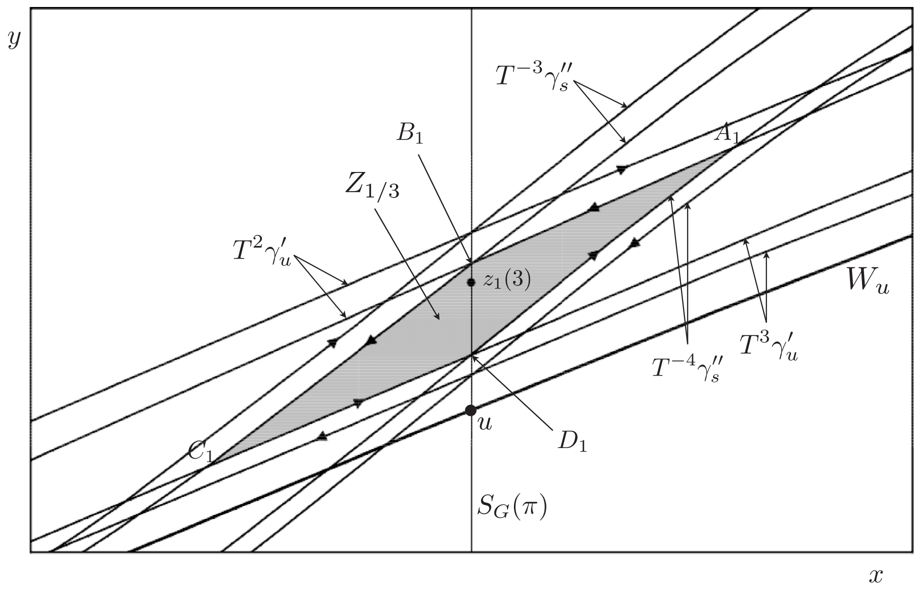

Fig. 12. A parallelogram $A_1B_1C_1D_1$ represents the hyperbolic region $Z_{1/3}$ on the universal cover.

Proof of Eq. (19)

In the limit $k \to \infty$, the orbital points $z_j(q_k)$ accumulate at u (see the proof of Proposition 3.6). Suppose that Eq. (19) does not hold. There exists an integer n such that $T^n S_H(0)$ intersects $W_u(P)$ in the vicinity of u. This implies that $S_H(0)$ intersects $W_u(P)$ in the vicinity of P. This is a contradiction. Thus, Eq. (19) is proved.

3.3 Proof of Theorem 1.1

We know that $\xi_u(u) < a/2 < \xi_s(u)$ for $a \geq 4/3$ (Yamaguchi and Tanikawa, 2000). We may have $\xi_u(u) > a/2 > \xi_s(u)$ for some $a < 4/3$. In that case we should have $\xi_u(u) = \xi_s(u) = a/2$ at some $a = a^*$ ($0 < a^* < 4/3$). We display the disposition of $W_u(P)$ and $W_s(P')$ for $a = a^*$ in Fig. 8. At $a = a^*$, the elliptic points of $1/q_k$-BEs accumulate at u since the relations $\xi_k(q_k) > a/2$ for $k \gg 1$ holds (see Propositions 2.3). In the following, we assume $\xi_u(u) = \xi_s(u) = a/2$ at $a = a^*$ and derive a contradiction.

Here, Ushiki's theorem (Ushiki, 1980) is introduced. In our proof, Ushiki's theorem is essential.

Theorem 3.8 (Ushiki's theorem). The biholomorphic map $f : \mathbf{C}^n \to \mathbf{C}^n (n \geq 2)$ can not have a one-dimensional compact smooth invariant manifold. Such a map defined in the plane can not have a homoclinic connection or a heteroclinic connection.

The standard map T defined on the universal cover is biholomorphic. Therefore, the saddle connection between P and P' does not exist. This implies that the unstable manifold $W_u(P)$ and the stable manifold $W_s(P')$ intersect at u transversely or non-transversely. We plot Fig. 8 taking into account Ushiki's theorem. The existence of arcs γ_s and γ_s is guaranteed by Ushiki's theorem (see Fig. 1).

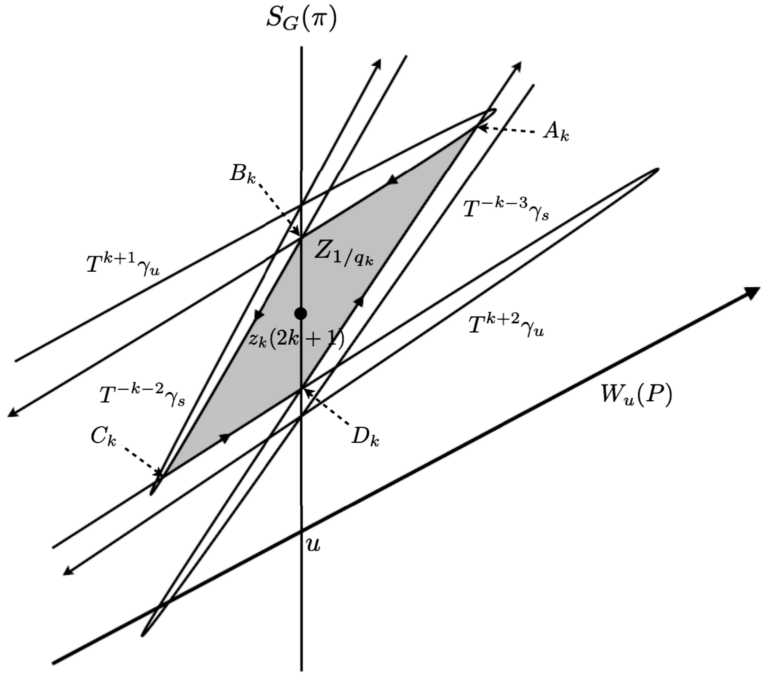

Fig. 13. Schematic illustration of hyperbolic region Z_{1/q_k} (gray region) on cylinder.

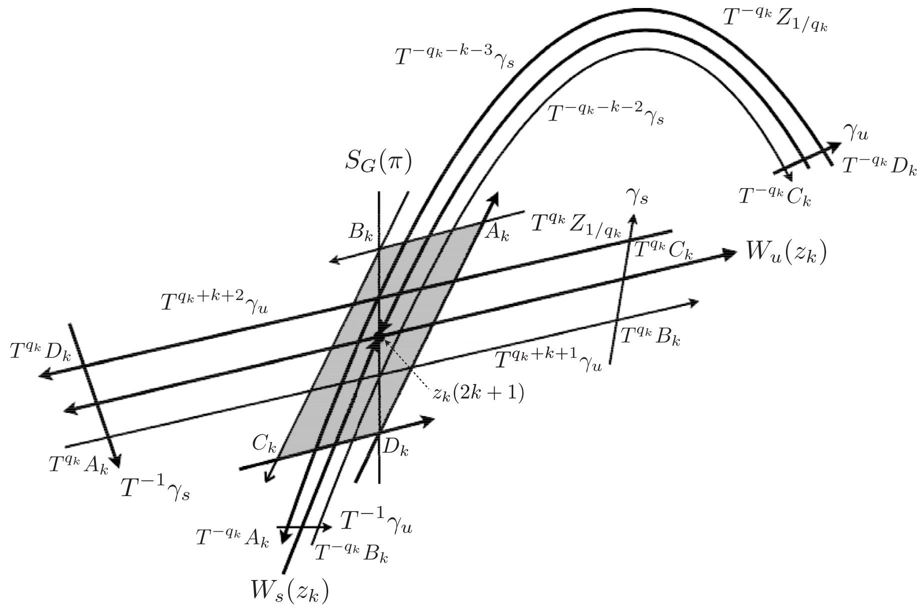

Fig. 14. Schematic illustration of the hyperbolic region Z_{1/q_k} (gray region), $T^{q_k}Z_{1/q_k}$ and $T^{-q_k}Z_{1/q_k}$ on cylinder.

In the following, we use the forward images of γ_u and the backward images of γ_s. These arcs are defined even if the intersections at u is not transverse. The information on the neighborhood of end points of γ_s and γ_u is not used. We remark that two arcs γ_u' and γ_s' exist in the region $-\pi < x < 0$ and γ_u'' and γ_s'' exist in the region $3\pi < x < 4\pi$ on the universal cover (see Fig. 9).

We first take a large $a > 0$, and gradually decrease the value. We consider the parameter region of a such that $T^{-1}\gamma_s$ intersects $S_H(0)$. We claim that point $z_k(2k+1)$ is sandwiched by the points $T^{k+1}\gamma_u' \cap S_G(\pi)$ and $T^{k+2}\gamma_u' \cap S_G(\pi)$ for $k \geq 1$. The situation for $k = 1, 2, 3$ is shown in Fig. 10.

In this situation, $T^{-2}\gamma_s$ also intersects $S_H(0)$ (see Fig. 11). The relations $HT^{-n}\gamma_s = T^n\gamma_u'$ and $H\gamma_s = \gamma_u'$ hold where the symmetry axis of H is $S_H(0)$. The intersection points $T^n\gamma_u' \cap T^{-n}\gamma_s$ ($n \geq 1$) locate on $S_H(0)$. Let one of intersection points of $T\gamma_u' \cap T^{-1}\gamma_s$ be α and one of intersection points of $T^2\gamma_u' \cap T^{-2}\gamma_s$ be β. These points are displayed in Fig. 11. The image $T\alpha \in \gamma_s$ locates in the region $\pi < x < 2\pi$ and $T\beta \in T^{-1}\gamma_s$ in the region $0 < x < \pi$. This implies that $\mathrm{Arc}[T\beta, T\alpha]_{TS_H(0)}$ intersects $S_G(\pi)$. This intersection point is the orbital point of 1/3-BE. The point is $z_1(3)$. Repeating the method used here, we obtain the relative configuration among $T^{k+1}\gamma_u'$ and $z_k(2k+1)$ for $k \geq 1$. Thus, our claim is proved.

Let us construct the hyperbolic region $Z_{1/3}$ of $z_1(3)$ (see Fig. 12). We consider the situation such that $T^2\gamma_u'$ intersects $T^{-4}\gamma_s''$. Let one of the intersection points be A_1 and the lower intersection point of $T^2\gamma_u'$ and $S_G(\pi)$ be B_1. We obtain the relation $T^{-3}\gamma_s'' \cap T^3\gamma_u' \neq \emptyset$ from $\emptyset \neq G(T^2\gamma_u' \cap T^{-4}\gamma_s'') = T^{-2}\gamma_s'' \cap T^4\gamma_u'$. Let one of the intersection points of $T^{-3}\gamma_s'' \cap T^3\gamma_u'$ be C_1 and the upper intersection point of $T^3\gamma_u'$ and $S_G(\pi)$ be D_1. As a result, a closed region $A_1B_1C_1D_1$ of the form of parallelogram (gray region in Fig. 12) is constructed. We name it the hyperbolic region $Z_{1/3}$ of $z_1(3)$.

In general, for a given $a > 0$, there is an integer $k_0 > 0$ such that $T^{k+1}\gamma_u'$ intersects $T^{-k-3}\gamma_s''$ for $k > k_0$. It is to be noted that we have $k_0 \to \infty$ when $a \to 0$. Consequently, we need to treat the hyperbolic regions for a large $k > 0$ if $a > 0$ is small. Now, $T^{k+1}\gamma_u'$ intersects $S_G(\pi)$ and the orbital point $z_k(2k+1)$ is sandwiched by $T^{k+1}\gamma_u'$ and $T^{k+2}\gamma_u'$. We can construct the hyperbolic region Z_{1/q_k} of $z_k(2k+1)$ on the universal cover. Rewriting γ_u' as γ_u and γ_s'' as γ_s, the hyperbolic region Z_{1/q_k} on cylinder is obtained (see Fig. 13).

Since the hyperbolic region Z_{1/q_k} includes $z_k(2k+1)$, the relations $Z_{1/q_k} \cap T^{q_k}Z_{1/q_k} \neq \emptyset$ holds. By T^{q_k}, Z_{1/q_k} is stretched in the longitudinal direction and is compressed to the vertical direction. It is noted that $\mathrm{Arc}T^{q_k}B_kT^{q_k}C_k$ is a portion of γ_s and $\mathrm{Arc}T^{q_k}D_kT^{q_k}A_k$ is that of $T^{-1}\gamma_s$. Comparing the original region $A_kB_kC_kD_k$ with image $T^{q_k}A_kT^{q_k}B_kT^{q_k}C_kT^{q_k}D_k$, we can confirm that the region in the vicinity of $z_k(2k+1)$ rotates by about 180 degree around $z_k(2k+1)$. The rotation is clockwise (see Fig. 3).

The region Z_{1/q_k} is stretched in the vertical direction and is compressed to the longitudinal direction under T^{-q_k}. It is noted that $\mathrm{Arc}T^{-q_k}A_kT^{-q_k}B_k$ is a portion of $T^{-1}\gamma_u$ and $\mathrm{Arc}T^{-q_k}C_kT^{-q_k}D_k$ is that of γ_u. The region in the vicinity of $z_k(2k+1)$ rotates by about 180 degree around $z_k(2k+1)$ counterclockwise. Summarizing the results, we have that the orbital point $z_k(2k+1)$ is the saddle point with reflection. The stable manifold $W_s(z_k)$ and the unstable manifold $W_u(z_k)$ are also displayed in Fig. 14 where $W_u(z_k)$ ($W_s(z_k)$) is an abbreviation of $W_u(z_k(2k+1))$ ($W_s(z_k(2k+1))$).

3.4 Proof of the transversality

Suppose that the non-transverse intersection appears at $a = a^*$ ($< 4/3$). At $a = a^*$, the elliptic points accumulate to u. In §3.3, the following (i) and (ii) are proved.

(i) The hyperbolic regions in the vicinity of u exist at $a = a^*$.

(ii) There exist an integer k such that the saddle point with reflection of $1/q_k$-BE exists in the vicinity of u at $a = a^*$.

As a result, the accumulation of elliptic points to u at $a = a^*$ contradicts (i) and (ii). There exists an integer n such that the relation $\xi_n(q_n) < a/2$ holds. From Eqs. (18) and (19), the relation $\xi_u(u) < a/2$ is derived. This implies the relations $\xi_u(u) < a/2 < \xi_s(u)$. The proof completes.

4. Conclusion

First, the monotone inclination property (Proposition 3.7) is derived. Next, using Ushiki's theorem, the hyperbolic region Z_{1/q_k} is constructed. The hyperbolic region includes the orbital point of $1/q_k$-BE which is the saddle point with reflection. Combining these facts, Theorem 1.1 is proved. In the proof, we use the properties of periodic points accumulating to u from the upper region of u on $S_G(\pi)$ and do not use the slope of $W_u(P)$ at u. In this sense, our proof is new.

Finally, we give the problems to be solved.

(1) The monotone inclination property plays an essential role in our proof. What phenomena happens in the map that the monotone inclination property does not hold?

(2) Elucidate the properties of periodic points accumulating to u from the lower region of u on $S_G(\pi)$. The orbits of these points rotate around Q.

(3) The transversality affects the stability of periodic points around the homoclinic point. Determine the relation of the transversality and the dominant axis property (Greene, 1979).

(4) What conditions are needed to appear the non-transverse intersection satisfying the relation $\xi_s(u) = \xi_u(u)$ or the situation satisfying the relation $\xi_s(u) < a/2 < \xi_u(u)$ (Tanikawa and Yamaguchi, 2001)?

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